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ON THE GENERALIZED BROUWERIAN AXIOMS

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After Oskar Becker[1] a modal thesis of the following form:[2]

B_n $\quad \mathfrak{C}pL^nMp$

for any $n > 1$, is called a generalized Brouwerian axiom. Since in Lewis' system S4 the following thesis

$M1$ $\quad \mathfrak{C}LpLLp$

holds, it is obvious that in S4 (and hence a fortiori in S5) every formula B_n, for any $n > 1$, is inferentially equivalent to the proper Brouwerian axiom, i.e. Lewis' thesis

$C12$ $\quad \mathfrak{C}pLMp$

On the other hand, it seems that in the field of some of Lewis' systems which are weaker than S4, a generalized Brouwerian axiom B_n, for any $n > 1$, is a stronger thesis than $C12$. For while, as far as I know, only the following definitive results concerning the addition of $C12$ to the systems weaker than S4 are obtained:

a) In [5], pp. 151-152, Parry has proved that the addition of $C12$ to S3 gives system S5 of Lewis.

and

b) In [8], pp. 56-58, I have shown recently that the same holds, if we add $C12$ as a new axiom either to S3° or to S3*.

and while the effect of the addition of $C12$ either to S1° or to S1 is not yet fully investigated,[3] in [2], pp. 78-81, it is proved by Churchman that the addition of B_n, for any $n > 1$, to S2 gives system S5.

In this note I shall investigate some properties of a generalized Brouwerian axiom, i.e. of formula B_n, for any $n > 1$. Namely:

1) In §1 a certain subsystem of S1 is defined. This system, called S1* is such that it is weaker than S1, it contains S1° and it is stronger than the latter system.

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2) In §2 I show that the addition of B_n to S1* (and hence a fortiori to S1) gives system S5. Thus, the result of Churchman is strengthened.

3) In §3 it is proved that the same holds, if we add B_n either to S3° or to S3*.

§1. We obtain system S1* by addition of the following new axiom

J1 ℭ*MpMMp*

to S1°. Group IV of Lewis-Langford[4] verifies S1*, and falsifies the proper axiom of S1, i.e. the thesis

G1 ℭ*pMp*

since for $p = 1$: *G1* = ℭ*1M1* = ℭ*12* = *3*. On the other hand, the following modification of Parry's matrix:[5]

K	0	1	2	3	4	5	6	7	N	M
0	0	0	0	0	0	0	0	0	7	1
1	0	1	0	1	0	1	0	1	6	5
2	0	0	2	2	0	0	2	2	5	7
3	0	1	2	3	0	1	2	3	4	7
4	0	0	0	0	4	4	4	4	3	7
5	0	1	0	1	4	5	4	5	2	7
* 6	0	0	2	2	4	4	6	6	1	7
7	0	1	2	3	4	5	6	7	0	6

verifies the axioms of S1° and Lewis' rules of procedure, but falsifies *J1*, since for $p = 6$: *J1* = ℭ*M6MM6* = ℭ*7M7* = ℭ*76* = *NMK7N6* = *NMK71* = *NM1* = *N5* = *2*. Thus, system S1*, which by the definition contains S1° and, obviously, is contained in S1, is stronger than the former system and weaker than the latter.

§2. Since S1* contains S1°, we have Lewis' axiom

A6 ℭ*K*ℭ*pq*ℭ*qr*ℭ*pr*

in this system.[6] And, obviously, S1° and *J1* imply

J2 ℭ*LLpLp*

Hence, if we add a generalized Brouwerian axiom

B_n ℭpL^nMp

for an arbitrary $n > 1$, to S1*, this last formula together with *J2* and *A6* gives

B_2 $\mathfrak{C}pLLMp$

and

B_1 $\mathfrak{C}pLMp$

Therefore, having S1° and B_1 we obtain

J3 $\mathfrak{C}MLpp$

without any difficulty.

On the other hand, since it is proved in [8], p. 59, that the addition of *C12*, i.e. B_1, to S1° generates a system, called $S1^+$, which contains S2°, we have at our disposal the so-called Becker's rule.[7] Hence, the application of this rule to B_2 gives at once

J4 $\mathfrak{C}MpMLLMp$

And, therefore, we have

C11 $\mathfrak{C}MpLMp$ [*A6*, p/Mp, $q/MLLMp$, r/LMp; *J4*; *J3*, p/LMp]

i.e. the proper axiom of S5. Since it is proved in [8], p. 58, that the addition of *C11* to S1° gives system S5, and since B_n, for any $n > 1$, is provable in this latter system, our proof is completed.

I have to note here that I do not know whether the addition of B_n, for any $n > 1$, to S1° gives system S5, although using the deductions analogous to the reasonings presented in [8], pp. 56-59, one can prove easily that:

a) The addition of an arbitrary Brouwerian formula B_n, for any $n \geq 1$, to S1° generates a system in which the following formula

N1 $\mathfrak{C}pM^{n+1}p$

holds,
and that:

b) The addition of B_n, for $n > 1$, to S1° generates a system in which besides *N1*, the formula

N2 $\mathfrak{C}M^{n+2}pMp$

is provable.

