Notre Dame Journal of Formal Logic
Volume III, Number 2, April 1962

INTUITIONISM RECONSIDERED

HUGUES LEBLANC and NUEL D. BELNAP, JR.

It has long been known that standard Gentzen rules of inference for PC_I, the intuitionist propositional calculus, will do for PC_C, the classical propositional calculus, once the intuitionist elimination rule for '$\sim$', namely:

NE_I: $A, \sim A \vdash B$,

is strengthened to read:

NE_C: $\sim \sim A \vdash A$.

It has recently been shown that the said rules will also do for PC_C once the intuitionist elimination rule for '$\supset$', namely:

HE_I: $A, A \supset B \vdash B$,

is strengthened to read:

HE_C: $A \supset B, (A \supset C) \supset A \vdash B$.[1]

We shall now show that the said rules will finally do for PC_C once the intuitionist elimination rule for '$\equiv$', namely:

BE_I: (a) $A, A \equiv B \vdash B$
(b) $A, B \equiv A \vdash B$,

is strengthened to read:

BE_C: (a) $A, (C \equiv A) \equiv (C \equiv B) \vdash B$
(b) $A, (C \equiv B) \equiv (C \equiv A) \vdash B$.

The debate between intuitionist logic and classical logic, a debate which originally centered around '$\sim$' and has more recently come to center around '$\supset$', can thus be made to center around '$\equiv$' as well. Details are as follows.

Let all five of '$\supset$', '$\sim$', '&', 'v', and '$\equiv$' serve as primitive connectives of PC_C; let 'A', 'B', and 'C' range over the well-formed formulas of PC_C; let '$A_1, A_2, \ldots, A_n \vdash B$', where $n > 0$, be short for 'B is deducible in PC_C

Received December 23, 1961

from the sequence made up of the wffs A_1, A_2, . . . , and A_n in that order'; let '$\vdash A$' be short for 'A is deducible in PC_C from the null sequence of wffs'; and, finally, let '$\vdash$' obey the following rules, where $n \geq 0$.

Structural rules

R: $A \vdash A$ (Reflexivity);

P: If $A_1, A_2, \ldots, A_{i-1}, A_i, A_{i+1}, A_{i+2}, \ldots, A_n, A_{n+1}, A_{n+2} \vdash B$, then $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, A_i, A_{i+2}, \ldots, A_n, A_{n+1}, A_{n+2} \vdash B$, where $i \leq n + 1$ (Permutation);

E: If $A_1, A_2, \ldots, A_n \vdash B$, then $A_{n+1}, A_1, A_2, \ldots, A_n \vdash B$ (Expansion);

S: If $A_1, A_2, \ldots, A_n, A_{n+1} \vdash B$ and $A_1, A_2, \ldots, A_n \vdash A_{n+1}$, then $A_1, A_2, \ldots, A_n \vdash B$ (Simplification).

Elimination and introduction rules

HE_I: $A, A \supset B \vdash B$ (Intuitionist elimination rule for '$\supset$');

HI: If $A_1, A_2, \ldots, A_n, A_{n+1} \vdash B$, then $A_1, A_2, \ldots, A_n \vdash A_{n+1} \supset B$ (Introduction rule for '$\supset$');

NE_I: $A, \sim A \vdash B$ (Intuitionist elimination rule for '$\sim$');

NI: If $A_1, A_2, \ldots, A_n, A_{n+1} \vdash B$ and $A_1, A_2, \ldots, A_n, A_{n+1} \vdash \sim B$, then $A_1, A_2, \ldots, A_n \vdash \sim A_{n+1}$ (Introduction rule for '$\sim$');

CE: (a) $A \,\&\, B \vdash A$, (b) $A \,\&\, B \vdash B$ (Elimination rule for '&');

CI: If $A_1, A_2, \ldots, A_n \vdash B$ and $A_1, A_2, \ldots, A_n \vdash C$, then $A_1, A_2, \ldots, A_n \vdash B \,\&\, C$ (Introduction rule for '&');

DE: If $A_1, A_2, \ldots, A_n, A_{n+1} \vdash B$ and $A_1, A_2, \ldots, A_n, A_{n+2} \vdash B$, then $A_1, A_2, \ldots, A_n, A_{n+1} \vee A_{n+2} \vdash B$ (Elimination rule for 'v');

