Notre Dame Journal of Formal Logic
Volume IV, Number 3, July, 1963

JACOBSON THEORY OF RINGOIDS

S. K. SEHGAL

In memory of Prof. Th. Skolem

1. *Introduction*: We discussed the structure of ringoids with minimum condition in [1] and proved the analogue of the Wedderburn theorem for simple rings. Now we plan to take up the analogue of the Jacobson theory. After the proper formulations of definitions and results the proofs mainly copy the well known proofs for rings (see for example [2]). However, in order to indicate how this is done some proofs are given in detail. In Section 6, we take up a few elementary results on the representations of Barratt ringoids.

2. We recollect the definitions given in [1]. We call a 'collection' $\mathbf{R}$ of elements with partially defined operations of addition and multiplication a ringoid, if it satisfies the following axioms, for any $a,b,c,$ in $\mathbf{R}$:

a) i) $a + (b+c) = (a+b)+c$, *if either side is defined.* (i.e. if one side is defined, the other is defined and equality holds.)
ii) *Given* $a \in \mathbf{R}$, *there exists* $o_a \in \mathbf{R}$ *such that* $a + o_a = a$ *and* $o_a + x = x$ *whenever for* $x \in \mathbf{R}$, $o_a + x$ *is defined.* (We shall write o for o_a whenever no confusion is possible.)
iii) *Given* $a \in \mathbf{R}$, *there exists* $b \in \mathbf{R}$ *such that* $a + b = b + a = o_a$.
iv) $a + b = b + a$, *if either side is defined.*

b) $a(bc) = (ab)c$ *if either side is defined.*

c) $$\left.\begin{array}{l} a(b + c) = ab + ac \\ (b + c)a = ba + ca \end{array}\right\} \quad \textit{when either side is defined.}$$

We further impose the conditions α), β) and δ) given below: Define for $a \in \mathbf{R}$, $\mathbf{L}(a) = \{x\colon\ xa \textit{ defined}\}$, $\mathbf{R}(a) = \{x\colon\ ax \textit{ defined}\}$.

α) *For every* $a \in \mathbf{R}$, $\mathbf{L}(a) \neq \phi \neq \mathbf{R}(a)$.

β) *For every* $a \in \mathbf{R}$, *there is an element* $b \neq a$ *of* $\mathbf{R}$ such that $a + b$ *is defined.*

δ) *If* $\mathbf{L}(a) \cap \mathbf{L}(b) \neq \phi$, $\mathbf{R}(a) \cap \mathbf{R}(b) \neq \phi$ *then* $a + b$ *is defined.*

We observe that in a ringoid if $a + b$ is defined then $o_a = o_b$ and o_a and additive inverse are unique. Suppose $a + b$ and xa are defined then

$$xa = x(a + o_b) = x(a + b - b), \text{ which implies } xb \text{ is defined.}$$

Received December 4, 1962

Thus we see that a ringoid also satisfies the property:

γ) *If* $a + b$ *is defined then* $\mathsf{L}(a) = \mathsf{L}(b)$ *and* $\mathsf{R}(a) = \mathsf{R}(b)$.

If $a + b$ is defined then $o_a = o_b = o_{a+b}$. Conversely if $o_a = o_b$ we can write as above $a = a + o_a = a + o_b = a + (b - b)$, which means $a + b$ is defined. Hence $a + b$ is defined if and only if $o_a = o_b$. This divides R into a union of disjoint abelian groups $\mathsf{R} = \bigcup_{i \in \mathsf{I}} \mathsf{A}_i$.

We also recall, a Barratt ringoid G is a 'collection' $(\mathsf{G}_{ij})_{i,j}$ of disjoint abelian groups, written additively, one for each ordered pair of symbols (i,j) in some indexing set, such that there is defined a bilinear product $\mathsf{G}_{ij} \circ \mathsf{G}_{j,k} \to \mathsf{G}_{ik}$, for every triple (i,j,k) and every G_{ii} has an element 1_i such that

$$1_i\, g_{ij} = g_{ij} = g_{ij}\, 1_j \quad \text{for} \quad g_{ij} \in \mathsf{G}_{ij}.$$

For a number of interesting examples of ringoids we refer the reader to [1].

