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A DECISION PROCEDURE FOR POSITIVE IMPLICATION

MICHAEL D. RESNIK

In this paper various formulations of the positive implicational calculus [**PIC**] will be studied. This partial propositional calculus is specified by the axiom schemes (Ax.1) and (Ax.2), given below, and has as a rule of inference, *modus ponens* or detachment. The usual truth-functional tests will not serve as decision procedures for **PIC** because there are pure implicational tautologies that are not theorems of **PIC**. A well known example is Peirce's law

$$((\mathbf{A} \supset \mathbf{B}) \supset \mathbf{A}) \supset \mathbf{A},$$

which, when added to (Ax.1) and (Ax.2), yields classical implication. Gentzen[1] and Wajsberg[2] have obtained decision procedures for **PIC** as corollaries to their decision procedures for the intuitionist propositional calculus. In this paper another decision procedure will be stated and justified. It will be formulated in the implicational fragment of Fitch's method of subordinate proofs [**FI**],[3] a variant of the implicational fragment of Gentzen's system for natural deduction the "Kalkül **LHJ**". The decision procedure offered appears to have two advantages over the other decision procedures: it is in some sense more "natural" because the proof structures in **FI** correspond to the proofs used by ordinary mathematicians; and it is more compact and faster to use (at least by hand). It would appear as if the second advantage would be of interest to those working in mechanical theorem proving, but the author has no information as to whether or not the process of programming the procedure will destroy the advantages it has for hand calculation. Using certain reductions the decision procedure for **PIC** can be extended to include parts of conjunction, disjunction, and negation.[4,5]

The axiom schemes for **PIC** are:

Ax.1. $\mathbf{A} \supset (\mathbf{B} \supset \mathbf{A})$.
Ax.2. $[\mathbf{A} \supset (\mathbf{B} \supset \mathbf{C})] \supset [(\mathbf{A} \supset \mathbf{B}) \supset (\mathbf{A} \supset \mathbf{C})]$.

On the other hand, **FI** has no axioms but only rules, and a notation that allows one to nest proofs within proofs. The rules of **FI** are the following:

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1. Implication introduction [$\supset$ **I**].

```
| |- A
| |  .
| |  .
| |  B
| A ⊃ B      ⊃ I
```

2. Implication elimination [$\supset$ **E**].

```
| A                          | A ⊃ B
| .                          | .
| .               or         | .
|_A ⊃ B                      |_A
| B          ⊃ E             | B          ⊃ E
```

3. Reiteration [reit].

```
| A                          | A
| .                          | .
| .                          | .
| | .             or         | A  reit
| | .
| | A  reit
```

It is easy to see that a derived rule of reiteration may be proved which will allow the reiteration of a formula through several levels of proofs at one time.

Using the deduction theorem for **PIC** one can show that **FI** and **PIC** have the same set of theorems, thus a decision procedure for **FI** serves as one for **PIC**. It is interesting to learn that **FI** has a decision procedure because one would usually think that $\supset$ **E** would preclude this.

The decision procedure for **PI** is stated as follows:

Let **A** be a candidate implicational formula. One can assume that **A** has the form **B** $\supset$ **C**, for a propositional variable alone can not be a theorem because it can not even be a tautology. Working "backwards" one tries to build an **FI** proof for **A** as follows:

1. Apply $\supset$ **I** as many times as possible.
2. After (1) has terminated an inner proof $\mathfrak{P}_o$ having a propositional variable as its conclusion will have been obtained. Reiterate into $\mathfrak{P}_o$ all formulas that occur above and without $\mathfrak{P}_o$.
3. Apply $\supset$ **E** as many times as possible to pairs of formulas in $\mathfrak{P}_o$ (the conclusion is excluded from this, and (**C**, **B**) = (**B**, **C**)). If the conclusion is obtained in this way stop, **A** is a theorem.
4. If formulas of the form **E** $\supset$ **F** appear in $\mathfrak{P}_o$ but **E** does not, try to prove **E** by returning to (1), except in (2) reiterate, in addition to the usual formulas, all the formulas thus far derived in $\mathfrak{P}_o$. If **E** is provable in

this way, insert it and its proof in $\mathfrak{P}_0$ and return to (3). If there are several such $\mathbf{E} \supset \mathbf{F}$'s try to prove the $\mathbf{E}$'s in all possible orders in case some $\mathbf{E}$ is needed in the proof of another $\mathbf{E}$. Trying to prove an $\mathbf{E}$ will be called "following a branch."

