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PROOF OF SOME THEOREMS ON RECURSIVELY ENUMERABLE SETS

TH. SKOLEM

In this paper I shall first define a class of functions which I call lower elementary, abbreviated l.el. functions in the sequel, and after some preliminary considerations prove that every recursively enumerable set of integers can be enumerated by a l.el. function. All variables and functions shall here take non-negative integers as values. L. Kalmár defined the notion elementary function (see [1]). These are the functions that can be constructed from addition, multiplication and the operation $\dot{-}$ by use of the general sums and products

$$\sum_{r=0}^{x} f(r) \quad \text{and} \quad \prod_{r=0}^{x} f(r),$$

where f may contain parameters, together with the use of composition. If we omit the use of general products, we get what I call the lower elementary functions. The definition is therefore:

Df 1. The l.el. functions are those which can be built by starting with the functions 0, 1, $x + y$, xy, $x \dot{-} y$ and using the summation $\sum_{r=0}^{x} f(r)$, where f may contain parameters, besides use of composition. By the way, instead of $x \dot{-} y$ one can choose $\delta(x, y)$, the Kronecker delta (see [2]). As to the summation schema it can be shown that it is sufficient to require its use in the case that f contains one parameter at most. Of course xy can be omitted as starting function.

Clearly every polynomial is an l.el. function. Further every l.el. function can be majorised by a polynomial. This is seen immediately to be true for the starting functions and it is easily seen to be hereditary with regard to summation and composition. If for example $f(x, y)$ is always $\leqq \varphi(x, y)$, where φ is a polynomial, then for all x and y

$$\sum_{r=0}^{x} f(r, y) \leqq \sum_{r=0}^{x} \varphi(r, y)$$

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and the right hand side here is again a polynomial. In order to prove that also composition leads from functions which can be majorised by polynomials to functions of this kind we may first suppose that $f(x, y)$ increases steadily for increasing x and y, that is $f(x, y) \leqq f(x', y')$ when $x \leqq x'$, $y \leqq y'$. Then we have for all x and y, supposing that f, g, h are majorised by the polynomials φ, γ, η,

$$f(g(x, y), h(x, y)) \leqq f(\gamma(x, y), \eta(x, y)) \leqq \varphi(\gamma(x, y), \eta(x, y))$$

and the last function is a polynomial. However, if $f(x, y)$ is not monotonous, we have in any case

$$f(x, y) \leqq \sum_{r=0}^{x} \sum_{s=0}^{y} f(r, s) \leqq \sum_{r=0}^{x} \sum_{s=0}^{y} \varphi(r, s) = \phi(x, y)$$

and the polynomial $\phi(x, y)$ is of course monotonous with regard to x and y. Therefore

$$f(g(x, y), h(x, y)) \leqq \phi(g(x, y), h(x, y)) \leqq \phi(\gamma(x, y), \eta(x, y)) ,$$

where the last function is a polynomial. Of course these proofs can be carried out just as well for functions of more variables.

Lemma 1. Let f and g be l.el. functions and always $f(x) > y$ when $x > g(y)$. Then the greatest x such that $f(x) \leqq y$ is a l.el. function of y.

Proof: As a matter of fact this greatest x can be expressed so:

$$x = \sum_{r=0}^{g(y)} r \cdot \overline{\mathbf{sg}}(f(r) \dot{-} y)\, \overline{\mathbf{sg}} \sum_{s=r+1}^{g(y)} s \cdot \overline{\mathbf{sg}}(f(s) \dot{-} y) ,$$

where $\overline{\mathbf{sg}}z$ as usual means $1 \dot{-} z$. Indeed, letting r take successively the values $g(y)$, $g(y) \dot{-} 1$, $g(y) \dot{-} 2$, . . . we will once for the first time reach an r such that $f(r) \dot{-} y$ is $= 0$ so that $r\overline{\mathbf{sg}}(f(r) \dot{-} y)$ is just $= r$ which is the desired x. Still $s\overline{\mathbf{sg}}(f(s) \dot{-} y)$ is $= 0$ for greater values of s than x so that $\overline{\mathbf{sg}} \sum_{s=x+1}^{g(y)} s\overline{\mathbf{sg}}(f(s) \dot{-} y) = 1$. For smaller values of r than x we have $\overline{\mathbf{sg}} \sum_{s=r+1}^{g(y)} s\overline{\mathbf{sg}}(f(s) \dot{-} y) = 0$. Therefore the value of the whole double sum is x as asserted.

Lemma 2. Putting for $r = 1, 2, \ldots, m$

$$x_r = \tau_r^{(m)} y$$

when $y = \mathfrak{p}_m(x_1, \ldots, x_m)$, where $\mathfrak{p}_m(x_1, \ldots, x_m)$ is the polynomial

$$\binom{x_1 + x_2 + \ldots + x_m + m - 1}{m} + \binom{x_1 + x_2 + \ldots + x_{m-1} + m - 2}{m - 1} + \ldots + \binom{x_1 + x_2 + 1}{2} + x_1 ,$$

the functions $\tau_r^{(m)}(y)$ are all of them l.el.