We do not explicity require that in a general ringoid if ab and bc are defined then $a(bc)$ is defined. We do not know if this property is a consequence of the other axioms. However, we have proved in([1], lemma 2-2) that if a^2, a^3, ab, bc are defined then $(ab)c$ is defined and hence $a(bc)$ is defined and $a(bc) = (ab)c$.

$\mathsf{I} \subset \mathsf{R}$ is said to be a left ideal in the ringoid R if

$$\mathsf{I} - \mathsf{I} = \{x - y\colon\ x, y \in \mathsf{I},\ x - y \textit{ defined}\} \subset \mathsf{I}$$

$$R\,\mathsf{I} = \{xi\colon\ x \in \mathsf{R},\ i \in \mathsf{I},\ xi \textit{ defined}\} \subset \mathsf{I}$$

One similarly defines right and two sided ideals. An ideal consisting of zeroes alone is called a null ideal and denoted by N.

The Wedderburn theorem for ringoids now reads:

Let R *be a simple ringoid, in the sense that*

1) R *contains no proper two sided ideals,*
2) R *satisfies the minimum condition for right ideals,*
3) $\mathsf{R}^2 = \mathsf{N}, \phi$.

Then R *is isomorphic to the matrix ringoid*

$\mathsf{M} = \bigcup_{m,n \in \mathsf{I}} \mathsf{M}_{a_m, a_n}(\mathsf{D})$, *where*

1) D *is a division ring,*
2) I *is a finite set,*
3) a_m, a_n *are integers,*

and

4) $\mathsf{M}_{a,b}$ (D) *stands for the set of all* $a \times b$ *matrices over* D.

It should be remarked that subscripts a_m, a_n are used instead of m,n in order to allow for matrices of same dimensions but of different colors. Then an $a_m \times a_n$ matrix is addible to an $a_u \times a_v$ matrix if and only if $m = n$, $n = v$ and an $a_m \times a_n$ matrix multiplied by an $a_n \times a_p$ matrix on the right gives an $a_m \times a_p$ matrix. Of course one easily sees that, conversely, the matrix ringoid M is simple.

3. In this section R will always denote a ringoid. We make a few definitions.

Definition 1. $\mathsf{M} = \bigcup_{i \in \mathsf{I}} \mathsf{M}_i$, *a disjoint union of additive groups is said to be an* R - *module if given* $r \in \mathsf{R}$, *there is a unique* $i, j \in \mathsf{I}$ *such that*

$$r\colon\ \mathsf{M}_i \to \mathsf{M}_j \quad i.e. \quad \mathsf{M}_i r \subset \mathsf{M}_j$$

and for $m, m' \in \mathsf{M}$, $r, s \in \mathsf{R}$ *we have*

$$\left.\begin{aligned} (m + m')r &= mr + m'r \\ m(rs) &= (mr)s \\ m(r+s) &= mr + ms \end{aligned}\right\} \quad \textit{if either side is defined.}$$

Definition 2. M' *is a submodule of* $\mathsf{M} = \bigcup_{i \in \mathsf{I}} \mathsf{M}_i$ if $\mathsf{M}' = \bigcup_{i \in \mathsf{I}} \mathsf{M}'_i$,

$$\mathsf{M}'_i \subset \mathsf{M}_i \textit{ and } \mathsf{M}' \textit{ is an } \mathsf{R} \textit{ - module.}$$

Then $\mathsf{M}/\mathsf{M}' = \bigcup_{i \in \mathsf{I}} \mathsf{M}_i/\mathsf{M}'_i$ becomes an R - module by setting $\overline{m}\, r = \overline{mr}$ when ever the right hand side is defined, for $r \in \mathsf{R}$, $m \in \mathsf{M}$.