5. In step (4) it is possible to get into infinite regresses; however, this can be predicted by finding a subproof $\mathfrak{B}$ that contains a subproof $\mathfrak{P}'$ which is like $\mathfrak{P}$ except that the number and order (except for the conclusion which is the same) of occurrences of like formulas in $\mathfrak{P}$ and $\mathfrak{P}'$ may differ. In this case, the particular branch will not work and one has to give it up. If (1) - (4) do not yield a proof of $\mathbf{A}$, then $\mathbf{A}$ is not a theorem.

To justify this procedure it will be shown that every **FI** theorem can be proved in the normal form generated by the algorithm. The procedure terminates because of the suformula property and the fact that in the case of non-theorems one eventually runs out of instructions.

It turns out that the required normal form theorem for **FI** can be proved by working through "sequenzen-kalkül". By doing this two normal form theorems are proved for these systems, which enables one to refine Gentzen's decision procedure for **PIC**. These theorems are proved for the implicational fragment of Gentzen's "Kalkül **LJ**," **LJI**, but it is easy to see that they also hold in the system **G3**(**I**) studied by Kleene.[6] Normal form proofs in **LJI** are converted into normal form proofs in an intermediate system **LJI***; these last proofs are then converted into **FI** proofs in the normal form required.

LJI has the usual structural rules of thinning, contraction, and interchange in the antecedent. As is well known, the 'cut' rule, which allows one to prove that **LJI** and **PIC** are equivalent, can be derived.[7]

Axiom scheme for **LJI**: $\mathbf{A} \vdash \mathbf{A}$.
Logical rules for **LJI**:

1. $\vdash \supset$

$$\frac{\Phi, \mathbf{A} \vdash \mathbf{B}}{\Phi \vdash \mathbf{A} \supset \mathbf{B}}$$

2. $\supset \vdash$

$$\frac{\Phi \vdash \mathbf{A} \qquad \Phi, \mathbf{B} \vdash \mathbf{C}}{\Phi, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{C}}$$

A *variant* of a sequent $\Phi \vdash \mathbf{A}$ is a sequent $\Phi' \vdash \mathbf{A}$ that is derivable from $\Phi \vdash \mathbf{A}$ by structural inferences alone.

Let $\mathfrak{P}$ be a derivation in **LJI**. Each sequent in $\mathfrak{P}$ is assigned a natural number called its *level* as follows:

1. Axioms and axiom variants are assigned *0*.
2. If $\Phi \vdash \mathbf{A}$ is assigned k and $\Phi' \vdash \mathbf{A}$ is derived from it by some structural inferences, then $\Phi' \vdash \mathbf{A}$ is assigned k.
3. If $\Phi \vdash \mathbf{A} \supset \mathbf{B}$ follows from $\Phi, \mathbf{A} \vdash \mathbf{B}$ by $\vdash \supset$ and $\Phi, \mathbf{A} \vdash \mathbf{B}$ is assigned k, then $\Phi \vdash \mathbf{A} \supset \mathbf{B}$ is assigned $k + 1$.
4. If $\Phi, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{C}$ follows from $\Phi \vdash \mathbf{A}$ (of level k) and $\Phi, \mathbf{B} \vdash \mathbf{C}$ (of level p), then the level of $\Phi, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{C}$ is $max\ (k, p) + 1$.

With the use of the concepts just defined the following theorems can be proved:

Theorem I. Let Φ, $\mathbf{A} \supset \mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}$ be a provable non-axiom (variant) sequent. Then (ignoring structural inferences) Φ, $\mathbf{A} \supset \mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}$ can be proved by $\vdash \supset$ in a cut free proof as follows:

$$\frac{\Phi, \mathbf{A} \supset \mathbf{B}, \mathbf{C} \vdash \mathbf{D}}{\Phi, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}.}\ \vdash \supset$$

Proof: So that the theorem is not trivially true, assume that Φ, $\mathbf{A} \supset \mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}$ is proved by $\supset \vdash$. It can also be assumed without loss of generality that this proof is cut free. Hence, it suffices to prove the sequent by $\vdash \supset$ without introducing a cut. That this can be done is shown by an induction on the level k of the sequent.

