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AN AXIOM-SYSTEM FOR $\{K;N\}$–PROPOSITIONAL CALCULUS RELATED TO SIMONS' AXIOMATIZATION OF S3

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In [4] Simons has shown that Lewis' system S3 can be axiomatized with six mutually independent axiom schemata and the rule of detachment for material implication. As I mentioned in [5], p. 52, it is clear that this formalization of Simons can be reformulated in such a way that instead of axiom schemata the analogous proper axioms

A1 *NMKpNKpp*
A2 *NMKKpqNq*
A3 *NMKKKrpNKqrNKpNq*
A4 *NKNMpNNp*
A5 *NMKpNMp*
A6 *NMKNMKpNqNNMKNMqNNMp*

are adopted together with the following two rules of procedure

I. The rule of substitution ordinarily used in the propositional calculus, but adjusted to the primitive functors "*K*", "*N*" and "*M*".

II. The rule of detachment adjusted to the primitive functors "*K*" and "*N*", viz.:

> *If the formulas* "$NK\alpha N\beta$" *and* "α" *are theses of the system, then formula* "β" *is also a thesis of this system.*

In this note I like to stress a rather interesting fact that the following four theses

B1 *NKpNKpp*
B2 *NKKpqNq*
B3 *NKKKrpNKqrNKpNq*
B4 *NKNKpNqNNKNqNNp*

i.e. the formulas which we can obtain by deleting the modal functor *M* in the axioms *A1*, *A2*, *A3* and *A6* of Simons, taken together with the rules of procedure I and II constitute an axiom-system for the complete classical $\{K; N\}$–

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propositional calculus. It should be noted that this axiomatization which possesses a certain peculiar property does not appear on the list assembled by Porte in [2] of the known axiom-systems for $\{K; N\}$–propositional calculus.

In order to prove that the discussed axiomatization constitute the complete classical $\{K; N\}$–propositional calculus we proceed as follows:

METARULE OF PROCEDURE **SI**

SI *If* $\vdash \alpha$ *and* $\vdash NK\alpha NNK\beta N\gamma$, *then* $\vdash NKN\gamma NN\beta$

Proof:

a)	$\vdash \alpha$	[The assumption]
b)	$\vdash NK\alpha NNK\beta N\gamma$	[The assumption]
c)	$\vdash NK\beta N\gamma$	[b; a]
d)	$\vdash NKN\gamma NN\beta$	[$B4$, p/β, q/γ; c]

Q. E. D.

$B5$	$NKNKpNqNNKKrpNKqr$	[$B4$, $p/KKrpNKqr$, $q/KpNq$; $B3$]
$B6$	$NKNpp$	[$B5$, q/Kpp, r/Np; $B1$; **SI**; $B2$, q/p]
$B7$	$NKNKprNNKrNNp$	[$B5$, p/NNp, q/p; $B6$, p/Np; **SI**]
$B8$	$NKpNNNp$	[$B7$, p/Np, r/p; $B6$]
$B9$	$NKNKppNNp$	[$B7$, $r/NKpp$; $B1$]
$B10$	$NKKrpNKNNpr$	[$B5$, q/NNp; $B8$]
$B11$	$NKNKNNprNNKrp$	[$B7$, p/Krp, $r/NKNNpr$; $B10$]
$B12$	$NKNNpNKNNpp$	[$B5$, $p/NKNNpp$, $q/NKpp$, r/NNp; $B11$, r/p; **SI**; $B9$]
$B13$	$NKNpNNp$	[$B5$, p/NNp, $q/KNNpp$, r/Np; $B12$; **SI**; $B2$, p/NNp, q/p]
$B14$	$NKpNp$	[$B5$, p/Np, q/Np, r/p; $B13$; **SI**; $B6$]
$B15$	$NKKpqNKqp$	[$B5$, p/q, r/p; $B14$, p/q]
$B16$	$NKNKqpNNKpq$	[$B4$, p/Kpq, q/Kqp; $B15$]

METARULE OF PROCEDURE **SII**

SII *If* $\vdash NK\alpha N\beta$ *and* $\vdash NK\beta N\gamma$, *then* $\vdash NK\alpha N\gamma$