DI: (a) $A \vdash A \vee B$, (b) $B \vdash A \vee B$ (Introduction rule for 'v');

BE_C: (a) $A, (C \equiv A) \equiv (C \equiv B) \vdash B$, (b) $A, (C \equiv B) \equiv (C \equiv A) \vdash B$ (Elimination rule for '$\equiv$');

BI: If $A_1, A_2, \ldots, A_n, B \vdash C$ and $A_1, A_2, \ldots, A_n, C \vdash B$, then $A_1, A_2, \ldots, A_n \vdash B \equiv C$ (Introduction rule for '$\equiv$').

The following lemmas are then provable:

Lemma 1: $A, A \equiv B \vdash B$ (BE_I(a)).

(1)	$A \vdash A$	(R)
(2)	$\vdash A \equiv A$	(BI, 1, 1)
(3)	$A \equiv B \vdash A \equiv A$	(E, 2)
(4)	$A \equiv B, A \equiv B \vdash A \equiv A$	(E, 3)
(5)	$A \equiv B \vdash A \equiv B$	(R)

(6)	$A \equiv A, A \equiv B \vdash A \equiv B$	$(E, 5)$
(7)	$A \equiv B, A \equiv A \vdash A \equiv B$	$(P, 6)$
(8)	$A \equiv B \vdash (A \equiv A) \equiv (A \equiv B)$	$(BI, 4, 7)$
(9)	$A, A \equiv B \vdash (A \equiv A) \equiv (A \equiv B)$	$(E, 8)$
(10)	$A, (A \equiv A) \equiv (A \equiv B) \vdash B$	$(BE_C$ (a))
(11)	$A \equiv B, A, (A \equiv A) \equiv (A \equiv B) \vdash B$	$(E, 10)$
(12)	$A, A \equiv B, (A \equiv A) \equiv (A \equiv B) \vdash B$	$(P, 11)$
(13)	$A, A \equiv B \vdash B$	$(S, 9, 12)$

Lemma 2: $A, B \equiv A \vdash B$ $(BE_I$(b)).

Similar proof, but using BE_C (b) instead of BE_C (a).

Lemma 3: If $A_1, A_2, \ldots, A_n \vdash B$ and $A_1, A_2, \ldots, A_n \vdash B \equiv C$, then $A_1, A_2, \ldots, A_n \vdash C$.

(1)	$B, B \equiv C \vdash C$	(Lemma 1)
(2)	$A_1, A_2, \ldots, A_n, B, B \equiv C \vdash C$	$(E, 1)$
(3)	$A_1, A_2, \ldots, A_n \vdash B \equiv C$	(Given)
(4)	$B, A_1, A_2, \ldots, A_n \vdash B \equiv C$	$(E, 3)$
(5)	$A_1, A_2, \ldots, A_n, B \vdash B \equiv C$	$(P, 4)$
(6)	$A_1, A_2, \ldots, A_n, B \vdash C$	$(S, 2, 5)$
(7)	$A_1, A_2, \ldots, A_n \vdash B$	(Given)
(8)	$A_1, A_2, \ldots, A_n \vdash C$	$(S, 6, 7)$

Lemma 4: If $A_1, A_2, \ldots, A_n \vdash B$ and $A_1, A_2, \ldots, A_n \vdash C \equiv B$, then $A_1, A_2, \ldots, A_n \vdash C$.

Similar proof, but using Lemma 2 instead of Lemma 1.

Lemma 5: If $A_1, A_2, \ldots, A_n \vdash B$ and $A_1, A_2, \ldots, A_n \vdash \sim B$, then $A_1, A_2, \ldots, A_n \vdash C$.

Similar proof, but using NE_I instead of Lemma 1.

Lemma 6: $\vdash (\sim A \equiv \sim \sim A) \equiv (\sim A \equiv A)$.

