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POSTULATES FOR THE SUBSTITUTIVE ALGEBRA OF THE 2-PLACE FUNCTORS IN THE 2-VALUED CALCULUS OF PROPOSITIONS

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In recent papers ([1], [2] in the Bibliography at the end of this note), the algebra of functions [3] has been applied to the 2-valued calculus of propositions.

Eight of the sixteen 2-place functors have been denoted by A, B, C, D, E, I, J, and 1, according to the truth tables

	A	B	C	D	E	I	J	1
(0,0)	1	0	1	1	1	0	0	1
(0,1)	1	1	1	0	0	0	1	1
(1,0)	1	1	0	1	0	1	0	1
(1,1)	0	1	1	1	1	1	1	1

Moreover, F' is the functor such that $F'(x,y) \neq F(x,y)$ for every (x,y), where F stands for any of the aforementioned eight functors.

A is incompatibility; A' is conjunction; B is nonexlusive disjunction; C is implication; E is equivalence. I and J will be called (2-place) *selectors* and 1 is the *constant* 2-place functor of value 1. Instead of $1'$, one may also write 0. (In contrast to the truth-values, symbolized by 1 and 0, the constant functors are designated by *italic 1* and 0.)

The principal operation in the algebra of functors is substitution. The result of substituting the ordered pair of functors (G,H) into the functor F is defined* as the functor

$F(G,H)$ such that $F(G,H)(x,y) = F(G(x,y), H(x,y))$ for every (x,y). This operation is what in [1]-[3] has been called *superassociative*; that is to say,
(1) $F(G_1,G_2)(H_1,H_2) = F(G_1(H_1,H_2),G_2(H_1,H_2))$ for any five functors.

*Besides this "*parenthesis substitution*", there is a superassociative "*bracket substitution*" of the ordered pair G, H into F resulting in a 4-place functor $F[G,H]$ (cf. [1], [2], [3]).

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There are 16^3 different expressions $F(G,H)$. If each of the 4096 functors thus designated can be identified with one of the 16 functors A, B, ..., $1'$, then every functor resulting from the substitution of a functor pair into a functor can be so identified. In other words, every formula of the substitutive algebra of 2-place functors can be derived from 4096 *basic* formulae, $F(G,H) = K$. In [2], these basic formulae, in turn, have been reduced to a small number of formulae in terms of A, I, and J, from which the other 13 functors can be defined.

In the present paper we give a more transparent derivation of the 4096 basic formulae from 48 of them, which then will be further reduced. For this purpose we consider the set of the 16^2 ordered pairs (M,N) of 2-place functors. We call each such pair a *transformation* and define the substitution of any transformation (N_1,N_2) into any transformation (M_1,M_2) by

$$(M_1,M_2)(N_1,N_2) = (M_1(N_1,N_2),M_2(N_1,N_2)).$$

This binary operation is *associative* because of (1), that is, because of the superassociativity of the substitution in the realm of 2-place functors. Hence the 256 transformations constitute a semigroup, which will be denoted by $\mathfrak{S}$:

$\mathfrak{S}$ has a maximal subgroup $\mathcal{g}$, which is isomorphic with the octahedral group or the ("symmetric") group of all permutations of four objects (cf. [2]). $\mathcal{g}$ consists of the 24 transformations (G,H) such that

$$G, H = I, J, E, I', J', E \quad \text{and} \quad G \neq H \neq G'.$$

$\mathcal{g}$ can be generated by two transformations, e.g., by (J',I') and (I',E), which we shall denote by h and e, respectively; that is to say, for each transformation (G,H) in $\mathcal{g}$ there exist an integer m (which can always be chosen ≤ 6) and a sequence of m transformations $g_1, g_2, \ldots, g_m$ such that $g_i = h$ or e for i = 1,2,..., m and

$$(G,H) = g_1 g_2 \ldots g_m .$$

E.g., $(I,J) = hh = eeee$ and $(J,I) = eehee$.

