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A SYSTEM OF QUANTIFICATIONAL DEDUCTION[1]

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I. *Introduction.* This paper will describe a simple system of quantificational deduction and give proofs of its soundness and completeness. The latter proof will be put in terms of a device to be introduced in a later section, the quantifier game. This expository gambit, which leads to a rather simple proof, permits an independent treatment of key semantic principles prior to the details of their application. Elsewhere, the device may be put to work both as a pedagogical ploy and in theoretical arguments, which lends it an interest that extends beyond the scope of its uses here. The present completeness proof carries over to most systems of natural deduction, since the deductions permitted by the system whose completeness is proved are easily seen to have counterparts in these other systems. However, this system is a clumsy one to use, and so for practical purposes its equivalence to a more workable system will be sketched in a final section.

II. *The system of deduction.* This system is designed to prove the inconsistency of single quantificational formulas in prenex normal form. It also provides for a *reductio ad absurdum* proof that an argument is valid, since it may be used to prove that a prenex normal form version of a conjunction of the premises and the negation of the conclusion is inconsistent.

The system has just two rules of derivation, called **UI** and **EI**. Let *Fm* be any formula in which *m* occurs free and let *Fn* be like *Fm* except that *Fn* has free *n* everywhere that *Fm* has free *m*. Then **UI** and **EI** are the rules whereby we respectively pass from a formula of form $(m)Fm$ or $(\exists m)Fm$ to the corresponding formula of form *Fn*. Starting with a single formula in prenex normal form, these rules enable us to write down a sequence of lines each

1. This system, which relates closely to the "more economical one" of Quine [3], p. 254, and, more remotely, to Method A of Quine [4], derives from Herbrand's Theorem, as may be seen from the version of this given in Hilbert and Bernays [2], pp. 157-163. For more light on these historical roots, also see Dreben [1]. I am grateful to J. S. Ullian and P. J. S. Benacerraf, who read an earlier draft of this paper and made helpful suggestions.

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of which follows by **UI** or **EI** from some predecessor. But we impose the restriction that the variable n of the formula Fn gotten by an **EI** step mustn't be free in any previous line of the sequence.

We will say that a set of formulas is *derivable* from S if there is a sequence as above that has S as its top line and has each formula of the set as one of its lines. (It might be said here that the union of two such derivable sets may itself not be derivable from S, due to our **EI** restriction.) The system works on the principle that a formula S is inconsistent just in case a truth-functionally inconsistent set of quantifierless formulas is derivable from S. Thus in using the system, we try to derive such a set. E. g. suppose we wish to prove the validity of the syllogism that has $(x)(Fx \supset Gx)$ and $(\exists x)(Fx \,.\, Hx)$ as premises and $(\exists x)(Gx \,.\, Hx)$ as conclusion. The following derivation, whose top formula is a prenex normal form version of a conjunction of the premises and the negation of the conclusion, accomplishes this, since lines *(4)* and *(7)* are truth-functionally inconsistent.

(1) $(x)(\exists y)(z)[(Fx \supset Gx) \,.\, Fy \,.\, Hy \,.\, \sim(Gz \,.\, Hz)]$

(2) $(\exists y)(z)[(Fx \supset Gx) \,.\, Fy \,.\, Hy \,.\, \sim(Gz \,.\, Hz)]$

(3) $(z)[(Fx \supset Gx) \,.\, Fw \,.\, Hw \,.\, \sim(Gz \,.\, Hz)]$

(4) $(Fx \supset Gx) \,.\, Fw \,.\, Hw \,.\, \sim(Gw \,.\, Hw)$

(5) $(\exists y)(z)[(Fw \supset Gw) \,.\, Fy \,.\, Hy \,.\, \sim(Gz \,.\, Hz)]$

(6) $(z)[(Fw \supset Gw) \,.\, Fy \,.\, Hy \,.\, \sim(Gz \,.\, Hz)]$

(7) $(Fw \supset Gw) \,.\, Fy \,.\, Hy \,.\, \sim(Gz \,.\, Hz)$

III. *Interpretation and soundness.* As we will use the term, an *interpretation* of a quantificational formula S assigns a universe to the quantifiers of S, a set of ordered k-tuples of members of the universe to each k-adic predicate-letter of S, a particular member of the universe to each free variable of S, and finally a truth-value, either **T** or **F**, to each statement-letter of S.

