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RELATIONS IRREDUCIBLE TO CLASSES

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1. *Purpose.* Relations are usually considered as two-place predicates with variables ranging over a certain object domain. Every relation can be interpreted, then, as a class of ordered pairs of objects that satisfy a given two-place predicate. This approach follows Wiener's well-known idea and reduces relations to terms [14], namely, the class of ordered pairs just mentioned. Bradley ([2], [3], [4]) and Whitehead ([11], [12], [13]) have criticized the reduction of relations to terms. Explicitly, Bradley distinguished between "external relations" (e.g., the aforementioned variety of relations reducible to classes) and "internal relations," i.e., relations irreducible to terms. According to Bradley, every internal relation modifies the terms related, but the relation itself is not modified by the terms. Whitehead accepted Bradley's terminology, but took the position that internal relations and terms not only should be mutually irreducible but also should be mutually modifiable ([13], [5]). The purpose of this paper is to formalize Whitehead's concept of internal relation, presenting it as a kind of variable independent of the two-place predicate definition of relation.

2. *Logical symbols and formation rules.* Terms and internal relations (hence forth called "relations") will be considered different formal symbols. Following Kleene's symbolism and terminology (Cf. [7]), we may describe the formal system we intend to form as containing: (1) Logical symbols: $\supset$, &, **v**, $\neg$, $\forall$, $\exists$ (the propositional connectives and quantifiers). (2) Predicate symbols: $=$ (equals). (3) Function symbols: $+$, $.$, $'$ (plus, times, successor). (4) Individual symbols: 0, $\bar{0}$ (zeros). (5) Term variables: a_1, b_1, c_1, ... (5) Relation variables: a_2, b_2, c_2, ... (7) Parentheses: (,).

As subcategories of the formal expressions we can form with these symbols, we will distinguish among "terms," "relations," and "formulae" according to the following formation rules:

a1. 0 is a term. a2. A term variable is a term. a3 - a5. If x_1 and y_1 are terms, then $x_1 + y_1$, $x_1 \cdot y_1$, x_1' are terms. a6. If x_1 and y_1 are terms and x_2 is a relation, then $x_1 x_2 y_1$ is a term. a7. a1 to a6 determine the only terms.

b1. $\bar{0}$ is a relation. b2. A relation variable is a relation. b3 - b5. If

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x_2 and y_2 are relations, then $x_2 + y_2$, $x_2 \cdot y_2$, x_2' are relations. b6. If x_2 and y_2 are relations and x_1 is a term, then $x_2 x_1 y_2$ is a relation. b7. b1 to b6 determine the only relations.

(By a6 and b6, terms and relations play the role of function symbols in a manner analogous to +, ., '.)

c1. If x_1 and y_1 are terms, then $x_1 = y_1$ is a formula. c2. If x_2 and y_2 are relations, then $x_2 = y_2$ is a formula. c3 - c6. If A and B are formulae, then $A \supset B, A \& B, A \vee B, \neg A$ are formulae. c7 - c8. If x_1 is a term variable and A is a formula, then $\forall x_1 A$ and $\exists x_1 A$ are formulae. c9 - c10. If x_2 is a relation variable and A is a formula, then $\forall x_2 A$ and $\exists x_2 A$ are formulae. c11. c1 to c10 determine the only formulae.

Application of these formation rules results in the following formal expressions. Terms: $a_1, b_1, c_1, a_1 a_2 a_1$, $a_1(b_2\ 0\ c_2) b_1$, $a_1(b_2(d_1\ \bar{0}(a_1 e_2 a_1))b_2)c_1$, etc. Relations: a_2, $b_2(d_1 e_2((a_1 b_2 c_1))\ (e_2 a_1 b_2)$, etc. Formulae: $a_1 = b_1$, $a_1 a_2 a_1 = a_1$, $a_1(b_2(d_1 e_2(a_1 b_2(a_1))b_2)c_1 = a_1 a_2 b_1$, $a_2 = b_2 d_1 e_2$, etc.

(Note: Tarski's introduction of relations as relation variables in [9] has nothing to do with the present approach. His variables are intended to be external relations, their properties essentially coinciding with the properties of the two-place predicate definition of relation.)