Obviously, $\mathcal{g}$ cannot be generated by less than two transformations. Moreover, if s is any transformation not belonging to $\mathcal{g}$, and t is any transformation, then st and ts do not belong to $\mathcal{g}$. Hence $\mathfrak{S}$ cannot be generated by less than three transformations. But $\mathfrak{S}$ can be generated by h, e, and $a = (A,I')$ (see Appendix); that is to say, for each transformation (G,H) in $\mathfrak{S}$ there exists an integer n and a sequence of n transformations $t_1, t_2, \ldots, t_n$ such that $t_i = h$ or e or a (for i = 1,2,...,n) and

$$(G,H) = t_1 t_2 \ldots t_n .$$

It follows that, for any functor F,

$$F(G,H) = F(t_1 t_2 \ldots t_n) = (Ft_1)(t_1 t_2 \ldots t_n) = ((Ft_1)t_2)(t_3 \ldots t_n) = (\ldots((Ft_1)t_2)\ldots t_{n-1})t_n .$$

Hence, if for every functor M, one can identify Mh, Me, and Ma with one of the 16 functors, then one can identify $F(G,H)$ with one of them. For example from

$$(I,A) = eeha \quad \text{and} \quad Ae = B,\ Be = C,\ Ch = C,\ Ca = C$$

one can conclude

$$A(I,A) = A(e\,e\,h\,a) = (A\,e)(e\,h\,a) = B(e\,h\,a) = (B\,e)(h\,a) = C(h\,a) = (C\,h)a = C\,a = C.$$

In this way one can derive the 4096 formulae $F(G,H) = K$ from the 48 formulae tabulated below, which list the results of substituting $h = (J',I')$, $e = (I',E)$, $a = (A,I')$ into the sixteen functions.

(2)

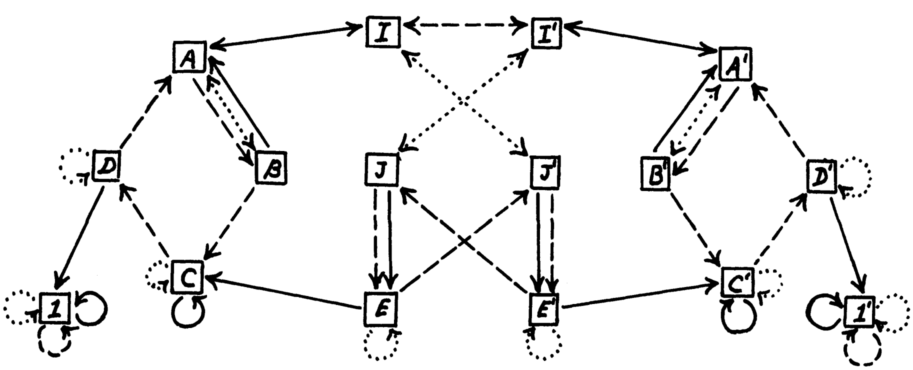
	h	e	a
A	B	B	I
B	A	C	A
C	C	D	C
D	D	A	1
E	E	J'	C
I	J'	I'	A
J	I'	E	I'
1	1	1	1

(2')

	h	e	a
A'	B'	B'	I'
B'	A'	C'	A'
C'	C'	D'	C'
D'	D'	A'	$1'$
E'	E'	J	C'
I'	J	I	A'
J'	I	E'	I
$1'$	$1'$	$1'$	$1'$

For example, $Ba = A$, $Ae = B$, $Ah = B$, $Bh = A$, and $Dh = D$.

The table may be replaced by the following graph in which solid, dashed, and dotted arrows indicate the substitution of a, e, and h, respectively. The aforementioned examples are expressed by the solid arrow from B to A, the dashed arrow from A to B, the dotted double arrow between A and B, and the dotted loop at D.



Starting from any of the 14 nonconstant functors in the graph, one can reach every other functor by a chain of arrows. Hence, *in terms of every nonconstant functor, every functor can be obtained by a sequence of transformations h, e, a.* For example, starting from A one obtains

$Ah = B$		$Aa = I$	
$Ahe = C$	$Aah = J'$	$Aae = I'$	
			$Aaea = A'$
$Ahee = D$	$Aahe = E'$		
	$Aahee = J$	$= Aaeh$	$Aaeae = B'$
$Aheea = 1$			
	$Aaheea = E$		$Aaeaee = C'$
			$Aaeaeee = D'$
			$Aaeaeeea = 1'$

Starting, say, from E, one would obtain $Ea = C$, $Eae = D$, $Eaee = A$ and thus could reach all the other functors.