Definition 3. Let M and M' be two R - modules. A map $\theta : \mathsf{M} \to \mathsf{M}'$ is called an R - homomorphism if for all m_1, m_2 in M, $r \in \mathsf{R}$ we have

$$\left.\begin{aligned} \theta(m_1 + m_2) &= \theta(m_1) + \theta(m_2) \\ \theta(m r) &= \theta(m) r \end{aligned}\right\} \quad \textit{if either side is defined.}$$

We observe that in the definition of M, and R - module, the uniqueness of i is equivalent to:

For $m, n \in \mathsf{M}$, $m + n$ *is defined* $\Leftrightarrow$ *there is an* $r \in \mathsf{R}$ *such that* $mr + nr$ *are defined.*

Thus in general a ringoid R is not an R - module. One may think that therefore the definition is unsatisfactory. But we shall see that the following theory does not work unless one makes such a definition. Moreover if R, in particular, is a ring then it is an R - module in the ringoid sense. We also have:

A minimal right ideal I *of a ringoid* R *is an* R - *module.*

Proof. We know $\mathsf{I} = x\mathsf{R} = \{xr\colon r \in \mathsf{R},\ xr \text{ defined}\}$, for some $x \in \mathsf{R}$. We have only to show that if two elements of I have a common right multiplier they can be added.

xlr, xmr defined $\Rightarrow l + m$ defined $\Rightarrow x(l + m)$ defined $\Rightarrow xl + xm$ defined. This completes the proof.

Remark - With a Barratt ringoid $\mathsf{R} = \bigcup_{i,j} \mathsf{G}_{ij}$, we can associate an R - module R^* in a natural way by setting $\mathsf{R}^* = \sum_k \mathsf{G}_k$, where $\mathsf{G}_k = \sum_i \mathsf{G}_{ki}$ and rg is defined in the natural way for $r \in \mathsf{R}$, $g \in \mathsf{G}_k$.

To continue the remark, in case **R** is a ring, $\mathbf{R}^* = \mathbf{R}$.

Lemma 1 (Schur). **R** - *endomorphisms of an irreducible* **R** - *module* **M** *form a division ring.*

Proof: Let $x \in \mathbf{M}$ and α, β be two **R** - endomorphisms of **M**. Since there exists an $r \in \mathbf{R}$ such that xr is defined and $\alpha(xr) = \alpha(x)\, r$ (if either side is defined), $\alpha(x)\, r$ is defined. Therefore by the definition of an **R** - module, $\alpha(x) + x$ is defined. Similarly $\beta(x) + x$ is defined and thus $\alpha(x) + \beta(x)$ is meaningful. Thus we can define

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x).$$

Now, as usual, since $\alpha(\mathbf{M}) = \mathbf{M}$ and kernal $\alpha = \mathbf{N}$, α is 1 - 1 and onto and has an inverse.

This proves Schur's lemma.

Definition 4. *A ringoid* **R** *is said to be primitive if it has a faithful irreducible* **R** - *module.*

Theorem 1. (Jacobson Density Theorem). *Let* **R** *be a primitive ringgoid with an irrdeucible faithful* **R** - *module* **M**. *By Schur's lemma, the* **R** - *endomorphisms of* **M** *form a division ring, say* Δ. *Suppose* $v_1, \ldots, v_n$ *are mutually addible elements of* **M** *and* $w_1, \ldots, w_n$ *are mutually addible elements of* **M**. *Further suppose* v_i *are linearly independent (in the vector space sense) with respect to* Δ. *Then there exists an element* r *of* **R** *such that*

$$v_i r = w_i \quad for \quad i = 1, 2, \ldots, n.$$

Proof: We first prove:

1) If **V** is a finite dimensional vector space over Δ, contained in **M** and $m \in \mathbf{M}$, $m \notin \mathbf{V}$ then there is an $r \in \mathbf{R}$ such that $\mathbf{V}r = o$, $mr \neq o$. We shall use induction on the dimension of **V**. We can write

$$\mathbf{V} = \mathbf{V}_0 + \Delta\, w,\ w \notin \mathbf{V}_0$$

$$\text{Let} \quad \mathcal{A}(\mathbf{V}_0) = \{r \in \mathbf{R} : \mathbf{V}_0 r = o\}.$$

Because of the induction assumption we have

$$v \quad \mathcal{A}(\mathbf{V}_0) = o \Longleftrightarrow v \in \mathbf{V}_0$$

Now suppose the assertion 1) is not true, i.e.

$$\mathbf{V}r = o \Longrightarrow mr = o.$$

Observe that $w \ \mathcal{A}\ (\mathbf{V}o) \subset \mathbf{M} \Longrightarrow w\ \mathcal{A}\ (\mathbf{V}o) = \mathbf{M}$.