Proof:

a)	$\vdash NK\alpha N\beta$	[The assumption]
b)	$\vdash NK\beta N\gamma$	[The assumption]
c)	$\vdash NKN\gamma\alpha$	[$B5$, p/α, q/β, $r/N\gamma$; a; **SI**; b]
d)	$\vdash NK\alpha N\gamma$	[$B16$, p/α, $q/N\gamma$; c]

Q. E. D.

$B17$	$NKNKpqNNKpNNq$	[$B16$, p/q, q/p; $B7$, p/q, r/p; **SII**]
$B18$	$NKNKpNNqNNKpq$	[$B16$, p/NNq, q/p; $B11$, p/q, r/p; **SII**]
$B19$	$NKNKqpNNKpNNq$	[$B16$; $B17$; **SII**]
$B20$	$NKNKqpKpNNq$	[$B18$, $p/NKqp$, $q/KpNNq$; $B19$]
$B21$	$NKKpNNqNKqp$	[$B16$, $p/KpNNq$, $q/NKqp$; $B20$]
$B22$	$NKKpqNp$	[$B15$; $B2$, p/q, q/p; **SII**]
$B23$	$NKKNKqrNNKrpNKpNq$	[$B21$, $p/NKqr$, q/Krp; $B3$; **SII**]
$B24$	$NKNKpNqNNKNKqrNNKrp$	[$B4$, $p/KNKqrNNKrp$, $q/KpNq$; $B23$]

Thus, we have the theses $B1$, $B22$, and $B24$, and, therefore, we obtained Rosser's axiom-system of $\{K;N\}$–propositional calculus, *cf.* [3], p. 12 and pp. 54-76. Hence, the proof is completed.

The argumentations concerning the independence of the formulas $H1$, $H2$ and $H3$ given by Simons, [4], pp. 314-315, show also that each of the axioms $B1$, $B2$ and $B3$ does not follow from the remaining postulates of our axiom-system. It is evident that $B4$ is independent of the other axioms, since $B1$-$B3$ and the rules of procedure I and II are such that they allow only: a) if α is one of the mentioned axioms, to deduce $K\alpha\alpha$ (by $B1$) and b) if $K\beta\beta$ is a formula already proved, to deduce either β (by $B2$) or $KK\beta\beta K\beta\beta$ (by $B1$). And, since no thesis being a consequence of $B1$-$B3$ can have a form $K\alpha N\beta$, axiom $B3$ cannot be used at all. Hence, our axioms are mutually independent.

As it was mentioned above the discussed axiom-system possesses a certain peculiar property. Namely, instead of $B4$ we can adopt, obviously, as an axiom, the following simpler thesis

$B4^*$ $NKNKpqNNKqNNp$

since $B4$ follows from $B4^*$ by a direct substitution ($B4^*$, q/Nq). Thus, we obtain two inferentially equivalent axiom-systems, viz. $\{B1;\ B2;\ B3;\ B4\}$ and $\{B1;\ B2;\ B3;\ B4^*\}$, such that the former implies $B4^*$ in a rather complicated way, while no use of $B1$-$B3$ is needed in order to deduce $B4$ in the latter system. This situation resembles the cases of the, so called, "generalizing deduction" analyzed by Łukasiewicz in [1].

BIBLIOGRAPHY

[1] Jan Łukasiewicz: Uwagi o aksjomacie Nicoda i o "dedukcji uogólniającej" (Remarks on Nocod's axiom and on "generalizing deduction"). Published in "*Księga pamiątkowa Polskiego Towarzystwa Filozoficznego we Lwowie.*" Lwów, 1931. Republished in Jan Łukasiewicz: Z zagadnień logiki i filozofii. Pisma wybrane. Warszawa, 1961. Pp. 164-177.

[2] Jean Porte: Schémas pour le calcul des propositions fondé sur la conjonction et la négation. *The Journal of Symbolic Logic*, v. 23 (1958), pp. 421-431.

[3] J. Barkley Rosser: *Logic for mathematicians.* McGraw-Hill Book Company, Inc. 1953.

[4] Leo Simons: New axiomatization of S3 and S4. *The Journal of Symbolic Logic*, v. 18 (1953), pp. 309-315.

[5] Bolesław Sobociński: A contribution to the axiomatization of Lewis' system S5. *Notre Dame Journal of Formal Logic*, v. III, pp. 51-60.

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