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A STIPULATION OF A MODAL PROPOSITIONAL CALCULUS IN TERMS OF MODALIZED TRUTH-VALUES

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The logical truths of a non-modal propositional calculus may be identified as the theorems following from certain axioms by certain rules. Alternatively, they may be identified by means of two-valued truth-tables as those propositions whose truth-tables have only **T's** in the main column. Stipulations of the logical truths of modal propositional calculi have been given in terms of axioms and rules, but not, as far as I know, in terms of what I will call the modalized truth-values (**MTVs**), viz. logical truth, contingent truth, contingent falsity, and logical falsity. I attempt the latter type of stipulation in the present note, using as a guide throughout the two-valued truth-value stipulation.[1]

Let the object-language consist of a non-modal propositional calculus with 'v' and '~' as primitive connectives, with propositional constants but no variables, to which is added the functor '4' with '4(p)' being well-formed if p is. The intended interpretation of '4(p)' is 'It is logically false that p' or '*It is logically impossible that p*'.

The definition of 'logical truth' will be given in terms of an alteration or rewriting of the usual tables of a four-valued logic. The four values will be represented by '1', '2', '3', and '4', corresponding respectively to logical truth, contingent truth, contingent falsity, and logical falsity. The basic matrices used are:

p	$\sim p$	$4p$
1	4	4
2	3	4
3	2	4
4	1	1

v	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

I submit that the content of these matrices is intuitively reasonable, except for the places (2,3) and (3,2) in the disjunction matrix, to which I will return below. By saying they are intuitively reasonable, I mean that, for example, to assign the value 4, logical falsity, to the denial of a proposition whose **MTV** is 1, logical truth, accords with our intuitive handling of these concepts.

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The following definitions of other modal functors in terms of '4' and the usual propositional connectives suggest themselves:

$$1p = df.\ 4{\sim}p$$
$$2p = df.\ p \cdot {\sim}1p$$
$$3p = df.\ {\sim}p \cdot {\sim}4p$$

'It is logically possible that p' may be identified with '$\sim 4p$'. Lewis' definition of strict implication is, in the present notation, $4(p \cdot {\sim}p)$, which has the same matrix as $1(p \supset q)$. His $p = q$ (strict equivalence) may be identified with $1(p \equiv q)$.

A four-valued table for a proposition p constructed according to these matrices and definitions I will call *the regular truth-table for p,* since it is the usual table of a four-valued logic based on these matrices. The table is to be constructed by writing 4^n rows for a proposition containing n propositional constants in such a way that all possible combinations of assignment of **MTVs** to the propositional constants are exhausted, the **MTVs** in the remaining places being determined according to the basic matrices.

Now intuitively a logical truth should have either a 1 or a 2 in every row of the main column of its truth-table, since a logical truth is a proposition which would be true in any logically possible circumstances. However, using the regular truth-tables and taking 1 and 2 as designated values will not suffice to formulate a reasonable criterion of logical truth, since it would be too restrictive. The regular truth-table (given below) for $1(A \vee {\sim}A)$, for example, has 4's in two rows of its main column.

An intuitively reasonable definition of logical truth can, however, be given in terms of the following alteration of the regular tables. Beginning with the smallest constituents in the given proposition p, rewrite any column of the regular truth-table for p which contains 1 or 2 in every row as all 1's and any column which contains 3 or 4 in every row as all 4's, leaving columns of other sorts unchanged; proceed step-wise in this way until the column for p itself is rewritten (if necessary). For example, the regular truth-table for $1(A \vee {\sim}A)$ is:

$1($	A	$\vee$	$\sim A)$
1	1	1	4
4	2	2	3
4	3	2	2
1	4	1	1

The only columns to be rewritten here are those for the constituent $A \vee {\sim}A$ and for the whole proposition; the rewriting gives:

$1($	A	$\vee$	$\sim A)$
1	1	1	4
1	2	1	3
1	3	1	2
1	4	1	1

Note that altering the regular tables in this way is intuitively reasonable, since a constituent with only 1's or 2's in its column is never false, i.e. is a logical truth, so that its value should really be 1 in every row; analogously with a constituent having only 3's or 4's. I will call a table altered in this way an *altered truth-table*.

A *logical truth* may now be defined as a proposition the altered truth-table for which has only 1's in its main column.

Now consider again the places (2,3) and (3,2) in the disjunction matrix above. The value assigned to $p \vee q$ in these positions, viz. 2, is intuitively incorrect only if whenever p is false q is true (e.g., $A \vee \sim A$). But in this case the column for $p \vee q$ in a regular truth-table will contain only 1's or 2's and hence the corresponding column of the altered truth-table will contain only 1's. Thus the intuitive discrepancy in the disjunction matrix does not matter, being rectified by the rewriting.

Note that the substitution of a well-formed formula for a propositional constant in a logical truth will not always yield a logical truth. For example, $1(A \vee B) = (1A \vee 1B)$ is a logical truth, while $1(A \vee \sim A) = (1A \vee 1 \sim A)$ is not. This situation contrasts with the non-modal propositional calculus, as well as with Lewis' modal systems.

The present stipulation contains the non-modal propositional calculus[2] and is a modal logic in the sense of Łukasiewicz.[3]

Finally, in view of the high degree of intuitive reasonableness of this stipulation, it would I think be an argument in favor of a modal logic stipulated axiomatically that it was equivalent to the present logic.

REFERENCES

[1] As will be seen, the present stipulation is not a truth-value stipulation in the sense of J. B. Rosser and A. R. Turquette's *Many-Valued Logics* (Amsterdam, 1952), p. 27. In the latter no rewriting of the truth-tables is involved.

[2] This may be seen by considering the conjunctive normal form criterion of tautology in the propositional calculus.

[3] See his "A System of Modal Logic," *The Journal of Computing Systems*, I (1953), p. 111 ff., Section 1. He requires certain propositional schemata (e.g., $p \supset 1p$) to be rejected, which they are if any substitution-instance of them is. Substituting 'A' for 'p' and writing the tables as above shows the present stipulation rejects them. He further requires of any modal logic that certain schemata (e.g., $1p \supset p$) be accepted along with their substitution instances. All of these involve only one propositional variable and in this special case substitution preserves logical truth as determined by the above criterion. With 'A' for 'p' the above tables will thus show that all the propositions required to be accepted are logical truths in the present calculus.

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