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QUANTIFICATION AND Ł-MODALITY

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1. *The Formula* $\Sigma aK\Delta\Theta 1a\nabla\mathbf{L}1a$ *with* Δ *and* ∇ *as Variable Functors.* In his paper "Arithmetic and Modal Logic",[1] Łukasiewicz drew attention to an odd theorem which is deducible when certain arithmetical laws are subjoined to his Ł-modal calculus, namely the theorem (with "Θab" for "$a = b$" and "$\mathbf{L}ab$" for "$a < b$")

$$5.4 \quad \Sigma aK\Delta\Theta 1a\nabla\mathbf{L}1a.$$

What is odd about this theorem is that it holds despite the fact that, according to Łukasiewicz, there exists no positive integer a for which $K\Delta\Theta 1a\nabla\mathbf{L}1a$ is true. But is this really so?

It is noteworthy that while Łukasiewicz's proof of the theorem 5.4 is perfectly rigorous and formal, his proof that there is no positive integer a for which $K\Delta\Theta 1a\nabla\mathbf{L}1a$ holds is not, but depends on the interpretation of Δ and ∇ as *constant four-valued* truth-operators, and on certain truth-value calculations based on this interpretation. If, on the contrary, we interpret Δ and ∇ as *variable two-valued* functors with their range restricted to V and S, with ∇ taking the opposite value to Δ in any given formula,[2] we obtain a different result. For suppose that in the formula 5.4 we assign to Δ the value S and consequently to ∇ the value V. Then if $a > 1$, $K\Delta\Theta 1a\nabla\mathbf{L}1a = KS0V1 = K01 = 0$, but if $a = 1$, $K\Delta\Theta 1a\nabla\mathbf{L}1a = KS1V0 = K11 = 1$; so that with this assignment of values to Δ and ∇, there *is* at least one positive integer, namely 1, for which $K\Delta\Theta 1a\nabla\mathbf{L}1a$ is true. Again, if we assign to Δ the value V and consequently to ∇ the value S, then if $a = 1$, $K\Delta\Theta 1a\nabla\mathbf{L}1a = KV1S0 = K10 = 0$, but if $a > 1$, $K\Delta\Theta 1a\nabla\mathbf{L}1a = KV0S1 = K11 = 1$. Hence for this assignment of values also, there is at least one positive integer, namely any greater than 1, for which $K\Delta\Theta 1a\nabla\mathbf{L}1a$ is true. Hence on both possible assignments of values to Δ and ∇, the formula 5.4 is true in its natural sense, and its appearing as a logical law, i.e. as true for all possible values of its free variables, presents no difficulties.

2. *The Formula* $\Sigma aK\Delta\Theta 1a\ \mathbf{L}1a$ *with* Δ *and* ∇ *as Constant Functors.* It remains true, however, that the Δ and ∇ of the $\mathbf{L}$-modal system *may* be interpreted, not as above, but as constant four-valued functors; and if they are

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so interpreted, the puzzle mentioned by Łukasiewicz does arise. But if we adopt a logic of more than two truth-values, we must be prepared to find that not only truth-operators but also quantifiers do not behave exactly as they do in two-valued systems, and Łukasiewicz's paradox will be sufficiently explained if we can see what broader features of his system, considered as a four-valued system, give rise to it.

In a many-valued system there will always be several different operators which we might with good reason identify with the existential quantifier; some resembling in one way, and some in another, but none resembling in all ways, the existential quantifier of classical logic. Łukasiewicz has chosen to symbolise by "Σ", his usual symbol for the existential quantifier, an operator which preserves the following two rules:

$\Sigma 1$: $\vdash C\,\varphi x\,\alpha \rightarrow \vdash C\,\Sigma\,x\,\varphi x\,\alpha$, for x not free in α.
$\Sigma 2$: $\vdash C\,\varphi\,\varphi x \rightarrow \vdash C\,\Sigma\,x\,\varphi x$,

when C is defined by the matrix

C	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

His paradox is, in the end, a consequence of this choice.