Basis case. $k = 1$ and the proof has the form

$$(\alpha) \qquad \frac{\Phi \vdash \mathbf{A} \qquad \Phi, \mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}}{\Phi, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}}\ \supset \vdash$$

Φ, $\mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}$ has the level 0 and is an axiom (variant). $\mathbf{C} \supset \mathbf{D}$ is not in Φ, for otherwise Φ, $\mathbf{A} \supset \mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}$ would be an axiom (variant). Hence, $\mathbf{B}$ is $\mathbf{C} \supset \mathbf{D}$. Thus a proof ($\beta$) can be built in proper form:

$$(\beta) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Phi \vdash \mathbf{A}}{\Phi, \mathbf{C} \vdash \mathbf{A}} \qquad \dfrac{\Phi, \mathbf{C} \vdash \mathbf{C} \qquad \Phi, \mathbf{C}, \mathbf{D} \vdash \mathbf{D}}{\Phi, \mathbf{C}, \mathbf{C} \supset \mathbf{D}[\mathbf{B}] \vdash \mathbf{D}}\ \supset \vdash}{\Phi, \mathbf{C}, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{D}}\ \supset \vdash}{}}{\Phi, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}.}\ \vdash \supset$$

Inductive case. The level k is such that $1 < k$. In this case Φ, $\mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}$ is proved by $\vdash \supset$ (ignoring structural inferences). Hence by a lemma of Kleene[8] the $\vdash \supset$ inference and the $\supset \vdash$ inference can be permuted to yield a proof in proper form.

[Alternatively: The fact that the inductive hypothesis applies to Φ, $\mathbf{B} \vdash \mathbf{C} \supset \mathbf{D}$ enables one to build a proof like (β) in the required form. If $\mathbf{D}$ is not a propositional variable some modifications in (β) are needed.]

Theorem II. Let Φ, $\mathbf{A} \supset \mathbf{B} \vdash \dot{\mathbf{C}}$ be a provable non-axiom (variant) sequent, where $\dot{\mathbf{C}}$ indicates that $\mathbf{C}$ is a propositional variable, and let the proof of this sequent be in the normal form of theorem I. Then if $\mathbf{A}$ is in Φ and in the antecedent(s) of the premiss(es) of the inference leading to the sequent Φ, $\mathbf{A} \supset \mathbf{B} \vdash \dot{\mathbf{C}}$, the sequent can be proved by $\supset \vdash$ as follows:

$$\frac{\Phi \vdash \mathbf{A} \qquad \Phi, \mathbf{B} \vdash \dot{\mathbf{C}}}{\Phi, \mathbf{A} \supset \mathbf{B} \vdash \dot{\mathbf{C}}}\ \supset \vdash$$

Moreover, this proof will have the normal form of theorem I.

Proof: Again assume that the proof is not in proper form but in the form given by theorem I. Two important cases arise: either the sequent follows thinning or by $\supset \vdash$. [The only other cases are contraction or interchange which have no bearing here.]

Case I. Thinning. The proof has the form

$$\frac{\Phi', \mathbf{A} \supset \mathbf{B} \vdash \dot{\mathbf{C}}}{\Phi', \mathbf{D}, \mathbf{A} \supset \mathbf{B} \vdash \dot{\mathbf{C}}}$$

where $\mathbf{A}$ is in Φ', and can be handled by the obvious inductive argument.

Case II. $\supset \vdash$. An induction is performed on the level k.

Basis case. The level $k = 1$, and the proof has the form:

$$(\alpha_o) \qquad \frac{\Phi', \mathbf{A} \supset \mathbf{B} \vdash \mathbf{D} \qquad \Phi', \mathbf{A} \supset \mathbf{B}, \mathbf{E} \vdash \dot{\mathbf{C}}}{\Phi', \mathbf{D} \supset \mathbf{E}, \mathbf{A} \supset \mathbf{B} \vdash \dot{\mathbf{C}}} \supset \vdash$$

Here, of course $\mathbf{A}$ is in Φ'.