What it means for a formula to be *true* on an interpretation, and thus to be readable as a true statement about the members of the assigned universe, may be seen in terms of an example. The formula '$p \vee (x)\,[(\exists y)\, Fxy \supset Gzx]$' is true on an interpretation if either the statement-letter 'p' is assigned the truth-value **T** or else for each member of the universe X, if a member Y exists such that the pair X, Y is in the set of ordered pairs assigned to the predicate-letter 'F', then the pair Z, X is in the set of ordered pairs assigned to the predicate letter 'G', where Z is the member of the universe assigned to the free variable 'z'.

To prove soundness, we must establish that if a formula S is consistent, then every set of quantifierless lines derivable from S is truth-functionally consistent. This makes the system sound in the sense that it never leads us to call a consistent formula inconsistent or an invalid argument valid.

Suppose that a formula S is consistent, and thus comes out true on some interpretation, whose universe let us call U. Then S may be read as a true statement about the members of U, along the lines just sketched.

But if S is of form $(m)Fm$, then a line Fn derivable from S by **UI** is obviously also a true statement, if we take the free variable n to name some member of U. Moreover, if S is of form $(\exists m)Fm$, then a line Fn obtained from S by **EI** is a true consequence of S if we take n to name one of the members of U whose existence is asserted by S. The same arguments apply to a line derived by **UI** or **EI** from a line derived from S, and so on as the derivation grows. Hence on this reading of their free variables as names, every quantifierless line in a set derivable from S becomes a true statement about the members of U.

It might seem that a set of truth-functional formulas which all qualify as true on such a reading must be truth-functionally consistent. In fact this principle can fail if the truth of the formulas depends upon homonymy. E.g. the two formulas 'Fx' and '$\sim Fx$' might both be true statements, granted a homonymous use of the variable 'x'. However, it is clear that the principle holds when homonyms don't occur, for then the reading on which all the formulas are true, like a truth-table analysis, assigns the same truth-value to each occurrence of any given atomic component.

Finally, let us note that the quantifierless formulas of a set derivable from S don't depend for their truth upon homonymy. A line Fn gotten by **UI** will be true no matter what member of the universe we take n to name, so this free variable needn't be read as a homonym when we read the line as a truth. Moreover, due to our **EI** restriction, the free variable n in a line Fn obtained by **EI** will be a new name and thus not a homonym on the reading that makes this formula true. But since homonyms don't arise, we see by the principle stated above that the set is truth-functionally consistent. This completes the proof that the system is sound.

IV. *The quantifier game.* The equipment used for a quantifier game is a quantificational formula S in prenex normal form, an interpretation of S in a finite or denumerable universe, and a set of symbols M the same size as the universe. These symbols, which must be distinct from the variables of S, will be used as one–one names of the members of the universe. Much as the board is set up for a chess game, each free variable of S is replaced before the game begins by the symbol of M that names the member of the universe assigned to it by the interpretation. This new formula is then transformed by stages as the game goes on.

The game is played by two players, P (for positive) and N (for negative). It is P's turn to move whenever the formula at hand is of form $(\exists m)Fm$. P's move is to write down a formula Fn that follows from this one by **EI**, where n is a symbol of M. In parallel fashion, N's move, which comes whenever the formula at hand is of form $(m)Fm$, is to write down a formula Fn derivable from this one by **UI**, where n is again a symbol of M. The earlier restriction on **EI** steps won't be imposed in this context.

The game ends when neither P nor N has a move, the formula then at hand being a truth-function whose atomic components are statement-letters and k-adic predicate-letters followed by rows of k symbols of M. The interpretation determines a truth-value for each of these atoms, in obvious fashion, so the truth-value of the final formula may now be worked out. P wins the game if this formula comes out true, while N wins if it comes out false.

The sequence of four formulas at the right below marks the stages of a quantifier game played with the topmost formula, an interpretation in a universe with two members, and a set M that contains the two symbols 'a' and 'b'. The sets assigned to the predicate-letters 'F' and 'G' are here shown by the truth-values assigned to the atomic formulas listed at the left.

p	**F**	Gaa	**T**	$(x)(\exists y)[p \vee (Fx \supset Gyz)]$
z	b	Gab	**F**	$(x)(\exists y)[p \vee (Fx \supset Gyb)]$
Fa	**T**	Gba	**T**	$(\exists y)[p \vee (Fa \supset Gyb)]$
Fb	**F**	Gbb	**T**	$p \vee (Fa \supset Gbb)$

P wins this game since the final formula comes out true on this assignment of truth-values.

In general, either P or N may be able to win a given game if the other plays badly enough. But while the mere outcome of a game thus tells us little, the quantifier game is governed by the following two useful principles:

(1) If N plays optimally and P wins, then the given formula comes out true on the interpretation at hand.