Given any propositional function φx in a four-valued system, the following eight possibilities exist:

1. φx may sometimes take the value *1* (whether or not it ever takes other values also).
2. φx may take no value but *2*.
3. φx may take no value but *3*.
4. φx may take no value but *4*.
5. φx may sometimes take the value *2* and sometimes *3*, but no others.
6. φx may sometimes take the value *2* and sometimes *4*, but no others.
7. φx may sometimes take the value *3* and sometimes *4*, but no others.
8. φx may sometimes take the value *2* and sometimes *3* and sometimes *4*, but never *1*.

Let us consider what value $\Sigma x\varphi x$ must have under each of these conditions if the rules $\Sigma 1$ and $\Sigma 2$ are both to be preserved.

Case 1. If φx sometimes takes the value *1*, then α might be capable of taking any value and $C\alpha\varphi x$ still be a law (for $C\ \alpha\ 1 = 1$ for any value of α, and when $\varphi x \neq 1$, α might be so connected with φx - since x might occur freely in α also - that it always takes a value for which $C\alpha\varphi x = 1$). But under these conditions, the only value for $\Sigma x\varphi x$ which will guarantee that $C\alpha\Sigma x\varphi x$ will also be a law (as the rule $\Sigma 2$ requires) will be *1*, since it is only when $\beta = 1$ that $C\alpha\beta = 1$ no matter what the value of α may be. Hence if φx sometimes takes the value *1*, $\Sigma x\varphi x = 1$.

Case 2. If φx takes no value but *2*, $C\varphi x\, \alpha$ will be a law only if α can take no value but *1*, if it can take no value but *2*, or if it can take either *1* or *2* but no others ($C2\alpha$ is a law only if $\alpha = 1$ or $\alpha = 2$). $C\Sigma x\varphi x\, \alpha$ will be a law (as $\Sigma 1$ requires) under all of these conditions, only if $\Sigma x\varphi x = 2$ or *4*. (For only if $\beta = 2$ or *4* do we have both $C\beta 1$ and $C\beta 2$). But it can be shown by similar considerations that $\Sigma 2$ is preserved only if $\Sigma x\varphi x = 1$ or *2*. Hence, $\Sigma 1$ and $\Sigma 2$ will both be preserved, for this sort of φx, only if $\Sigma x\varphi x = 2$.

Case 5. If φx sometimes takes the value *2* and sometimes *3* but no others, then α might be capable of taking any value and $C\alpha\varphi x$ still be a law. For $C12 = 1$, $C22 = 1$, $C33 = 1$, $C42 = 1$ and $C43 = 1$. But under these conditions, the only value for $\Sigma x\varphi x$ which will guarantee that $C\alpha\Sigma x\varphi x$ will also be a law (as $\Sigma 2$ requires) is *1*.

We need not consider other cases in detail, but the results of an examination of this sort will be found to be as follows:

(a) If φx sometimes takes the value *1*, or if it sometimes takes the value *2* and sometimes *3* but no others, or if it sometimes takes *2*, sometimes *3*, and sometimes *4*, but never *1*, then $\Sigma x\varphi x = 1$.
(b) If φx always takes the value *2*, or the values *2* and *4* only, $\Sigma x\varphi x = 2$.
(c) If φx always takes the value *3*, or the values *3* and *4* only, $\Sigma x\varphi x = 3$.
(d) If φx always takes the value *4*, $\Sigma x\varphi x = 4$.

Only if the values assigned to $\Sigma x\varphi x$ under these various conditions are as above, will the rules $\Sigma 1$ and $\Sigma 2$ be preserved for the C of the Ł-modal system.

If we describe a proposition with value *1* as "plain true", one with value *2* as "nearly true", one with value *3* as "nearly false" and one with value *4* as "plain false", we may say, in view of the above results, that "Something φ's" is plain false if and only if φx is plain false for every value of x; and so far, this is in agreement with the ordinary two valued use of this form. But it must also be observed that "Something φ's", in Łukasiewicz's sense, is plain true not only when φx is plain true for some value of x, but also when φx is not plain true for any value of x, but is nearly true for some values and nearly false for others; and here the preservation of the rules $\Sigma 1$ and $\Sigma 2$ with this 4-valued C has had to be paid for by a departure from the behaviour of the quantifier in its ordinary two-valued context. Łukasiewicz's arithmetical paradox is simply an illustration of this point.