Since Φ', $\mathbf{A} \supset \mathbf{B} \vdash \mathbf{D}$ and Φ', $\mathbf{A} \supset \mathbf{B}, \mathbf{E} \vdash \dot{\mathbf{C}}$ are axiom (variants), $\mathbf{D}$ is in Φ' and $\mathbf{E}$ is $\dot{\mathbf{C}}$. Hence a (β_o) can be built:

$$(\beta_o) \qquad \cfrac{\Phi', \mathbf{D} \supset \mathbf{E} \vdash \mathbf{A} \qquad \cfrac{\cfrac{\Phi' \vdash \mathbf{D}}{\Phi', \mathbf{B} \vdash \mathbf{D}} \qquad \cfrac{\mathbf{E} \vdash \dot{\mathbf{C}}}{\Phi', \mathbf{B}, \mathbf{E} \vdash \dot{\mathbf{C}}}}{\Phi', \mathbf{D} \supset \mathbf{E}, \mathbf{B} \vdash \dot{\mathbf{C}}} \supset \vdash}{\Phi, \mathbf{A} \supset \mathbf{B} \vdash \dot{\mathbf{C}}} \supset \vdash$$

Inductive case. $1 < k$. By applying the inductive hypothesis to Φ', $\mathbf{A} \supset \mathbf{B}$, $\mathbf{E} \vdash \dot{\mathbf{C}}$ in a proof of the form (α_o), this sequent can be proved by an $\supset \vdash$ inference whose principle formula is $\mathbf{A} \supset \mathbf{B}$ and whose side formulas are $\mathbf{A}$ and $\mathbf{B}$. But this $\supset \vdash$ inference and the one whose principle formula is $\mathbf{D} \supset \mathbf{E}$ can be permuted according to the Kleene lemma cited for theorem I to give a proof in the required form.

The intermediate system **LJI*** is now considered. It is just like **LJI** except for having an additional logical rule $\supset \vdash^*$:

$$\frac{\Phi, \mathbf{A}, \mathbf{B} \vdash \mathbf{C}}{\Phi, \mathbf{A}, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{C}} \supset \vdash *$$

LJI and **LJI*** are obviously equivalent since $\supset \vdash^*$ is a derived rule in **LJI**. Normal form proofs obtained from theorem II in **LJI** can be converted into normal form proofs in **LJI*** quite easily. For, let $\mathfrak{P}$ a normal **LJI** proof obtained by theorem II. Replace every $\supset \vdash$ inference whose conclusion has the form Φ, **A**, **A** ⊃ **B** |– $\dot{\mathbf{C}}$ by a ⊃ |–* inference and drop the left branch terminating in Φ, **A** |– **A**. This yields a proof $\mathfrak{P}_0$ in **LJI*** which has the following normal form: Starting from the end sequent and building a proof in this form |– ⊃ takes preference over ⊃ |–* and ⊃ |–, and ⊃ |–* take preferences over ⊃ |–. This does not mean, of course, that, e.g., all |– ⊃ inferences occur below all ⊃ |–* inferences.

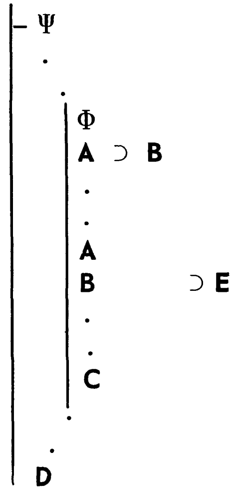
It can also be required that non-atomic axioms of **LJI*** be given normal form proofs, for theorem I obviously can be extended to sequents of the form **A** ⊃ **B** |– **A** ⊃ **B**. Assuming this, **LJI*** normal proofs are converted into **FI** normal proofs as follows: Working upwards:

```
                                        |– Ψ
                                        |  .
                                        |    .
                                        |      .
 Φ, A |– B                              |      |– Φ
 ----------- |– ⊃                       |      |     |– A
 Φ |– A ⊃ B             goes into       |      |     |  Φ    reit
 -----------                            |      |     |  .
 . . . . . .                            |      |     |  .
 -----------                            |      |     |  B
   Ψ |– C                               |      |  A ⊃ B   ⊃ I
                                        |      .
                                        |    .
                                        |  .
                                        | C
```

```
                                          |– Ψ
                                          |  .
                                          |    .
   Φ, A, B |– C                           |      .
 ----------------- ⊃ |–*                  |      | Φ
 Φ, A, A ⊃ B |– C                         |      | A
 -----------------      goes into         |      | A ⊃ B
 . . . . . . . . .                        |      | B          ⊃ E
 -----------------                        |      | .
     Ψ |– D                               |      | .
                                          |      | C
                                          |    .
                                          |  .
                                          | .
                                          | D
```