Define $\theta : \mathbf{M} \to \mathbf{M}$ as follows:

If $x = wa$, $a \in \mathcal{A}\ (\mathbf{V}o)$, set $\theta(x) = ma$.

θ is well defined as $wa = wb \Longrightarrow w\ (a - b) = o \Longrightarrow \mathbf{V}\ (a - b) = o \Longrightarrow m(a - b) = o$

θ is an **R** - endomorphism of **M** i.e. $\theta \in \Delta$.

For $a \epsilon \mathcal{A}(\mathbf{V}o)$, $ma = \theta(wa) = \theta(w)a \Rightarrow (m - \theta(w))\, a = o$ i.e. $(m - \theta(w))$ $\mathcal{A}(\mathbf{V}o) = o$

Therefore $m - \theta(w) \epsilon \mathbf{V}o$ and $m \epsilon \mathbf{V}_o + \Delta w = \mathbf{V}$ which is a contradiction, proving the assertion 1).

2) Now to complete the proof, suppose r is got by 1)

$$mr \neq o \Rightarrow mr\mathbf{R} \neq \mathbf{N} \Rightarrow mr\mathbf{R} = \mathbf{M}.$$

Thus there exists $s \epsilon \mathbf{R}$ with mrs arbitrary but $\mathbf{V}rs = $ o.
Therefore choose w_i such that

$$v_j w_i = o \quad j \neq i \text{ and } v_i r_i = w_i\,,\ i = 1, \ldots, n$$

then

$$v_i(r_1 + \ldots + r_n) = v_i r_i = w_i$$

($r_1 + \ldots + r_n$ is defined because of the distributive law).
This proves the theorem.

4. Now we shall deduce the Wedderburn theorem from the Density theorem.

Lemma 2. *Let* $\mathbf{R}$ *be a simple ringold with descending chain condition on right ideals then* $\mathbf{R}$ *is primitive.*

Proof: We only have to exhibit a faithful irreducible $\mathbf{R}$ - module. By the chain condition, $\mathbf{R}$ contains a non null minimal right ideal, say $\mathbf{I}$. Then $\mathbf{I}$ is an irreducible $\mathbf{R}$ - module. We claim that $\mathbf{I}$ is also faithful. Let

$\mathbf{J} = \{a \epsilon \mathbf{R} : \mathbf{I}a \subset \mathbf{N}\}$.
$\mathbf{J}$ is a two sided ideal of $\mathbf{R}$.
$\mathbf{I} \subset \{x : x\mathbf{J} \subset \mathbf{N}\}$, which implies that $\{x : x\,\mathbf{J} \subset \mathbf{N}\} = \mathbf{R}$

Therefore $\mathbf{J} \neq \mathbf{R}$ because otherwise $\mathbf{R}^2 \subset \mathbf{N}$. Hence $\mathbf{J} = \mathbf{N}$ i.e., $\mathbf{I}$ is faithful.

Theorem 2 (Wedderburn) *Let* $\mathbf{R}$ *be a simple ringoid with descending chain condition on right ideals. Then* $\mathbf{R}$ *is isomorphic to*

$$\{\mathbf{M}_{a_m, a_n}(\mathbf{D}) : m, n \epsilon \mathbf{I}\},$$

where $\mathbf{I}$ *is finite,* a_i *are integers and* $\mathbf{D}$ *is a division ring.*

Proof: Let $\mathbf{D}$ be the division ring of $\mathbf{R}$ - endomorphisms of a faithful irreducible $\mathbf{R}$ - module $\mathbf{M}$ (which exists by the lemma 2). Then $\mathbf{M} = \bigcup_{i \epsilon \mathbf{I}} \mathbf{V}_i$, where $\mathbf{V}_i$ are vector spaces over $\mathbf{D}$. By theorem 2, $\mathbf{R}$ consists of all linear transformations $\theta : \mathbf{V}_i \to \mathbf{V}_j$ for all $i, j \epsilon \mathbf{I}$.