3. *The Existential Quantifier and Alternation in the Ł-Modal System.* —It has often been observed that the form "Something φ's" is equivalent to an indefinitely continued alternation of the form "Either a φ's or b φ's or c φ's or d φ's, etc." In the Ł-modal system the alternation operator A, defined as an abbreviation for CN, has the matrix

A	*1*	*2*	*3*	*4*
1	*1*	*1*	*1*	*1*
2	*1*	*2*	*1*	*2*
3	*1*	*1*	*3*	*3*
4	*1*	*2*	*3*	*4*

If we regard existential quantification as a continued alternation with this matrix, we obtain the same results as before. For

1. If any of the φx's = 1, the whole alternation $\Sigma x\varphi x = 1$ (for $A1\alpha = 1$, for any α).
2. If every $\varphi x = 2$, the whole alternation $\Sigma x\varphi x = 2$ (for $A22 = 2$, hence $AA222 = A22 = 2$, etc.)
3. If every $\varphi x = 3$, $\Sigma x\varphi x = 3$ (for $A33 = 3$, hence $AA333 = A33 = 3$, etc.)
4. If every $\varphi x = 4$, $\Sigma x\varphi x = 4$ (proved similarly).
5. If the φx's include 2's and 3's, and these only, $\Sigma x\varphi x = 1$ (for $A23 = 1$, hence $AA23\alpha = A1\alpha = 1$).
6. If the φx's include 2's and 4's, and these only, $\Sigma x\varphi x = 2$ (for $A24 = A42 = A22 = 2$).
7. If the φx's include 3's and 4's, and these only, $\Sigma x\varphi x = 3$ (proved similarly)
8. If the φx's include 2's, 3's and 4's, and these only, $\Sigma x\varphi x = 1$ (proof as for case 5).

The "odd" cases 5 and 8 are thus connected with the "oddity" of this 4-valued alternation, which is such that "Either p or q" may be plain true even when neither p nor q is plain true, since it is plain true also when one of the alternants is nearly true and the other nearly false. And this property of this alternation is used by Łukasiewicz in his proof of 5.4; one step in his proof being to establish the proposition $A\Delta L11\nabla L11$, which is a theorem of the system despite the fact that neither $\Delta L11$ nor $\nabla L11$ has the value 1 (the former has the value 3 and the latter the value 2).

This peculiarity of Ł-alternation also accounts for the fact (noted by Anderson[3]) that the system contains a thesis, $A\Delta p\nabla q$, which is "unreasonable" in the sense of Halldén, being an alternation of which neither alternant is a thesis although the two alternants have no variable in common. This "unreasonableness" disappears, however, if Δ and ∇ are interpreted as variable functors with a limited range. For on this interpretation, when $A\Delta p\nabla q$ is expanded by the definition of ∇ to $A\Delta pC\Delta qq$, the alternants do contain a common variable, namely Δ; moreover, each of the possible substitutions for Δ turns one of the alternants into a thesis (the substitution $\Delta/'$ turns the whole into $ApCqq$, and Δ/C'' turns it into $ACppCCqqq$).

4. *Rejection and Existential Quantification of the Contradictory in the Ł-Modal System* - Earlier in the same paper, Łukasiewicz has a discussion of the proposition "If it is possible that a should not equal b, then it is a fact that a does not equal b". He argues against this that if the number a has been thrown with a die, it is possible that the next number thrown, b, will be different from a, but it does not follow that b will in fact be different from a, for "it is possible to throw the same number twice".[4] I do not wish here to dispute the force of this argument, but it should be pointed out that, whatever its force in itself, it is not an argument which can be consistently used by an advocate of the Ł-modal system, considered as a logic of necessity and possibility in the ordinary sense. For the supposition made is that

it is possible for b to be different from a, and also possible for it not to be different; but in the Ł-modal system, Δp and ΔNp are never true together. (In his earlier paper, Łukasiewicz is rightly emphatic about this, and even argues that this peculiarity of Δ is in accord with out intuitive notions of "possibility"[5]).