Thus *for any nonconstant functor M and any functor N there is at least one transformation transforming M into N.* We will select one of these transformations and denote it by t_{MN} . Then

$$Mt_{MN} = N \text{ for every } M \neq 1, 0 \text{ and every } N.$$

The 24 formulae (2') can be derived from the corresponding formulae (2) provided that one introduces a unitary functor (which we shall denote by $\sim$) satisfying the following assumptions

a) $\sim P = P'$ for $P = A, B, \ldots, 1$;

b) $(\sim\sim)P_0 = P_0$ for at least one of the seven nonconstant functors $A, B, \ldots, J$; for example, $(\sim\sim)A = A$.

Moreover, substitution of functors into $\sim$ is assumed to be *associative*; that is to say,

$\sim(Fg) = (\sim F)g$ and $\sim(\sim F) = (\sim\sim)F$ for every functor F and every transformation g.

It follows that, if P stands for any of the eight unprimed functors, then (3) $\sim(Pg) = (\sim P)g = P'g$.

In a formula (2), the functor on the right side is either primed or unprimed. In the first case, in view of (3) and a), from $Pg = Q$ it follows that $P'g = Q'$. In the second case, $P'g = \sim Q'$. Then, from $Q = A\,t_{AQ}$ in view of b) it follows that

$$\sim Q' = \sim(\sim(At_{AQ})) = \sim((\sim A)t_{AQ}) = (\sim(\sim A))t_{AQ} = ((\sim\sim)A)t_{AQ} = A\,t_{AQ} = Q.$$

In both cases, the formula (2) implies the corresponding formula (2').

Summarizing one can say that *the substitutive algebra of the* 2-*place functors can be derived from the* 48 *formulae* (2) *and* (2') *or from the* 24 *formulae* (2) *in conjunction with* a) *and* b).

APPENDIX ON THE SEMIGROUP $\mathfrak{S}$.

That the transformations $h, e,$ and a generate the semigroup $\mathfrak{S}$ was first verified by Miss Eva L. Menger and Mr. H. Ian Whitlock, who expressed each of the 256 transformations (G,H) in terms of $h, e,$ and a. Then the possibility of generating $\mathfrak{G}$ was shown by Mr. Richard Peters at Illinois Institute of Technology as follows. Each of the four pairs (x,y) = (0,0), (0,1), (1,0), (1,1) may be represented by the number 2x + y, which is 0, 1, 2, 3, respectively; and similarly, one may represent the value of the transformation (G,H) for (x,y), that is, the pair $(G(x,y),H(x,y))$, by the number 2 $G(x,y) + H(x,y)$. Accordingly, one may represent each transformation (G,H) by the ordered quadruple $[i_0,i_1,i_2,i_3]$ of the values that (G,H) assumes for 0, 1, 2, 3, respectively. (For this representation, cf. also [2].) For example, $h = (J',I')$, $e = (I',E)$ and $a = (A,I')$ are represented by

[3, 1, 2, 0], [3, 2, 0, 1], and [3, 3, 2, 0],

respectively. Each of the 24 transformations belonging to the maximal subgroup $\mathcal{g}$ of $\mathfrak{G}$ is represented by a permutation. Each of the 144 transformation that assume exactly three different values, as does a, is represented by $g_1 a g_2$ for some transformations g_1 and g_2 in $\mathcal{g}$. The 48 transformations assuming one value exactly three times can be obtained in the form $g_1 a e a g_2$. The 36 transformations assuming two values twice can be obtained in the form $g_1 a e h a g_2$. The four constant transformations (0,0),...(1,1) can be obtained in the form $g a e a h e e a$ for some g in $\mathcal{g}$. Thus all 256 transformations in $\mathfrak{G}$ are accounted for.

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