It remains to prove

1) All the $\mathbf{V}_i$ have finite dimension, and
2) $\mathbf{I}$ is finite.

To prove 1), suppose one of the $\mathbf{V}_i$ say $\mathbf{V}_1$ has an infinity of linearly independent elements say, $u_1, u_2, u_3, \ldots\ldots$ then $\mathcal{A}(u_1) \supset \mathcal{A}(u_1, u_2) \supset \mathcal{A}(u_1, u_2, u_3) \supset \ldots\ldots$ where $\mathcal{A}(u_1, \ldots, u_k) = \{r \epsilon \mathbf{R} : u_i r = o$

$i = 1, \ldots, k\}$ is an infinite chain of ideals and containment is proper at each step by theorem 2. This is forbidden.

To prove 2) suppose I contains a countable infinite set say Z, the natural numbers, then

$$\mathsf{R} \supset \bigcup_{m,n \in Z} \mathsf{M}_{a_m,\, a_m}(\mathsf{D}).$$

Then again there is an infinite chain of ideals:

$$\bigcup_{\substack{m,n \in \mathsf{I} \\ m \neq 1}} \mathsf{M}_{a_m,\, a_n}(\mathsf{D}) \supset \bigcup_{\substack{m,n \in \mathsf{I} \\ m \neq 1,2}} \mathsf{M}_{a_m,\, a_n}(\mathsf{D}) \supset \bigcup_{\substack{m,n \in \mathsf{I} \\ m \neq 1,2,3}} \mathsf{M}_{a_m,\, a_n}(\mathsf{D}) \supset \ldots\ldots,$$

which is not permitted.

This completes the proof of the theorem.

5. Definition 5. *Let* I,J *be two right ideals of the ringoid* R. *Suppose for every* $r \in \mathsf{R}$,

i) *either* $r = i + j$, $i \in \mathsf{I}$, $j \in \mathsf{J}$
   or $r = i$, $i \in \mathsf{I}$
   or $r = j$, $j \in \mathsf{J}$

ii) $\mathsf{I} \cap \mathsf{J} \subset \mathsf{N}$.

Then we say R *is a direct sum of* I *and* J *and write* $\mathrm{R} = \mathrm{I} + \mathrm{J}$.

Definition 6. If M is an R - module, $\mathcal{A}(\mathsf{M}) = \{x \in \mathsf{R} : \mathsf{M}x \subset \mathsf{N}\}$. Clearly $\mathcal{A}(\mathsf{M})$ is a two sided ideal of R.

Definition 7. The *radical* of R, $\mathsf{J}(\mathsf{R})$, is $\cap\, \mathsf{A}(\mathsf{M})$, where this intersection runs over all irreducible R - modules M. If R has no irreducible R - modules, we define $\mathsf{J}(\mathsf{R}) = \mathsf{R}$. The radical is thus the annihilator of all irreducible R - modules.

Definition 8. *We shall say* R *is semisimple if* $\mathsf{J}(\mathsf{R}) \subset \mathsf{N}$.

One can easily prove now, following the standard arguments, with slight modifications, of for example [2]:

1) $\mathsf{J}(\mathsf{R}/\mathsf{J}(\mathsf{R}))$ *is null.*
   (Factor ringoid is defined in [1])
2) $\mathsf{J}(\mathsf{R})$ = *intersection of its primitive ideals.*

Some other characterisations of the radical in the ring sense also hold for ringoids.

3) *If* R *is a semisimple ringoid with descending chain condition on two sided ideals then* R *is a direct sum of simple ringoids.*

6. In this section we formulate the elementary concepts of the matrix representations of a ringoid and a ringoid from now on stands for a Barratt ringoid.