What is a little confusing at this point is that although it can be shown that no proposition of the form $K\Delta p\Delta Np$ is true, yet the negation of this form, $NK\Delta p\Delta Np$, is not asserted but rejected in the system (it is equivalent to $NK\Delta pN\Gamma p$, and so to $C\Delta p\Gamma p$, which must be rejected since the weaker $C\Delta pp$ is rejected). This is confusing, and even paradoxical, for the following reason: To say that a formula, e.g. $NK\Delta p\Delta Np$, is asserted is to say that it is true for all values of its variables; and in fact if $NK\Delta p\Delta Np$ were a thesis we could derive from it, by Lukasiewicz's rules for Π, the further thesis $\Pi pNK\Delta p\Delta Np$.[6] It would seem, therefore, that to say that this formula is not asserted but rejected, is to say that for at least one value of p its opposite is true; that is, the rejection $\dashv NK\Delta p\Delta Np$ would appear to be equivalent to the existentially quantified assertion $\vdash \Sigma pK\Delta p\Delta Np$. (The possibility of thus dispensing with rejection in favour of existential quantification of the contradictory form was elsewhere suggested by Łukasiewicz himself.[7]) But in fact the form $\Sigma pK\Delta p\Delta Np$ is not asserted but rejected—even with the peculiar sense which "Some" here bears. For it can be shown that for any p the formula $K\Delta p\Delta Np$ will have the value 3, so that its existential quantification will have the value 3 (see last section).

This result, like that considered earlier, is capable of two alternative explanations. If we treat Δ as a constant functor in a 4-valued logic, we may say that α may be asserted if and only if $\Pi pqr\ldots\alpha$ is plain true, but that this may fail to be plain true (and α may in consequence be rejected) not only when $N\Pi pqr\ldots\alpha$, i.e. $\Sigma pqr\ldots N\alpha$, is plain true, but also when both of them have intermediate values. If, on the other hand, we treat Δ as a variable functor (with restricted range) in a 2-valued logic, $\dashv\alpha$ *will* be equivalent to $\vdash\Sigma pq\ldots N\alpha$, and $\dashv N\alpha$ to $\vdash\Sigma pq\ldots\alpha$, provided that *all* the variables in α, *including* Δ, are existentially bound at the beginning. For instance, although the rejection $\dashv NK\Delta p\Delta Np$ is not equivalent to the assertion $\vdash\Sigma pK\Delta p\Delta Np$, it *is* equivalent to the assertion $\vdash\Sigma p\Sigma\Delta K\Delta p\Delta Np$. And this assertion, "For some p and Δ, $K\Delta p\Delta Np$", can easily be seen to be true, for when $\Delta = V$, $K\Delta p\Delta Np = KVpVNp = K11 = 1$.

There are similar alternative explanations of the fact that although the formulae Δp and $N\Gamma p$ are rejected, no proposition of the form $N\Delta\alpha$ or $\Gamma\alpha$ is ever true by the matrices.

This discussion illustrates one point—so far as I can see, it is the only point—at which it makes a *formal* difference, i.e. a difference in the symbolic system itself, when we interpret Δ as a restrictedly variable 2-valued functor instead of a constant 4-valued one. If we introduce the quantifiers Π and Σ as capable of binding any variables of the system, the forms $\Pi\Delta\alpha$ and $\Sigma\Delta\alpha$ (where α is a statement-form) will be well-formed on the one interpretation but not on the other, since there is no such thing as the binding of a constant.[8]

REFERENCES

[1] J. Łukasiewicz, "Arithmetic and Modal Logic," *The Journal of Computing Systems*, Vol. 1, No. 4 (1954), pp. 217-8.

[2] *Cf.* A. N. Prior, "The Interpretation of Two Systems of Modal Logic," *Ibid*, pp. 203 ff.

[3] A. R. Anderson, "On the Interpretation of a Modal System of Łukasiewicz," *ibid.*, pp. 209-210.

[4] Łukasiewicz, *op. cit.* p. 214.

[5] J. Łukasiewicz, "A System of Modal Logic," *The Journal of Computing Systems*, Vol. 1, No. 3 (1953), pp. 135-6.

[6] *Cf.* J. Łukasiewicz, *Aristotle's Syllogistic*, (Oxford, 1951), pp. 86-7.

[7] *Ibid.*, p. 95.

[8] Variables with a restricted range are also necessary, it may be noted, if Łukasiewicz's formalisation of Aristotle's syllogistic is to be treated as a segment of Leśniewski's ontology, in which ordinary term-variables may stand for empty terms.

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