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BASES FOR S4 AND S4.2 WITHOUT ADDED AXIOMS

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In this paper I shall show how the calculi S4 and S4.2 may be formulated without the use of axioms beyond those of the **CPC**. I shall make use of two of the rules given by A. N. Prior for S5 without added axioms[1]—rewording the proviso on the second of these rules for our purpose—and the definition '$M = NLN$', all to be subjoined to any base sufficient for the **CPC**.

The rules, as we shall use them are these:

RL1: $\vdash C\alpha\beta \rightarrow \vdash CL\alpha\beta$

RL2: $\vdash C\alpha\beta \rightarrow \vdash C\alpha L\beta$, if α is completely modalized.

As we shall see, the definition of "complete modalization" will depend upon the specific calculus with which we are working. If we say that α is completely modalized if and only if each of its propositional variables (**PV**) is either the whole or part of a formula beginning with 'L' or 'M', the system yielded is S5; if we appropriately vary the meaning of "complete modalization," we will find that the rules can yield S4 and S4.2 as well.

I.

We shall thus define "complete modalization" for S4:

"α is completely modalized if and only if either:

a.—it is a law of the system, every **PV** of which is part of or the whole of a formula beginning with 'L' or 'M', or

b.—it is of the form '$KL\gamma KL\delta \ldots L\nu$', with '$\alpha = L\gamma$' as a limiting case.

Before we go further, we may note that the way we have chosen to view the systems in question sees them as derived from the **PC** by a common set of rules of inference. The systems are distinguished as we state them not by characteristic axioms, but by characteristic "concepts of modalization." Modality in S5 is seen as a function of propositional variables alone, whereas in the other systems it is seen to depend to some extent upon the truth-

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functional operators of a formula as well: as clause (b) above suggests, 'K' is the only such operator which does not affect the modality of S4 formulae in which it is involved.

I shall now show that the rules **RL1** and **RL2** with complete modalization defined as above, and df: $M = NLN$ is equivalent to Lemmon's basis for S4.[2]

PC	Cpp		(1)
1, **RL1**	$CLpp$	—Lemmon's axiom 1—	(2)
PC	$CCpqCpq$		(3)
3, **RL1**, **PC**	$CKLCpqLpq$		(4)
4, **RL2**	$CKLCpqLpLq$		(5)
5, **PC**, **RL2**	$CLCpqLCLpLq$	—Lemmon's ax. 2—	(6)
Hyp.	$\vdash\alpha$		(7)
1, subst p/Lp	$CLpLp$		(8)
7, 8, **PC**	$\vdash CCLpLp\alpha$		(9)
9, **RL2**	$\vdash CCLpLpL\alpha$		(10)
8, 10, **PC**	$\vdash L\alpha$		(11)

Hence we have as a rule ("**RL**"):

7, 11	$\vdash\alpha \rightarrow \vdash L\alpha$.		(12)

which, with the two axioms proven above, constitutes Lemmon's basis for S4.

Now, working from Lemmon's basis and calling his axioms '*1' and '*2', and his rule '**RL**', we have:

Hyp.	$\vdash C\alpha\beta$	(1)
1, **RL**	$\vdash LC\alpha\beta$	(2)
2, *2, *1, **PC**	$\vdash CL\alpha L\beta$	(3)
3, *1, **PC**	$\vdash CL\alpha\beta$	(4)

Hence, derivable as a rule in Lemmon's basis is our **RL1**:

1, 4	$\vdash C\alpha\beta \rightarrow \vdash CL\alpha\beta$	(5)

Now we state as an additional hypothesis:

Hyp. α is fully modalized, which is to say either:

a.—it is a law of the system, all **PV** of which are modalized in Prior's sense.

b.—it is of the form '$KL\gamma KL\delta \ldots Lv$', with '$L\gamma$' as a limiting case. (6)

If 6a:	$\vdash\alpha$	(7)
1, 7, **PC**	$\vdash\beta$	(8)
8, **RL**	$\vdash L\beta$	(9)
9, **PC**	$\vdash C\alpha L\beta$	(10)

So, if 6a is satisfied, we have as a rule:

1, 10	$\vdash C\alpha\beta \rightarrow \vdash C\alpha L\beta$	(11)

And if 6b is satisfied, we have:

1, **RL**	$\vdash LC\alpha\beta$	(12)
12, *2, *1. **PC**	$\vdash CL\alpha L\beta$	(13)

Since both '$ELpLLp$' and '$ELKpqKLpLq$' are provable in Lemmon's basis:

6b, **PC**, **RL**	$\vdash LELa\alpha$	(14)

and by the interchangability of strict equivalents (**ISE**):

13. 14, **ISE**	$\vdash C\alpha L\beta$	(15)

So we have, for the definition of completely modalized as stated in 6:

1, 15	$\vdash C\alpha\beta \rightarrow \vdash C\alpha L\beta$,	if α is completely modalized,

which is, in fact, our **RL2**. **RL1** and **RL2**, then, with the stated proviso on **RL2**, are equivalent to Lemmon's basis for S4.