Definition 9. *Let* $\mathsf{G} = \bigcup_{i,j} \mathsf{G}_{ij}$ *be a ringoid we say* $\mathsf{M} = \bigcup_{i \in \mathsf{I}} \mathsf{M}_i$ *is a representation (space) of* G *if each* M_i *is a finite dimensional vector space over a field* F *and for every* $g \in \mathsf{G}$, *there is a unique pair* i,j *such that a product*

ug is defined for all $u \in \mathsf{M}_i$ and $ug \in \mathsf{M}_j$ and has the following properties for $u_1, u_2 \in \mathsf{M}$, $g \in \mathsf{G}$, $\alpha \in \mathrm{F}$.

$$\left.\begin{array}{l} 1)\ (u_1 + u_2)g = u_1 g + u_2 g \\ 2)\ (\alpha u)\, g = \alpha(ug) \\ 3)\ u(g_1 g_2) = (ug_1)g_2 \\ 4)\ u(g_1 + g_2) = ug_1 + ug_2 \end{array}\right\} \quad \textit{when either side is defined.}$$

We remark that if $\mathsf{M} = \bigcup_{i \in I} \mathsf{M}_i$ is a representation space of G, suppose $g : \mathsf{M}_i \to \mathsf{M}_j$ and $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ are F - bases of M_i and M_j. Suppose $u_i g = \sum_{\alpha = 1} a_{ij}(g)\, u_j$. $i = 1, \ldots, n$. Then $g \to \Delta(g) = (a_{ij})_{m, n}$ has the properties:

$$\left.\begin{array}{ll} \Delta(g_1 + g_2) & = \Delta(g_1) + \Delta(g_2) \\ \Delta(g_1 g_2) & = \Delta(g_1)\,\Delta(g_2) \end{array}\right\} \quad \begin{array}{l}\text{when either side} \\ \text{is defined,}\end{array}$$

i.e. Δ is a matrix representation. Conversely, having a matrix representation one can construct a G - module M as in Definition 9, having properties 1) to 4).

Definition 10. *Let $\mathsf{M} = \bigcup_i \mathsf{M}_i$ be a representation space of $\mathsf{G} = \bigcup_{i,j,} \mathsf{G}_{ij}$. Then $m = \bigcup_i m_i$, is said to be an invariant subspace if each m_i is a subspace of M_i and m is a representation space (with respect to the induced multiplication).* In this case we write $\mathsf{M} \supset m$.

Let $\mathsf{M} = \bigcup_i \mathsf{M}_i$, a representation of $\mathsf{G} = \bigcup_{i,j} \mathsf{G}_{ij}$ contain an invariant subspace $m = \bigcup_i m_i$. Then $\mathsf{M}/m = \bigcup_i \mathsf{M}_i/m_i$ becomes a representation space by defining $\bar{u}g = \overline{ug}$, $g \in \mathsf{G}$, $\bar{u} \in \mathsf{M}_j/m_j$.

Definition 11: *We shall say two representation spaces $\mathsf{M} = \bigcup_i \mathsf{M}_i$ and $m = \bigcup_i m_i$ of $\mathsf{G} = \bigcup_{i,j} \mathsf{G}_{ij}$ are equivalent if for each i, there is a vector space isomorphism $\theta_i : \mathsf{M}_i \to m_i$* and the diagrams

$$\begin{tikzcd} \mathsf{M}_j \arrow[r, "\theta_j"] \arrow[d, "g"'] & m_j \arrow[d, "g"] \\ \mathsf{M}_i \arrow[r, "\theta_i"'] & m_i \end{tikzcd} \qquad g \in \mathsf{G}$$

commute whenever they can be drawn.

In other words if we define the map $\theta = \{\theta_i\}$ on M, it is required that

$$(\theta u)g = \theta(ug),$$

whenever either side is defined.