3. *Term formulae, relation formulae, and term-relation formulae.* The system we intend to form is a kind of two-sorted predicate calculus; i.e., a calculus whose predicate formulae are defined in two categories of fundamental objects, terms and relations. These will be represented respectively by two kinds of informal variables: $x_1, y_1, z_1, \ldots$ (terms); $x_2, y_2, z_2, \ldots$ (relations). The general category of terms has a subcategory of terms that are obtained by applying formation rules a1 - a5 only. Let us represent this subcategory of terms with the symbol T^*. Analogously, the symbol R^* will represent the subcategory of relations obtained by applying rules b1 - b5 only. If terms are of T^*, they will be represented by the symbols $m_1, n_1, p_1, \ldots$ ($m_2, p_2, q_2, \ldots$ for relations of R^*). The symbol x_1 may then represent either a formal term variable a_1 or a term m_1 of T^*; or, to be more general, a term obtained by applying rules a1 - a6. This is similarly true for relations (indicated by "s.f.r." for the rest of the paper).

According to this classification of terms and relations, predicate formulae can be divided into these five subcategories. (i) Formulae with only terms ranging in T^*, with the use of rules c1, c3-c6, c7-c8 only (term formulae). (ii) Formulae with only relations, these relations ranging in R^*, with the use of rules c2, c3-c6, c9-c10 only (relation formulae). (iii) Formulae with terms only or relations only, these terms or relations ranging respectively beyond T^* or R^*. (iv) Formulae with terms and relations ranging in T^* and R^* respectively. (v) Formulae with terms and relations ranging one or both beyond T^* and R^*. Formulae of kinds (iii), (iv) and (v) will be called "term-relation formulae." Examples of term-relation formulae follow: $\forall x_1\ \exists x_2 (x_1 = x_1 x_2 x_1)$; $\exists x_2(x_2 + y_2 = y_2' \cdot (y_2 x_1 z_2))$; $\exists x_1\ \forall x_2\ \neg(x_2 x_1 x_2 = x_2)$, etc.

4. *The term-relation number theory.*

The predicate calculus for term formulae is the same as the predicate calculus for relation formulae. For the purpose of outlining such calculi, we shall assume Kleene's postulates 1 - 12 (see [7], p. 82), which will apply independently to term formulae on one hand, and relation formulae on the other. Let us also include Kleene's postulates 13 - 21, so that the system of term formulae and the system of relation formulae will be formally equivalent (respectively) to one and the same number-theoretic formal system. For a term-relation number theory, let us now add the following definitions, postulates, and theorems.

For postulates **P1** - **P21**, see Kleene's [7], p. 82. (From P22 on, even-numbered postulates—also definitions and theorems—will refer to terms and odd-numbered ones to relations.)

P22. $x_1\bar{0}x_1 = x_1$. **P23.** s.f.r.

T0 (Theorem 0). Every term of the form $x_1\bar{0}\,x_1\bar{0}\,x_1 \ldots x_1\bar{0}\,x_1$ with any arrangement of the corresponding parentheses, but with no left parenthesis immediately preceding any occurrence of $\bar{0}$, equals x_1.

Proof: By iteration of **P22.** **T1.** s.f.r.

By applying postulates **P22** and **P23,** terms or relations in which expressions of the forms $x_1\bar{0}\ x_1$ or $x_2 0\ x_2$ appear can be simplified by substituting x_1 and x_2, respectively, for such expressions. This leads to the following definition:

D0. A term will be said to be written in the *reduced form* if and only if it does not contain any expression of the form $x_1\bar{0}\,x_1$, or the form $x_2 0\ x_2$. **D1.** s.f.r.

D2. Two terms are called equal if and only if after writing them in the reduced form they are identical formal expressions, i.e., if they are formed by occurrences of the same logical symbols in the corresponding places in each one of both expressions.

Hence, for example, $a_1a_2b_1 = a'_1a'_2b'_1$ if $a_1 = a'_1$, $a_2 = a'_2$, $b_1 = b'_1$. $a_1a_2((b_1b_2(c_1c_2d_1))\ d_2e_1) = a'_1a'_2((b'_1b'_2(c'_1c'_2d'_1))d'_2e'_1)$ if $a_1 = a'_1$, $a_2 = a'_2$, etc., with the parentheses occurring in the corresponding places. (These examples use equality of terms in T^*, the latter equality now being a particular case of equality of terms in general.) **D3.** s.f.r.

