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STRUCTURAL RULES OF INFERENCE

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On many occasions the following three rules:

R: $A \vdash A$ (Reflexivity),

E: *If* $A_1, A_2, \ldots, A_n \vdash B$, *then* $A_1, A_2, \ldots, A_n, C \vdash B$ (Expansion),

P: *If* $A_1, A_2, \ldots, A_{i-1}, A_i, A_{i+1}, A_{i+2}, \ldots, A_n, A_{n+1}, A_{n+2} \vdash B$, *then* $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, A_i, A_{i+2}, \ldots, A_n, A_{n+1}, A_{n+2} \vdash B$, *where* $i \leq n + 2$ (Permutation),

are appointed as structural rules of inference for the propositional calculus;[1] on others, **P** and the following generalization of **R**:

GR: $A_1, A_2, \ldots, A_n, A_{n+1} \vdash A_i$, *where* $i \leq n + 1$ (Generalized Reflexivity),

are made to serve in that capacity.[2] I examine here the impact of this switch from **R** and **E** to **GR** upon the proving and deriving of rules of inference for the said calculus.

Let P be a (pure) propositional calculus with '$\sim$' and '$\supset$' as primitive connectives. Let 'A', 'B', and 'C' range in the metalanguage MP of P over the wffs of P. Let (meta)statements of MP of the form '*B is implied in P by* (*or deducible in P from*) $A_1, A_2, \ldots,$ *and* A_n' be abbreviated to read '$A_1, A_2, \ldots, A_n \vdash B$' and called turnstile statements or, for short, *T*-statements. Let the following four rules serve as intelim rules for '$\sim$' and '$\supset$':

NI: *If* $A_1, A_2, \ldots, A_n, B \vdash C$ *and* $A_1, A_2, \ldots, A_n, B \vdash \sim C$, *then* $A_1, A_2, \ldots, A_n \vdash \sim B$,

NE: *If* $A_1, A_2, \ldots, A_n \vdash \sim\sim B$, *then* $A_1, A_2, \ldots, A_n \vdash B$,

HI: *If* $A_1, A_2, \ldots, A_n, B \vdash C$, *then* $A_1, A_2, \ldots, A_n \vdash B \supset C$,

HE: *If* $A_1, A_2, \ldots, A_n \vdash B$ *and* $A_1, A_2, \ldots, A_n \vdash B \supset C$, *then* $A_1, A_2, \ldots, A_n \vdash C$.

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Let a finite column of T-statements qualify as a derivation in MP from $p\,(p \geq 0)$ T-statements $T_1, T_2, \ldots,$ and T_p if every T-statement in the column is one of $T_1, T_2, \ldots,$ and T_p, or is of the form **GR**, or follows from previous T-statements in the column by application of **P**, **NI**, **NE**, **HI**, or **HE**. Let a T-statement be said to be derivable in MP from $p\,(p \geq 0)$ T-statements $T_1, T_2, \ldots,$ and T_p if it comes last in a derivation in MP from $T_1, T_2, \ldots,$ and T_p. Let a finite column of T-statements qualify as a proof in MP if it qualifies as a derivation in MP from zero T-statements. Finally, let a T-statement be said to be provable in MP if it comes last in a proof in MP.

It is easily shown that:

Theorem 1: *If a given T-statement* $A_1, A_2, \ldots, A_n \vdash B$ *is provable in MP, so is the corresponding T-statement* $A_1, A_2, \ldots, A_n, C \vdash B$, *where C is any wff of P.*

Proof: Let

$$\begin{array}{l} A_{1_1}, A_{1_2}, \ldots, A_{1_{n_1}} \vdash B_1, \\ A_{2_1}, A_{2_2}, \ldots, A_{2_{n_2}} \vdash B_2, \\ \cdot \\ \cdot \\ A_{q_1}, A_{q_2}, \ldots, A_{q_{n_q}} \vdash B_q, \end{array}$$

constitute the proof of $A_1, A_2, \ldots, A_n \vdash B$ in MP. The result of inserting '$,C$' to the left of '$\vdash$' in each one of the T-statements in question either qualifies or can be so supplemented as to qualify as a proof of $A_1, A_2, \ldots, A_n, C \vdash B$ in MP. For suppose that $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}} \vdash B_j$ is of the form **GR**; then $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}}, C \vdash B_j$ is likewise of the form **GR**. Or suppose that $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}} \vdash B_j$ follows from $A_{h_1}, A_{h_2}, \ldots, A_{h_{n_h}} \vdash B_h$, where $h < j$, by application of **P** or **NE**; then $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}}, C \vdash B_j$ likewise follows from $A_{h_i}, A_{h_2}, \ldots, A_{h_{n_h}}, C \vdash B_h$ by application of **P** or **NE**. Or suppose that $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}} \vdash B_j$ follows from $A_{h_1}, A_{h_2}, \ldots, A_{h_{n_h}} \vdash B_h$ and $A_{i_1}, A_{i_2}, \ldots, A_{i_{n_i}} \vdash B_i$, where $h < j$ and $i < j$, by application of **HE**; then $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}}, C \vdash B$ will likewise follow from $A_{h_1}, A_{h_2}, \ldots, A_{h_{n_h}}, C \vdash B_h$ and $A_{i_1}, A_{i_2}, \ldots, A_{i_{n_i}}, C \vdash B_i$ by application of **HE**. Or suppose that $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}} \vdash B_j$ – B_j being of the form $\sim A_{h_{n_h}}$ and $\sim A_{i_{n_i}}$ – follows from $A_{h_1}, A_{h_2}, \ldots, A_{h_{n_h}} \vdash B_h$ and $A_{i_1}, A_{i_2}, \ldots,$