(2) If P plays optimally and N wins, then the given formula comes out false on the interpretation at hand.

These two statements are easily established by proving two others that are equivalent to them in light of truth-functional equivalences and the dualities of truth and falsity and winning and losing. Omitting a similar proof of (1), we will prove (2) in this manner, the equivalent statement being the following:

(3) If the given formula comes out true on the interpretation at hand, then P wins if P plays optimally.

If we suppose that the formula comes out true, we may then regard it as a true statement about the members of the universe of the interpretation. Moreover, this same truth remains after we replace free variables. But at any later stage of the game, if the formula at hand is a true statement of form $(\exists m)Fm$, then P will be able to pick a member n of the universe such that the formula Fn that results from his move is also a true statement, while if the formula at hand is a true statement of form $(m)Fm$, then N will be unable to select a member of the universe n such that the formula Fn that he writes down as his move is a false statement. Thus the truth of the original formula is preserved at every stage of the game, granted that P plays optimally, and P will win the game, since the final formula will come out true on the assignment of truth-values fixed by the interpretation.

V. *The completeness of the system.* We show that if a formula S in prenex normal form is inconsistent, then some finite set of quantifierless lines derivable from S will be truth-functionally inconsistent. Hence the system is complete in the sense that it allows us in every case to prove that an inconsistent formula is inconsistent or that a valid argument is valid.

The first stage of the proof is most perspicuous if we use a quasi-example. Suppose that S is of form $(w)(\exists x)(y)(\exists z)Fwxyz$. Then where V is an infinite set of variables which contains all free variables of S but none of the bound variables of S, it is plain that there is a set D of quantifierless formulas derivable from S which meets the following condition:

(4) For every variable a of V there is a variable b of V such that for every variable c of V there is a variable d of V such that the formula $Fabcd$ is in D.

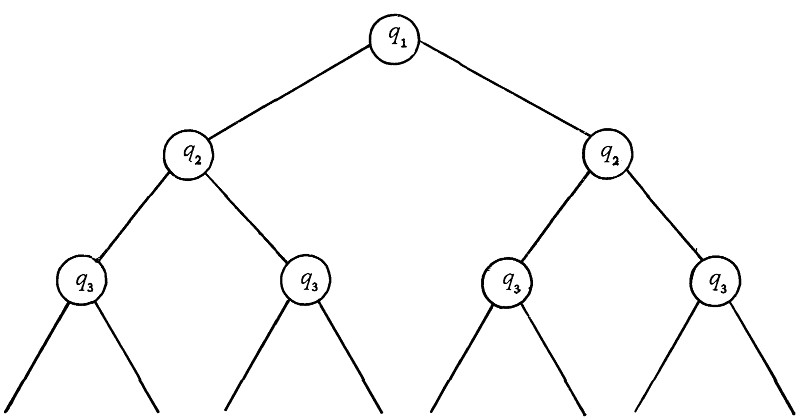
Now suppose that all the formulas of D become true on some assignment of truth-values to their component atoms. Then where V, which is obviously denumerable, serves as the set M, it turns out that however well N plays, P can win a quantifier game using S and an interpretation that corresponds to this assignment. By quantifier game principle (1), therefore, S is consistent.

P wins by playing in such a fashion that the final formula of the game is a formula of D, which, by our supposition, comes out true on the interpretation at hand. That P can manage this is evident, since (4) entails the following statement:

(5) For every first move by N there is a first move by P such that for every second move by N there is a second move by P such that the final formula of the game is in D.

This argument shows that if S is inconsistent then the set D is truth-functionally inconsistent. As a second stage of the proof in progress, we show that in this case some finite subset of D is also truth-functionally inconsistent. However, what we prove won't be just this, since our argument also establishes a more general law. Let R be an infinite set of truth-functional formulas whose atomic components are q_1, q_2, ... This law then tells us that if R is truth-functionally inconsistent then some finite subset of R is also truth-functionally inconsistent.

The proof of this will be carried out in terms of a binary tree with labelled nodes. The top node is labelled q_1 and each node labelled q_i branches into two nodes each labelled q_{i+1}. A finite top part of this tree is shown below:



By a *path* in this tree will be meant the sequence of nodes encountered if we start at the top node and move downward along the lines that connect the nodes, going either left or right at every juncture. A path is finite if this descent stops at some stage and is infinite otherwise. If we now construe going left and going right from a node labelled q_i as the assignments of a **T** and an **F** respectively to q_i, each path in the tree becomes an assignment of truth-values to the q's—an infinite path assigns truth-values to all the q's, on this construction, and a finite path with $n+1$ nodes assigns truth-values to $q_1, \ldots, q_n$.