D4. A term will be called of *order* n if and only if written in the reduced form it reads: $p_1^{(1)}\ p_2^{(1)}\ q_1^{(2)}\ q_2^{(2)} \ldots s_1^{(n-1)}\ s_2^{(n-1)}\ t_1^{(n)}$ (with some arrangement of the corresponding parentheses). **D5.** s.f.r.

As a consequence of D4, terms of T^* are all of the first order, 0 included. $a_1\bar{0}a_1$ is an example of a term of the first order not originally in T^*. $(a_1\bar{0}a_1)\ \bar{0}\ ((a_1a_2b_1)\ \bar{0}b_1)$ is of order four, provided that $a_1 \neq b_1$, etc.

D6. $T^{(n)}$ represents the category of terms of order n. **D7.** $R^{(n)}$ represents the category of relations of order n. (T^* and R^* are subcategories of $T^{(1)}$ and $R^{(1)}$ in principle. But because of the reduction of terms and relations that is possible with P22 and P23, T^* coincides with $T^{(1)}$, and R^* coincides with $R^{(1)}$.)

D8. Given a term $x_1x_2y_1$, x_1 and y_1 will be called *components* of $x_1x_2y_1$. If, furthermore, either (i) $x_2 \neq \bar{0}$ or (ii) $x_1 \neq y_1$ whenever $x_2 = \bar{0}$, then x_1 and y_1 will be called *proper components*. **D9.** s.f.r.

D10. A term $x_1x_2y_1$ will be called *normal* if and only if its components are proper. **D11.** s.f.r.

T2. If x_1 is a term of order m, y_1 a term of order n, x_2 a relation of order p $(m, n, p > 1)$, and $x_1x_2y_1$ is not normal, then the order of $x_1x_2y_1$ is $m = n$.

Proof: **D4, D10** and **P22. T3.** s.f.r.

T4. If $x_1 = y_1x_2z_1$ is normal of order n', then the components y_1 and z_1 of x_1 are of order k, h respectively, with $k, h \leq n$.

Proof: This is a corollary of **D4** and **T2. T5.** s.f.r.

P24a. If $a_1, b_1, \ldots$ are terms (of T^*), and $a_2, b_2, \ldots$ are relations (of R^*), then $a_1a_2b_1 + c_1b_2d_1 = (a_1 + c_1)(a_2 + b_2)\_(b_1 + d_1)$. **P25a.** s.f.r.

In particular, in P24a, if $c_1 = d_1$ and $b_2 = \bar{0}$, then $a_1a_2b_1 + c_1 = (a_1 + c_1)$ $a_2(b_1 + c_1)$.

P24b. If $x_1x_2y_1$ is a normal term of order m' and $z_1y_2w_1$ is a normal term of order n', with x_2 a normal relation of order p' $(m \geq p' \geq 1)$ and y_2 a normal relation of order q' $(n \geq q' \geq 1)$, such that both x_2 and y_2 are either in R^* or are of the form $s_2s_1t_2$, $u_2u_1v_2$, with s_1 a term of order h $(p \geq h \geq 1)$, and u_1 a term of order k $(q \geq k \geq 1)$, then

$$x_1x_2y_1 + z_1y_2w_1 = (x_1 + z_1)(x_2 + y_2)(y_1 + w_1),$$

where $x_1 + z_1$ and $y_1 + w_1$ are sums of terms x_1, y_1 of order less than or equal to m and terms z_1, w_1 of order less than or equal to n, and

$$x_2 + y_2 = s_2s_1t_2 + u_2u_1v_2 = (s_2 + u_2)(s_1 + u_1)(t_2 + v_2),$$

where $s_2 + u_2$ and $t_2 + v_2$ are sums of relations s_2 and t_2 of order less than or equal to p and relations u_2, v_2 of order less than or equal to q, and where $s_1 + u_1$ is the sum of terms s_1 (of order less than or equal to $p \leq m$) and u_1 (of order less than or equal to $q \leq n$). **P25b.** s.f.r.