$A_{i_{n_i}} \vdash B_i$, where $h < j$ and $i < j$, by application of **NI**; then $A_{h_1}, A_{h_2}, \ldots,$ $C, A_{h_{n_h}} \vdash B_h$ follows from $A_{h_1}, A_{h_2}, \ldots, A_{h_{n_h}}, C \vdash B_h$ by application of **P**, $A_{i_1}, A_{i_2}, \ldots, C, A_{i_{n_i}} \vdash B_i$ follows from $A_{i_1}, A_{i_2}, \ldots, A_{i_{n_i}}, C \vdash B_i$ by application of **P**, and $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}}, C \vdash B_j$ follows from A_{h_1}, $A_{h_2}, \ldots, C, A_{h_{n_h}} \vdash B_h$ and $A_{i_1}, A_{i_2}, \ldots, C, A_{i_{n_i}} \vdash B_i$ by application of **NI**. Or suppose that $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}} \vdash B_j$ – B_j being of the form $A_{h_{n_h}}$ $\supset B$ – follows from $A_{h_1}, A_{h_2}, \ldots, A_{h_{n_h}} \vdash B_h$, where $h < j$, by application of **HI**; then $A_{h_1}, A_{h_2}, \ldots, C, A_{h_{n_h}} \vdash B_h$ follows from $A_{h_1}, A_{h_2}, \ldots, A_{h_{n_h}},$ $C \vdash B_h$ by application of **P**, and $A_{j_1}, A_{j_2}, \ldots, A_{j_{n_j}}, C \vdash B_j$ follows from $A_{h_1}, A_{h_2}, \ldots, C, A_{h_{n_h}} \vdash B_h$ by application of **HI**. Hence Theorem 1.

It is easily shown also that:

Theorem 2: *If a given T-statement $A_1, A_2, \ldots, A_n, C \vdash B$ is provable in MP, then it is derivable in MP from the corresponding T-statement $A_1, A_2, \ldots, A_n \vdash B$.*

Proof: The column of T-statements made up of $A_1, A_2, \ldots, A_n \vdash B$, followed by the proof of $A_1, A_2, \ldots, A_n, C \vdash B$ qualifies by definition as as a derivation in *MP* from $A_1, A_2, \ldots, A_n \vdash B$. Hence Theorem 2.

Theorem 2 is trivial enough. I include it, though, to throw into relief Theorem 3, according to which a given T-statement $A_1, A_2, \ldots, A_n, C \vdash B$ is not derivable in *MP* from the corresponding T-statement $A_1, A_2, \ldots, A_n$ $\vdash B$ unless $A_1, A_2, \ldots, A_n, C \vdash B$ is, as required in Theorem 2, provable in *MP*.

Theorem 3: *If a given T-statement $A_1, A_2, \ldots, A_n, C \vdash B$ is not provable in MP, then it is not derivable in MP from the corresponding T-statement $A_1, A_2, \ldots, A_n \vdash B$.*

Proof:

Part one: Consider (1) any column, call it C_1, of T-statements which qualifies as a derivation in *MP* from $p\,(p \geq 0)$ T-statements $T_1, T_2, \ldots,$ and T_p, and (2) the column, call it C_2, which results from C_1 when all the T-statements in C_1 exhibiting fewer wffs of P than the last T-statement in C_1, have been deleted from C_1. C_2 qualifies by definition as a derivation in *MP* from those T-statements among $T_1, T_2, \ldots,$ and T_p, call them T'_1, $T'_2, \ldots,$ and T'_m, where $m \leq p$, which figure in C_2. For suppose that a given T-statement from C_1 which figures in C_2 happened to be one of $T_1, T_2, \ldots,$ and T_p; that statement will now be one of $T'_1, T'_2, \ldots,$ and T'_m. Or suppose that a given T-statement from C_1 which figures in C_2 happened to be of the form **GR**; that statement will still be of the form **GR**. Or suppose that a given T-statement from C_1 which figures in C_2 happened to follow from one or two previous T-statements in C_1 by application of **P**, **NI**, **NE**, **HI**, or **HE**; the one or two T-statements from which that statement followed are bound to figure in C_2[3] and the statement will still follow from them by application of **P**, **NI**, **NE**, **HI**, or **HE**.