Definition 12. *By a reduction of a representation space* M *of a ringoid we mean a sequence of representation spaces*:

$$\mathsf{M} = \mathsf{M}_0 \supset \mathsf{M}_1 \supset \ldots \supset \mathsf{M}_n = 0$$

We call the spaces $\mathsf{M}_i/\mathsf{M}_{i+1}$ the components of the representation.

Definition 13. We say two reductions

$$\mathsf{M} = \mathsf{M}_0 \supset \mathsf{M}_1 \supset \ldots \supset \mathsf{M}_n = 0$$

$$\mathsf{M} = m_0 \supset m_1 \supset \ldots \supset m_k = 0$$

of a representation space M *of a ringoid* R *are isomorphic if there exists a one to one correspondence between the representation spaces* $\mathsf{M}_i/\mathsf{M}_{i+1}$ *and* m_i/m_{i+1} *such that the corresponding representations are equivalent.*

Theorem 3. Jordan-Hölder-Schreier Theorem--*The irreducible components of a representation* $\mathsf{M} = \bigcup_{i \in \mathsf{I}} \mathsf{M}_i$ *of a ringoid* $\mathsf{G} = \bigcup_{i,j} \mathsf{G}_{ij}$ *are unique up to order and equivalence.*

Proof: Let

$$\mathsf{M} = \mathsf{M}^0 \supset \mathsf{M}^1 \supset \mathsf{M}^2 \supset \ldots \supset \mathsf{M}^n = 0$$

$$\mathsf{M} = m^0 \supset m^1 \supset m^2 \supset \ldots \supset m^k = 0$$

where $\mathsf{M}^r = \bigcup_i \mathsf{M}_r^i,\ m^s = \bigcup_i m_s^i,\ \mathsf{M}_r^i \subset \mathsf{M}_i,\ m_s^i \subset \mathsf{M}_i,$

be two reductions of M. In other words,

$$\mathsf{M}_1 = \mathsf{M}_0^1 \supset \mathsf{M}_1^1 \supset \ldots \supset \mathsf{M}_n^1 = 0 \qquad \mathsf{M}_1 = m_0^1 \supset m_1^1 \supset \ldots \supset m_k^1 = 0$$

$$\mathsf{M}_2 = \mathsf{M}_0^2 \supset \mathsf{M}_1^2 \supset \ldots \supset \mathsf{M}_n^2 = 0 \qquad \mathsf{M}_2 = m_0^2 \supset m_1^2 \supset \ldots \supset m_k^2 = 0$$

As is customary, we shall exhibit isomorphic refinements of these reductions. Set

$$\mathsf{M}_{ij}^r = \mathsf{M}_i^r \ (\mathsf{M}_{i-1}^r \cap m_j^r) \qquad i = 1, 2, \ldots, n$$

$$m_{ij}^r = m_j^r \ (m_{j-1}^r \cap m_i^r) \qquad j = 1, \ldots, k$$

$$r \in \mathrm{I}.$$

We claim $\mathsf{M}_{ij} = \bigcup_r \mathsf{M}_{ij}^r$ and $m_{ij} = \bigcup_r m_{ij}^r$ are isomorphic reductions.

We know by the Zassenbaus Lemma, there exist vector space isomorphisms

$$\overset{r}{\underset{i,j}{\theta}} : \mathsf{M}_{i,j-1}^r / \mathsf{M}_{i,j}^r \to \mathsf{M}_{i-1,j}^r / m_{i,j}^r$$

It suffices for us to prove the commutativity of the diagram

$$\begin{array}{ccc} M^r_{i,j-1}/M^r_{i,j} & \xrightarrow{\overset{r}{\theta}_{i,j}} & m^r_{i-1,j}/m^r_{i,j} \\ \downarrow g & & \downarrow g \\ M^s_{i,j-1}/M^s_{i,j} & \xrightarrow[\underset{i,j}{\overset{s}{\theta}}]{} & m^s_{i-1,j}/m^s_{i,j} \end{array} \qquad g \in G.$$

$$\overset{r}{\theta}_{i,j}((a+c)g) = \bar{c}g = \overline{cg} \qquad a \in M_i,\ c \in M_{i-1} \cap m_j.$$

$$\overset{r}{\theta}_{i,j}\ \overline{(a+c)}g = \overset{r}{\theta}_{i,j}(\overline{ag+cg}) = \overline{cg} \text{ (because } ag \in \overset{s}{M}_i).$$

This completes the proof.