P24 determines addition of terms by induction on the maximum order of their components first, and on the maximum order of the components of the relations x_2 and y_2 second. Similarly, this is also the case with **P25.** After these postulates have been stated, we can compute the sum of any given pair of normal terms or normal relations through a finite number of steps. Addition of normal terms of m'-th and n'-th order, for instance, is recursively computable through m or n steps at most. Let us consider the following sum, for example: $(a_1\bar{0}b_1)\ a_2c_1 + c_1(a_20(a_2d_1b_2))e_1$. This is of the form $x_1x_2y_1 + z_1y_2w_1$ witn $x_1x_2y_1$ of order 3 and $z_1y_2w_1$ of order 4, both terms normal. Recursively we obtain:

$((a_10b_1) + c_1)\ (a_2 + (a_20(a_2d_1b_2))(c_1 + e_1)$ (first step),
$((a_1 + c_1)\ \bar{0}\ (b_1 + c_1))((a_2 + a_2)\ 0\ (a_2 + (a_2d_1b_2)))(c_1 + e_1)$ (second step),
$((a_1 + c_1)\ \bar{0}\ (b_1 + c_1))((a_2 + a_2)\ 0\ ((a_2 + a_2)\ d_1(a_2 + b_2)))(c_1 + e_1)$ (third and last step).

Every term and relation in the final normal term, which is of order less than or equal to 5, is directly computable in T^* and R^*.

If terms and relations are not normal, they can be reduced from forms $x_1 0 x_1$, $x_2 0 x_2$ to forms x_1, x_2 respectively. If, in turn, x_1 and x_2 are not normal, or if some of their components are not normal, we can reduce every non-normal term or relation to a term or relation that is normal or, eventually, to a term of T^* or a relation of R^*. There is then no lack of generality is assuming in our postulates **P24** and **P25** that every term and relation we operate upon is normal.

T6. The sum of a normal term of order n and a term of order 1 (or vice versa) is a term of order n.

Proof: By induction on n. We assume the statement to be true for every $h < k'$, and take the terms $x_1x_2y_1$ of order k' and z_1 of order 1. In $x_1x_2y_1$, x_1 is of order $m \leq k$ and so is y_1. By P24, $x_1x_2y_1 + z_1 = (x_1 + z_1)$ $x_2(y_1 + z_1)$, where $x_1 + z_1$ is the sum of a term of order $m \leq k$ and z_1, and where $y_1 + z_1$ is the sum of a term of order $p \leq k$ and z_1. **T7.** s.f.r.

T8. The sum of normal terms of order m' and n' is a term of order k such that $1 \leq k \leq (m+n)'$.

Proof: a) That $1 \leq k$ is easily inferred from the case $a_1\bar{0}b_1 + b_1\bar{0}a_1 = a_1 + b_1$. b) That k may assume the value $(m+n)'$ is shown by T6. c) That k cannot be greater than $(m+n)'$ can be proved in the following indirect way. Let us assume in $x_1x_2y_1$, x_1 of order e, x_2 of order k, and y_1 of order h; also, in $z_1y_2w_1$, z_1 of order s, y_2 of order t, and w_1 of order v. By **D2** and **D3**, $e + k + h - 1 = m'$ and $s + t + v - 1 = n'$. Now for $(x_1 + z_1)$ $(x_2 + y_2)$ $(y_1 + w_1)$ to be of order $(m+n)''$ (hypothesis of the indirect proof), either the order of $x_1 + z_1$ has to be greater than $e + s - 1$, or the order of $y_1 + w_1$ greater than $h + t - 1$, or the order of $x_2 + y_2$ greater than $k + v - 1$. We already know by b) that the order of $x_1 + z_1$ may be $e + s - 1$ and that the order of $x_1 + w_1$ may be $h + v - 1$. By similar reasoning carried out for relations, we can show that the order of $x_2 + y_2$ may be $k + t - 1$. Let us assume now, for instance, that the order of $x_1 + y_1$ is $(e + s - 1)'$. This is the same as the hypothesis of the indirect proof, but for orders $e < m'$ and $s < n'$. Repeating this process we arrive at the sum of a pair of terms or relations of the form $\alpha_1 + \beta_1$ or $\alpha_2 + \beta_2$, where either α_1 or β_1 on ohe hand, or α_2 or β_2 on the other, will be of order 1. If α_1 is of order ν and β_1 of order 1, for instance, the order of $\alpha_1 + \beta_1$ cannot be ν' (T6); or if α_2 is of order η and β_2 of order 1, the order of $\alpha_2 + \beta_2$ cannot be η' (**T7**). Therefore, some of the remaining terms or relations must satisfy the hypothesis of the indirect proof. Repeating the analysis as many times as necessary, we see that the hypothesis necessarily contradicts theorems **T6** and **T7.** Hence, $k \leq (m+n)'$. **T9.** s.f.r.