$$\begin{array}{c} \Phi \vdash A \qquad \Phi, B \vdash C \\ \hline \Phi, A \supset B \vdash C \\ \hline \cdots\cdots\cdots\cdots \\ \hline \Psi \vdash D \end{array} \supset\vdash$$

goes into



The structural rules are ignored, for ⊃ **E** is set up so that interchange is not needed, and thinning corresponds to an unused hypothesis. Contraction may cause some doubt since steps of the form

$$\frac{\Phi, A \supset B \vdash A \qquad \Phi, A \supset B, B \vdash C}{\Phi, A \supset B \vdash C} \supset\vdash$$

may be needed in a **LJI** (**LJI***) proof to avoid cut; but in the converted proof this corresponds to repeating a formula. By using the derived reiteration rule these **FI** proofs can be given the form generated by the algorithm.

Step (5) of the decision procedure concerning an infinite regress needs some justification. This can be done in two ways. First, if $\mathfrak{P}$ does contain $\mathfrak{P}'$ in the way described, the algorithm, if not stopped, would lead to an infinite regress and thus not to a proof. But the formula, if provable, does have a proof in the form generated by the algorithm thus the proof does not come following the branch that leads to a regress. Second, observe that the treatment given here could have been applied to **G**3(**I**), but every theorem of **G**3(**I**) can be given an irredundant proof. Thus in converting we can expect the same for **FI** theorems given normal forms proofs.

In step (4) one would like to advoid reiterating **E** ⊃ **F**, but [**A** ⊃ [((**A** ⊃ **B**) ⊃ **C**) ⊃ **B**]] ⊃ [[(**A** ⊃ **B**) ⊃ **C**] ⊃ **C**] is a counterexample.

Dr. James Guard has communicated to the author an argument that allows on to carry out the justification of the decision procedure directly in **FI**. It seems as if this argument can be extended to the intuitionistic and classical propositional calculi to yield some sort of decision procedures for these systems in their subproof formulations.

NOTES AND REFERENCES

1. G. Gentzen, "Untersuchungen über das logische Schliessen," *Math. Zeit.* 39 (1936-37), pp. 176-210 and pp. 405-431.

2. M. Wajsberg, "Untersuchungen über den Aussagenkalkül von A. Heyting," *Wiadomości Matematyczne*, 46, pp. 45-101.
3. F. B. Fitch, *Symbolic Logic* (1952). Familiarity with Fitch's and Gentzen's work will be presupposed in this paper.
4. See, for instance, H. Arnold Schmidt, *Mathematische Gesetze der Logik, Vol. I* (1960).
5. This is the minimal calculus of Johanson, see, Johanson, "Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus," *Comp. Math., 4,* pp. 119-136 (1936).
6. See, Kleene, *Introduction to Metamathematics*, p. 479.
7. The cut rule is:

$$\frac{\Phi \vdash \mathbf{A} \qquad \Phi', \mathbf{A} \vdash \mathbf{B}}{\Phi, \Phi' \vdash \mathbf{B}.}\ \text{Cut}$$

One then gets detachment as follows:

$$\frac{\Phi \vdash \mathbf{A} \qquad \dfrac{\Phi \vdash \mathbf{A} \supset \mathbf{B} \qquad \dfrac{\mathbf{A} \vdash \mathbf{A} \quad \mathbf{B} \vdash \mathbf{B}}{\mathbf{A}, \mathbf{A} \supset \mathbf{B} \vdash \mathbf{B}}\ \supset\vdash}{\Phi, \mathbf{A} \vdash \mathbf{B}}\ \text{Cut}}{\Phi \vdash \mathbf{B}}\ \text{Cut}$$

8. Lemma 4 in Kleene, "The permutability of inferences in Gentzen's calculi *LK* and *LJ*," *Memoirs of the Am. Math. Soc., 10*, (1952).

Air Force Cambridge Research Laboratories
Bedford, Massachusetts