Now suppose that R is inconsistent. This means that every infinite path, regarded as an assignment of truth-values to the q's, falsifies some formula of R. Given any one of these paths and a formula it falsifies, moreover, it is plain that a finite front end of the path falsifies the formula. For every formula is such that for some index r, $q_{r+1}, q_{r+2}, \ldots$ don't occur in it, and whether it comes out true or false on a given assignment is determined just by the truth-values assigned to $q_1, \ldots, q_r$. On this basis, we now erase as much as possible of each path still leaving it a path that falsifies some formula of R. The result of these erasures will be a tree every path in which is finite.

It remains to prove that this tree is *finite* in the sense that it has a finite number of nodes. This means that for some integer s, no path has more than $s+1$ nodes, which tells us that every assignment of truth-values to $q_1, \ldots, q_s$ falsifies one or another formula of R. For if some such assignment falsified no formula of R, then by the manner of its construction, some path in the tree at hand would have more than $s+1$ nodes. It follows from this, however, that a subset of R that contains at most 2^s formulas is truth-functionally inconsistent.

As the last part of this argument, then, we establish that if a tree as above is infinite in the sense defined then it has an infinite path. The finiteness of the tree at hand will then follow from the fact that every path in this tree is finite.

Let a node be called *productive* if the subtree that it tops is infinite. In these terms, we define a particular path P in a tree labelled as above as the sequence of nodes met in a descent from the top node which is guided by the following instructions:

(6) Descend to the left from a node labelled q_i to a node labelled q_{i+1} if the latter node is productive,

(7) Otherwise, descend to the right from the node labelled q_i to a node labelled q_{i+1} if the latter node is productive,

(8) Stop if neither node labelled q_{i+1} is productive.

Suppose now that the tree in which P is a path is infinite. Since the top node of this tree is then productive, we see from the definition of P that every node of P is productive. But it follows from this and the definition that P is infinite. For P will have a last node just in case neither node into which this node branches is productive. In such a case, however, the last node itself won't be productive, which is impossible.[2]

On the basis of these soundness and completeness proofs, it is easy to establish Löwenheim's Theorem, which states that if a formula is consistent, then it is true on some interpretation in a denumerable universe. Since the system is sound, we know that if a formula S in prenex normal form is consistent, then the set D, which is derivable from it, is truth-functionally consistent. In that case, however, as was seen in the completeness proof, S comes out true on an interpretation in a universe no larger than the denumerable set of variables V.

VI. *An equivalent system of deduction.* We next present a more workable system of deduction than the one defined in Section II. The new system differs from the old in that it allows us to use a finite set of formulas in prenex normal form as premises in a derivation. The rules of inference are the same as before, and again the object is to derive a truth-functionally inconsistent set of quantifierless lines, which will indicate that the set of premises is inconsistent. Thus an argument may be proved valid, in general with less effort than before, by establishing that a set of formulas that comprises prenex normal form equivalents of its premises and the negation of its conclusion is inconsistent.

Let $S_1, \ldots, S_n$ be formulas in prenex normal form and let S be a prenex normal form version of a conjunction of these. The new and old systems will be shown to be equivalent in the sense that an inconsistent set of quantifierless formulas is derivable from $S_1, \ldots, S_n$ in the new system just in case another such set is derivable from S in the old system.

The soundness proof of Section III applies *mutatis mutandis* to the new system. Hence $S_1, \ldots, S_n$ form an inconsistent set if a truth-functionally inconsistent set of quantifierless formulas is derivable from them. But in this case, S is inconsistent too, which means that another such set is derivable from S in the old system, since the old system is complete.

To establish the converse implication, suppose we are given a derivation in the old system which proves S to be inconsistent. Our argument remains fully general if we assume that no variable of $S_1, \ldots, S_n$ occurs free here. With this derivation as a guide, however, it is easy to construct a derivation from $S_1, \ldots, S_n$ in the new system whose quantifierless lines, put into conjunctions of n terms each in a suitable fashion, become the quantifierless lines of the given derivation. The trick here is simply to copy **UI** and **EI** steps in the obvious sense. Thus a truth-functionally inconsistent set of quantifierless formulas turns out also to be derivable from $S_1, \ldots, S_n$ in the new system.

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2. The general law here proved is essentially the nontrivial half of what is called the Law of Infinite Conjunction in Quine [3], pp. 254,5. The fact used in proving it, that if every path in a tree as above is finite then the tree is finite, is a special case of the Brouwer Fan Theorem. What has just been proved is the contrapositive of this, a special case of König's Lemma.

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