Multiplication of terms and relations can be introduced in a way that is analogous to the introduction of addition in postulates **P24** and **P25.** The general forms for multiplication will be:

P26. $(x_1x_2y_1)\ (z_1y_2w_1) = (x_1z_1)\ (x_2y_2)\ (y_1w_1)$.
P27. $(x_2x_1y_2)\ (z_2y_1w_2) = (x_2z_2)\ (x_1y_1)\ (y_2w_2)$.
T10. $x_1x_2y_1 + z_1 = (x_1 + z_1)\, x_2\, (y_1 + z_1)$.

Proof: **P24, T0. T11.** s.f.r.

T12. $x_1x_2y_1 \cdot z_1 = (x_1z_1)\ \bar{0}\ (y_1z_1)$. **T13.** s.f.r.
T14. $x_1 + y_1 = y_1 + x_1$. **T15.** s.f.r.

T16. $x_1 \cdot y_1 = y_1 \cdot x_1$. **T17.** s.f.r.
T18. $(x_1 + y_1) + z_1 = x_1 + (y_1 + z_1)$. **T19.** s.f.r.
T20. $(x_1 \cdot y_1)z_1 = x_1(y_1 \cdot z_1)$. **T21.** s.f.r.
T22. $(x_1 + y_1)z_1 = x_1z_1 + y_1z_1$. **T23.** s.f.r.
T24. $(x_1x_2y_1)' = x_1'x_2y_1'$. **T25.** s.f.r.
T26. $(x_1x_2y_1 + z_1)' = (x_1' + z_1)\, x_2\, (y_1' + z_1)$. **T27.** s.f.r.
T28. $(x_1x_2y_1)' \cdot z_1 = (x_1z_1)\,\bar{0}\,(y_1z_1) + z_1$. **T29.** s.f.r.
T30. $((x_1x_2y_1)y_2z_1)' = (x_1'x_2y_1')y_2z_1'$. **T31.** s.f.r.

Proofs of theorems **T14** - **T23** can be obtained by double induction (on the maximum order of terms and on the maximum order of relations), since in T^* and R^* commutative, associative, distributive, and well-ordering laws are all valid. Proofs for **T24** - **T31** are straightforward. In general, there is no well-ordering law for either terms or relations beyond $T^{(1)}$ and $R^{(1)}$, for it is clear that the trichotomy law does not apply in such cases.

If we now write 0, 1, 2, 3, ... for terms, and $\bar{0}$, $\bar{1}$, $\bar{2}$, $\bar{3}$, ... for relations in T^* and R^*, respectively, we can present the following examples of the arithmetic of the term-relation number theory:

E1. $4 = 4\bar{0}4 = (4\bar{0}(4\bar{0}4))\bar{0}4 = 4\bar{0}((4\bar{0}4)\bar{0}4) = 4\bar{0}(4\bar{0}(4\bar{0}4))$, etc.
E2. $1\bar{2}3.4 = 4\bar{0}12.$
E3. $4\bar{7}29.6\bar{8}1 = 24\overline{56}29.$
E4. $0 = 0\bar{0}0.$
E5. $2\bar{0}2 > 2\bar{0}1.$
E6. $2\bar{0}1$ and $1\bar{0}2$ are not comparable in the sense of order. In general, it is not true that $a_1a_2(b_1b_2(c_1c_2d_1)) = a_1a_2((b_1b_2c_1)c_2d_1) = ((a_1a_2b_1)b_2c_1)c_2d_1$, and operations involving any two of these three terms will produce, in general, different results. For example:

E6. $2\bar{3}4 + 4\bar{5}(6\bar{1}2) = 6\bar{8}(10\bar{1}6)$; $2\bar{3}4 + (4\bar{5}6)\bar{1}2 = (6\bar{5}8)\bar{4}6$.
E7. $2\bar{3}4 \cdot 4\bar{5}(6\bar{1}2) = 8\overline{15}(24\bar{0}8)$; $2\bar{3}4 \cdot (4\bar{5}6)\bar{1}2 = (8\bar{0}12)\bar{2}8$.