Note that whatever is completely modalized in the sense in which we use that expression in S4 will be completely modalized in S5 as well. Consistency with the modalization requirements of S5 is the reason for the way clause (a) of the S4 definition of 'complete modalization' is stated. The laws of S4 as we have stated it, then are also all laws of S5.

II.

We shall now show that the system S4.2[3] one of those intermediate between S4 and S5, may be handled in a similar manner. To obtain S4.2 ordinarily, we subjoin to a basis sufficient for S4 Geach's axiom:

*3. *CMLpLMp*.

We shall use this in conjunction with the Lemmon S4 basis and call the result the "Lemmon-Geach S4.2 basis." Since S4.2 is a stronger system than S4, we would expect that in formulating it with **RL1** and **RL2** that the S4 proviso on **RL2** must be extended to admit more forms as fully modalized. We shall do this by adding to our S4 definition the clause:

> c.—"α is fully modalized if it is of the form '$\phi\gamma$', where 'ϕ' is a prefix composed of alternating '*N*'s' and '*L*'s' (no '*N*' to follow an '*N*', or '*L*' to follow an '*L*'), beginning with '*N*', and containing *at least two* '*L*'s'.

Lemmon's *1, *2, and **RL** are provable in the system thus stated just as they were in our formulation of S4. Geach's axiom is provable thus:

*1, **PC**	*CNpNLp*		(1)
1, **RL1**, **RL2**	*CLNpLNLp*		(2)
PC, **RL2** (clause 'c')	*CNLNLpLNLNp*		(3)
df. *M*	*CMLpLMp*	--Geach's axiom--	(4)

In like manner, **RL1** and **RL2** for the S4 clauses of its proviso are provable in the Lemmon-Geach basis, as above for S4. For the added clause of the **RL2** proviso:

Hyp. α is of the form '$\phi\gamma$', where 'ϕ' is a prefix composed of alternating '*N*'s' and '*L*'s', beginning with '*N*', and containing at least two '*L*'s'. (1)

Hyp.	$\vdash C\alpha\beta$	(2)
2, **RL**, *2, *1, **PC**	$\vdash CL\alpha L\beta$	(3)
*1, subst p/α	$\vdash CL\alpha\alpha$	(4)

Since α is of the form '$\phi\gamma$' as in (1), there is a form '$ML\delta$' such that:

1, df. M	$\vdash LE\alpha ML\delta$	(5)
*3, subst p/$L\delta$	$\vdash CMLL\delta LML\delta$	(6)

But since S4.2 contains '$LELpLLp$',

6, **ISE**	$\vdash CML\delta LML\delta$	(7)
5, 7, **ISE**	$\vdash C\alpha L\alpha$	(8)
4, 8, **PC, RL**	$\vdash LE\alpha L\alpha$	(9)
3, 9, **ISE**	$\vdash C\alpha L\beta$	(10)

Hence the rule:

$\vdash C\alpha\beta \rightarrow \vdash C\alpha L\beta$, if α is completely modalized

is provable in the Lemmon-Geach basis for S4.2 from the S4.2 characteristic clause of the **RL2** proviso, and **RL1** and **RL2** with its proviso properly stated are equivalent to the Lemmon-Geach S4.2.

It may be noted in concluding that the **RL2** proviso for S5 may be stated in terms of the S4 and S4.2 provisos. If we change the clause of the proviso characteristic of S4.2 to read ". . . and containing at least *one* 'L'," the proviso as a whole becomes equivalent to Prior's "if all the variables in α are modalized"—for '$CNLpLNLp$' is then in the system.

REFERENCES

[1] *Cfr.* A. N. Prior: *Formal Logic*, 2nd ed. (Oxford, 1962) p. 312.

[2] *Cfr.* E. J. Lemmon, New Foundations for Lewis Modal systems. *The Journal of Symbolic Logic*, v. XXII (1957), pp. 176-88.

[3] *Cfr.* M. A. E. Dummett and E. J. Lemmon: Model Systems Between S4 and S5. *Zeitschrift für matematische Logik und Gründlagen der Mathematik*, v. V(1959), pp. 250-264.

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