(1)	$A, \sim A \equiv A \vdash \sim A$	(Lemma 2)
(2)	$\sim A \equiv A, A \vdash \sim A$	$(P, 1)$
(3)	$A \vdash A$	(R)
(4)	$\sim A \equiv A, A \vdash A$	$(E, 3)$
(5)	$\sim A \equiv A \vdash \sim A$	$(NI, 2, 4)$
(6)	$\sim A \equiv A \vdash \sim A \equiv A$	(R)
(7)	$\sim A \equiv A \vdash A$	(Lemma 3, 5, 6)
(8)	$\sim A \equiv A \vdash \sim A \equiv \sim \sim A$	(Lemma 5, 5, 7)
(9)	$\sim A, \sim A \equiv \sim \sim A \vdash \sim \sim A$	(Lemma 1)
(10)	$\sim A \equiv \sim \sim A, \sim A \vdash \sim \sim A$	$(P, 9)$
(11)	$\sim A \vdash \sim A$	(R)
(12)	$\sim A \equiv \sim \sim A, \sim A \vdash \sim A$	$(E, 11)$
(13)	$\sim A \equiv \sim \sim A \vdash \sim \sim A$	$(NI, 10, 12)$
(14)	$\sim A \equiv \sim \sim A \vdash \sim A \equiv \sim \sim A$	(R)

(15) $\sim A \equiv \sim\sim A \vdash \sim A$ (Lemma 4, 13, 14)
(16) $\sim A \equiv \sim\sim A \vdash \sim A \equiv A$ (Lemma 5, 13, 15)
(17) $\vdash (\sim A \equiv \sim\sim A) \equiv (\sim A \equiv A)$ (*BI*, 8, 16)

Lemma 6, plus the three rules E, BE_C, and S, now lead to the promised result:

Theorem 1: $\sim\sim A \vdash A$ (NE_C).

(1) $\vdash (\sim A \equiv \sim\sim A) \equiv (\sim A \equiv A)$ (Lemma 6)
(2) $\sim\sim A \vdash (\sim A \equiv \sim\sim A) \equiv (\sim A \equiv A)$ (E, 1)
(3) $\sim\sim A, (\sim A \equiv \sim\sim A) \equiv (\sim A \equiv A) \vdash A$ (BE_C)
(4) $\sim\sim A \vdash A$ (S, 2, 3)

The fourteen rules R, P, E, S, HE_I, HI, NE_I, NI, CE, CI, DE, DI, BE_I, and BI provide for all inferences in PC_I.[2] The self-same rules with NE_C in place of NE_I provide, on the other hand, for all inferences in PC_C.[3] We thus conclude in the light of Theorem 1 (and Lemmas 1 and 2) that standard Gentzen rules of inference for PC_I will do for PC_C once the intuitionist elimination rule for '$\equiv$' is strengthened to read like BE_C. We also conjecture, by the way, that any structural rule which holds in PC_C also holds in PC_I, that any elimination or introduction rule for '&' and 'v' which holds in PC_C also holds in PC_I, and hence that the only way of turning standard Gentzen rules of inference for PC_I into rules for PC_C is to strengthen the elimination or introduction rules for '$\sim$', or those for '$\supset$', or those for '$\equiv$'. We cannot, however, address ourselves to that problem here.[4]

NOTES

1. See E. W. Beth and H. Leblanc, "A note on the intuitionist and the classical propositional calculus," *Logique et Analyse*, vol. 3, no. 11-12 (1960), pp. 174-176. Rule HE_C was suggested to Professor Leblanc by Professor Stig Kanger.
2. That the rules in question provide for all inferences in PC_I can be shown by matching them against the axioms and rules of inference of P_S^i in A. Church, *Introduction to Mathematical Logic, Volume I*, pp. 141-142.
3. That the rules in question provide for all inferences in PC_C can be shown by matching them against the axioms and rules of inference of P_H in Church, *loc. cit.*, pp. 140-141.
4. We require that a rule of inference for PC_I or for PC_C be a structural, an elimination, or an introduction rule, that a structural rule exhibit no connective, and that an elimination or introduction rule exhibit only the connective which it serves to eliminate or to introduce. It is possible, on the other hand, to pass from PC_I to PC_C by adding to the rules of inference of PC_I such rules as $\vdash A \vee \sim A$, $\vdash \sim\sim A \supset A$, $\vdash \sim\sim A \equiv A$, and so on.

Bryn Mawr College
Bryn Mawr, Pennsylvania
and
Yale University
New Haven, Connecticut