§3. Since in system S3* Lewis' rule of detachment for strict implication holds as a metarule of procedure, and since in both systems S3° and S3* the theses

P1 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}LpLq$
P2 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}MpMq$
P3 $\mathfrak{C}pNNp$
P4 $\mathfrak{C}NNpp$
P5 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}NqNp$

$P6$ $\mathfrak{C}pCqp$
$P7$ $\mathfrak{C}LCpq\mathfrak{C}pq$
$P8$ $\mathfrak{C}\mathfrak{C}pqLCpq$
$P9$ $\mathfrak{C}\mathfrak{C}pCqr\mathfrak{C}qCpr$
$P10$ $\mathfrak{C}Kpqq$
$P11$ $\mathfrak{C}NMKrNNpNMKrp$
$P12$ $NMKNMKKrpqNNMKKpqr$
$P13$ $\mathfrak{C}MKpqKMpMq$

and the following metarule of procedure

PI *If the formulas* $\mathfrak{C}\,\alpha\,\beta$ *and* $\mathfrak{C}\,\beta\,\gamma$ *are provable in the system, then also formula* $\mathfrak{C}\,\alpha\,\gamma$ *is provable in the system*

are provable,[8] the addition of a generalized Brouwerian axiom B_n, for any $n > 1$, as a new axiom either to S3° or to *S3** allows us to make the following easy deductions:

$S1$	$\mathfrak{C}M^nLpp$	[Follows from B_n; $P1$; $P2$; $P3$; $P4$; $P5$ and **PI**]
$S2$	$\mathfrak{C}LpLCqp$	[$P1$; $P6$]
$S3$	$\mathfrak{C}LpLCLqLp$	[**PI**; $P1$; $P7$; $P8$; $S2$]
$S4$	$\mathfrak{C}M^nLpM^nLCLqLp$	[$P2$; $S3$]
$S5$	$\mathfrak{C}M^nLpCLqLp$	[**PI**; $S4$; $S1$]
$S6$	$\mathfrak{C}LqCM^nLpLp$	[$P9$; $S5$]
$S7$	$\mathfrak{C}LLq\mathfrak{C}M^nLpLp$	[$P1$; $S6$; **PI**; $P7$]
$S8$	$\mathfrak{C}q\mathfrak{C}M^nLpLp$	[**PI**; B_n, p/q; $S7$; since in B_n: $n > 1$]
$S9$	$\mathfrak{C}M^nLpLp$	[$S8$; $P3$[9]]
$S10$	$\mathfrak{C}KrM^nLpLp$	[**PI**; $P10$; $S9$]
$S11$	$NMKKrM^nLpMNp$	[$P11$; $S10$]
$S12$	$NMKKM^nLpMNpr$	[$P12$; $P11$]
$S13$	$\mathfrak{C}MKM^{n-1}LpNpr$	[**PI**; $P13$; $S12$]
$S14$	$\mathfrak{C}NNrNMKM^{n-1}LpNp$	[$S13$; $P5$]
$S15$	$\mathfrak{C}M^{n-1}Lpp$	[$S14$; $P3$]
B_{n-1}	$\mathfrak{C}pL^{n-1}Mp$	[$S15$; $P1$; $P2$; $P3$; $P4$; $P5$; **PI**]

Now, it is obvious that

a) if $n - 1 = 1$, B_{n-1} is $C12$, i.e. the proper Brouwerian axiom,
b) if $n - 1 > 1$, then using entirely the same deductions which allowed us to obtain B_{n-1} from B_n we can deduce

B_{n-2} $\mathfrak{C}pL^{n-2}Mp$

from B_{n-1} .

Hence $C12$ follows from B_n, for any $n > 1$, in the fields of both systems, S3° and S3*. And, therefore, since in [8], pp. 56-58, it was proved that the addition of $C12$ either to S3* or to S3° gives S5, our proof is completed.

NOTES

1. *Cf.* [1], [2] and [7].

2. In this note instead of the original symbols of Lewis I use a modification of Łukasiewicz's symbolism which is described in [8], p. 52. In particular, the formulas

$$M^n\alpha \quad \text{and} \quad L^n\alpha$$

where n is an arbitrary natural number, will have here the following meanings

a) if $n = 1$, then $M^n\alpha = M\alpha$ and $L^n\alpha = L\alpha$
b) if $n > 1$, then $M^n\alpha = MM^{n-1}\alpha$ and $L^n\alpha = LL^{n-1}\alpha$.

It has to be noted that

1) Throughout this paper symbols C, L, $\mathfrak{C}$ and $\mathfrak{E}$ are used as the abbreviations.

and that

2) The definitions of the systems S1°, S2°, S3° and S3* discussed in this note are given in [8], pp. 52-53.

Moreover, in this paper the term "thesis" means: a formula which is true in the system under consideration.

3. The addition of $C12$ to S1° generates a system which contains S2°. Hence, obviously, the addition of $C12$ to S1 gives a system which contains S2. *Cf.* [8], p. 59.

4. *Cf.* [4], p. 494.

5. *Cf.* [6] and [4], p. 507.

6. *Cf.* [4], p. 493, [3], p. 483, and [8], p. 52.

7. I.e. the following metarule of procedure:

If the formula $\mathfrak{C}\alpha\beta$ is provable in the system, then also $\mathfrak{C}M\alpha M\beta$ is provable in the system.

This metarule is proved in S2 by Churchman, *cf.* [2], pp. 79-80, but it can be proved easily in S2°, *cf.* [3], p. 491, and [8], p. 58.

8. It follows clearly from the proofs given in [8], pp. 53-54 and pp. 57-58, that the theses $P1$-$P13$ and the mentioned metarules of procedure are provable in S3° and S3*. Since we do not have the first rule of substitution of Lewis in S3* and a proof that an analogous metarule holds in this system is not given in [8], all deductions given in this paragraph are conducted in such a manner that this rule (or metarule) is not used. The rule of adjunction of Lewis holds, obviously, as an analogous metarule in S3*.

9. *Cf.* the proof of $S9$ given here with the deductions given by Parry in [5], pp. 151-152.

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