Definition 14. *A representation space* $M = \bigcup_i M_i$ *of a ringoid* R *is said to be a direct sum of representation spaces* $M^a = \bigcup_i M_i^a$ $a = 1, 2$ (written $M = M^1 + M^2$) *if each*

$$M_i = M_i^1 + M_i^2$$

Definition 15. *A representation* $M = \bigcup_i M_i$ *of* $G = \bigcup_{i,j} G_{ij}$ *is said to be unital if for each identity* $e_i \in G_{ii}$ *we have*

$$ue_i = u \text{ for } u \in M_i.$$

Let a representation space M of a ringoid G be the direct sum

$$M = M^1 + M^2.$$

Let $u = u_1 + u_2, u \in M,\ u_1 \in M^1,\ u_2 \in M^2$,

define

$$\theta_1(u) = u_1,\ \theta_2(u) = u_2.$$

Then θ_i are operator homorphisms in the sense that

$$\theta_i(u+v) = \theta_i(u) + \theta_i(v)$$

$$\theta_i(ug) = \theta_i(u)g \quad g \in G$$

when either side is defined.

Theorem 3. *Every unital representation* $M = \bigcup_i M_i$ *of a semisimple ringoid* $G = \bigcup_{i,j} G_{ij}$ *is fully decomposable.*

Proof: Let $G = L_1 + \ldots + L_r$ be a decomposition of G into minimal right ideals. Suppose $m_{j1}, m_{j2}, \ldots, m_{jk}$ is an F-basis of M_j. We write

$$\mathsf{M}_j = \langle m_{j1}, \ldots, m_{jk} \rangle$$

$$\mathsf{M} = \bigcup_j \langle m_{j1}, \ldots, m_{jk} \rangle$$

$$= \bigcup_j \langle \ldots m_{jl}\ \mathrm{G} \ldots \rangle$$

$$= \bigcup_{i,j,l} \langle m_{jl} \mathsf{L}_i \rangle$$

Since L_i is a minimal right ideal, so is $\mathsf{L}_i m_{jl}$. Therefore

$$m_j \mathsf{L}_i \cap m_{st}\ \mathsf{L}_r = \text{Empty or null for distinct } m_j \mathsf{L}_i \text{ and } \mathsf{L}_r m_{st}.$$

$$\mathsf{M} = \bigcup_j (m_{.k}\ \mathsf{L}_i \cap \mathsf{M}_j)$$

after omitting the ideals occuring more than once and reindexing the m_j. Observe

$$(m_k\ \mathsf{L}_i \cap \mathsf{M}_j) g \subset m_k\ \mathsf{L}_i \cap \mathsf{M}_r$$

$$m_k\ \mathsf{L}_i \cap \mathsf{M}_j \neq \theta \Longrightarrow m_k\ \mathsf{L}_i \cap \mathsf{M}_p \neq \theta \text{ for all } p.$$

Hence if we write

$$\mathsf{M}^{ik} = \bigcup_j (m_k\ \mathsf{L}_i) \cap \mathsf{M}_j,$$

M is the direct sum of representation spaces, M^{ik}. Since M^{ik} are minimal ideals of G, they are irreducible representation spaces. Hence the theorem.

Theorem 4 (Krull-Remak-Schmidt) *Let* M *be a representation space of* $\mathsf{G} = \bigcup_{i,j} \mathsf{G}_{ij}$ *such that*

$$\mathsf{M} = \mathsf{A}_1 + \ldots + \mathsf{A}_r = \mathsf{B}_1 + \ldots + \mathsf{B}_s$$

with each A_i, B_s *indecomposable representation spaces, then after rearrangement of indices each* A_i *is equivalent to* B_i *and* $r = s$.

The proof is identical with the standard proof.

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University of Notre Dame
Notre Dame, Indiana