0 is the identity element for addition of terms; hence, the totality $T^{(\infty)}$ of terms of any order and the operation of addition, form a semi-group with identity. However, 1 is not an identity for multiplication in $T^{(\infty)}$; $T^{(\infty)}$ and multiplication of terms form a semi-group without identity.

5. *Logic of two-sorted theories*. At this point, the question about the consistency of the system of term-relation predicate formulae arises. The general problem of consistency for systems of many-sorted theories has been studied by Herbrand [6], Schmidt [8], and Wang [10]. Following Wang's approach to this problem, let us call T_2 the theory outlined above, and $T_{1,(2)}$ the theory formed in the following way: we put together terms and relations as being parts of one single kind of variable x, and introduce two predicates, S_1 and S_2, such that x is a term if and only if $S_1(x)$, and x is a relation if and only if $S_2(x)$. Adding these two predicates to the predicates of T_2, we obtain a one-sorted logic, $L_{1,(2)}$, by requiring that for every i $(i = 1,2)$, $\exists\, xS_i(x)$ is a theorem. Since L_2 is the logic for theory T_2, every statement of L_2 can be translated in $L_{1,(2)}$ whereas, on the other hand, some statements of $L_{1,(2)}$ have translation in L_2. Under these circumstances (see

Wang [10]), we may make the following statements. (i) A statement in T_2 is provable if and only if its translation in the corresponding theory $T_{1,(2)}$ is provable. (ii) If T_2 is consistent, then $T_{1,(2)}$ is consistent. (iii) If $T_{1,(2)}$ is consistent, then T_2 is also consistent. (iv) Given a statement of T_2 and a proof for it in T_2, there is an effective way of finding a proof in $T_{1,(2)}$ for its translation in $T_{1,(2)}$; and conversely, given a statement of $T_{1,(2)}$ which has a translation in T_2, and given a proof for it in $T_{1,(2)}$, there is an effective way of finding a proof in T_2 for its translation in T_2 (Schmidt).

6. *Interpretation.* The totality of objects that satisfy a given term-predicate formula is usually called a set. Let us give the name of "net" to the totality of objects that satisfy a relation-predicate formula. Sets and nets have no formal differences apart from interpretation. With respect to the interpretation of term-relation predicate formulae, we may now introduce the following terminology. Let us apply the term "organism of order $m \times n$" (employing a word used by Whitehead in [12] that does not have too different a meaning, we believe) to the totality of terms of order m and relations of order n which satisfy a given term-relation predicate formula. With this definition, a theory of organisms could be developed that would contribute to the interpretation of the term-relation predicate calculus. Typical problems of this theory would be of this nature, for example: For a given set of terms of order m and a net of relations of order n, how many organisms of order $m \times n$ are possible? If T^* is $\{a_1, b_1\}$ and R^* is $\{a_2, b_2, c_2\}$, we should in this case form first the set of terms $T^{(\infty)} = \{a_1, b_1, a_1a_2b_1, a_1a_2a_1, b_1a_2a_1, \ldots, (a_1a_2b_1)\ a_2c_1$, etc.$\}$, and the corresponding net of relations $R^{(\infty)}$. A proper subset of $T^{(\infty)}$ is $T^{(m)}$, the collection of terms of order m; s.f.r. An arbitrary term-relation predicate formula $P(x_1,x_2)$ divides $T^{(m)}$ and $R^{(n)}$ into two subsets and subnets, those of terms of order m and those of relations of order n, which may or may not satisfy the predicate formula P. It is an elementary problem of combinatorial analysis to obtain in every case, for m and n fixed, the number of possible different organisms.

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