Part two: Suppose $A_1, A_2, \ldots, A_n, C \vdash B$ were derivable in MP from $A_1, A_2, \ldots, A_n \vdash B$. By virtue of part one, the derivation in MP from $A_1, A_2, \ldots, A_n \vdash B$ that closed with $A_1, A_2, \ldots, A_n, C \vdash B$ could be trimmed into a proof in MP closing with $A_1, A_2, \ldots, A_n, C \vdash B$, and hence $A_1, A_2, \ldots, A_n, C \vdash B$ would be provable in MP. Hence Theorem 3.

The fate of **E**, once **GR** is made to do duty for **R** and **E**, should now be clear. **E** will be forthcoming, in the presence of **P**, **NI**, **NE**, **HI**, and **HE**, under the provability form of Theorem 1; it will be forthcoming under the derivability form of Theorems 2 and 3 when and only when $A_1, A_2, \ldots, A_n, C \vdash B$ is already provable in MP and hence is trivially derivable in MP from $A_1, A_2, \ldots, A_n \vdash B$. This anomaly is reminiscent of the one recently brought out by Hiż and others in connection with Modus Ponens in axiomatic presentations of P.[4]

The first theorem, one's only excuse for switching from **R** and **E** to **GR**, would still hold if **NE** and **HE** were modified to read, as often happens:

NE': $A_1, A_2, \ldots, A_n, \sim\sim B \vdash B$,

HE': $A_1, A_2, \ldots, A_n, B, B \supset C \vdash C$.[5]

I suspect, however, that Theorem 1 would no longer hold if **NE** and **HE** were weakened to read, as often happens:

NE'': $\sim\sim A \vdash A$,

HE'': $A, A \supset B \vdash B$,

and **GR**, **P**, and the following rule.

S: *If* $A_1, A_2, \ldots, A_n, B \vdash C$ *and* $A_1, A_2, \ldots, A_n \vdash B$, *then* $A_1, A_2, \ldots, A_n \vdash C$ (Simplification),

were appointed to serve as structural rules of inference for P.[6] I also suspect that Theorem 1 would no longer hold if **GR**, **P**, **NI**, **NE**, **HI**, **HE**, and the following two intelim rules for '$\forall$':

$\forall$I: *If* $A_1, A_2, \ldots, A_n \vdash B$, *then* $A_1, A_2 \ldots, A_n \vdash (\forall W)\, B$, *where the individual variable* W *is not free in anyone of* $A_1, A_2, \ldots,$ *and* A_n,

$\forall$E: *If* $A_1, A_2, \ldots, A_n \vdash (\forall W)\, B$, *then* $A_1, A_2, \ldots, A_n \vdash B'$, *where* B' *is like* B *except for containing free occurrences of an individual variable* W' *at all the places where* B *contains free occurrences of* W,

were appointed as rules of inference for a (pure) quantificational calculus with '$\sim$' and '$\supset$' as primitive connectives and '$\forall$' as primitive quantifier letter.[7]

NOTES

1. See, for example, A. Church, *Introduction to Mathematical Logic*, pp. 214-215, where a further structural rule, (III), is easily shown to be redundant in the presence of **R**, **E**, **P**, **HI**, and **HE**. The three rules **R**, **E**, and **P** would seem to stem from G. Gentzen, who in his "Untersuchungen über das Logische Schliessen," *Mathematische Zeitschrift*, 1934, pp. 176-210 and 405-431, lays down similar structural rules for his so-called calculi LK.

2. See, for example, S. Jaśkowski, "On the Rules of Supposition in Formal Logic," *Studia Logica*, 1934. See also K. R. Popper, "New Foundations for Logic," *Mind*, 1947, pp. 193-235.

3. Note for proof that the one or two T-statements from which a given T-statement exhibiting r ($r \geq 1$) wffs of P follows by **P**, **NI**, **NE**, **HI**, or **HE** are bound to exhibit r or $r + 1$ wffs of P and hence to figure in C_2 if the said T-statement does.

4. See H. Hiż, "Extendible Sentential Calculus," *The Journal of Symbolic Logic*, 1959, pp. 193-202. See also H. Leblanc, "The Algebra of Logic and the Theory of Deduction," *The Journal of Philosophy*, 1961, pp. 553-558.

5. See, for example, S. Jaśkowski, *loc. cit.*

6. See, for example, E. W. Beth and H. Leblanc, "A Note on the Intuitionist and the Classical Propositional Calculus," *Logique et Analyse*, 1960, pp. 174-176.

7. My thanks go to Professor Nuel D. Belnap, Jr., with whom I discussed the results of this paper. I owe him, among other things, the distinction drawn in the text between the provability and the derivability version of **E**.

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