Proof: As is well known (see [3]) the equation $y = \mathfrak{p}_m(x_1, \ldots, x_m)$ yields the simplest one to one correspondence between the integers y and the m-tuples of integers $x_1, \ldots, x_m$. Putting for $r = 1, 2, \ldots, m$ $x_1 + x_2 + \ldots + x_r = \xi_r$ we get that ξ_m is the greatest value of z such that

$$\binom{z+m-1}{m} \leqq y\,.$$

Therefore according to lemma 1 ξ_m is a l.el. function $\sigma_m^{(m)}y$. Further ξ_{m-1} is the greatest z such that

$$\binom{z+m-2}{m-1} \leqq y \dot{-} \binom{\sigma_m^{(m)}y+m-1}{m} = y_1$$

so that according to lemma 1 ξ_{m-1} is a l.el. function of y_1. But y_1 is a l.el. function of y. Therefore $\xi_{m-1} = \sigma_{m-1}^{(m)}(y)$, $\sigma_{m-1}^{(m)}(y)$ being a l.el. function of y. This can be continued in an obvious way. We obtain for $r = 2, \ldots, m$

$$x_r = \xi_r \dot{-} \xi_{r-1} = \sigma_r^{(m)}(y) \dot{-} \sigma_{r-1}^{(m)}(y) = \tau_r^{(m)}(y) \text{ and } x_1 = \xi_1 = \tau_1^{(m)}(y)\,,$$

where all the $\tau_r^{(m)}(y)$ are l.el.

In the sequel $\mathbf{p}_n$ means the $(n+1)$ th prime and $\mathbf{e}(m, n)$ the exponent of the highest power of $\mathbf{p}_n$ dividing m.

Lemma 3. Both $\mathbf{p}_n$ and $\mathbf{e}(m, n)$ are l.el. functions.

Proof: According to a well known theorem of Tchebychef in elementary number theory one has that

$$\pi(x) > \mathbf{c}\,\frac{x}{\log x}\,,$$

where $\pi(x)$ is the number of primes $\leqq x$ and $\mathbf{c}$ some positive constant. It follows that for $x > g$, g some positive integer,

$$\pi(x) > x^{1/2}\,,$$

because g can be chosen such that $\mathbf{c}x^{1/2} > \log x$ for all $x > g$. Now if $\mathbf{p}_n$ is the largest prime $\leqq x$ so that $n + 1 = \pi(x)$, we get

$$x < (n+1)^2\,,$$

whence

$$\mathbf{p}_n < (n+1)^2\,.$$

This is certainly valid for all $n > g$, because $x > \pi(x) > n$ yields $x > g$.

Now $\mathbf{d}(a, b) = \sum_{r=1}^{b} \delta(ar, b)$ is 1 or 0 according as a divides b or not.

Writing $\bar{\delta}(x, y)$ instead of $1 - \delta(x, y)$ the function

$$\mathbf{P}(a) = \mathbf{sg} \sum_{r=1}^{a} \mathbf{d}(r, a)\,\bar{\delta}(r, 1)\,\bar{\delta}(r, a) + \delta(a, 1)$$

is $= 0$ or 1 according as a is a prime or not. Hence

$$\mathbf{p}_n = \sum_{t=o}^{\max(g,\,(n+1)^2)} t \cdot ((1 \dot{-} \mathbf{P}(t)) \cdot \delta(n, \sum_{s=o}^{t-1} (1 \dot{-} \mathbf{P}(s))) .$$

Thus $\mathbf{p}_n$ is a l.el. function of n.

That the function $\mathbf{e}(m, n)$ is l.el. as well can be proved easier. The l.el. function of a and n

$$\sum_{r=1}^{a} \sum_{s=1}^{a} (\mathbf{d}(r, a) \cdot \mathbf{d}(s, a) \cdot \bar{\mathbf{d}}(r, s) \cdot \bar{\mathbf{d}}(s, r)) + \bar{\mathbf{d}}(\mathbf{p}_n, a)$$

is $= 0$ or > 0 according as a is a power of $\mathbf{p}_n$ or not. Therefore the l.el. function

$$\mathbf{Q}(a, m, n) = \bar{\mathbf{d}}(\mathbf{p}_n, a) + \bar{\mathbf{d}}(a, m) + \sum_{r=1}^{a} \sum_{s=1}^{a} (\mathbf{d}(r, a) \cdot \mathbf{d}(s, a) \cdot \bar{\mathbf{d}}(r, s) \cdot \bar{\mathbf{d}}(s, r))$$

is $= 0$ if and only if a is a power of $\mathbf{p}_n$ and divides m. Then $\mathbf{e}(m, n)$ is just the number of these a for which $\mathbf{Q}(a, m, n) = 0$. Therefore

$$\mathbf{e}(m, n) = \sum_{a=1}^{m} (1 \dot{-} \mathbf{Q}(a, m, n))$$

and this is a l.el. function of m and n.

Df 2. A set shall be called l. el.enum. (that is lower elementary enumerable) if its elements can be enumerated by a l.el. function.

Theorem 1. Let $\nu(x, x_1, \ldots, x_m)$ *and* $\lambda_1(x), \ldots, \lambda_m(x)$ *be l.el. functions such that* $\lambda_r(x) \leqq x$ *for* $r = 1, 2, \ldots, m$. *Further let the function* f *be defined by the course of values recursion*

$$f(n+1) = \nu(n, f(\lambda_1(n)), f(\lambda_2(n)), \ldots, f(\lambda_m(n))), \quad f(0) = a .$$

Then the set of values of f *is l. el. enum.*

It ought to be remarked that f itself need not be l.el. which is already shown by the very simple example $f(n+1) = 2f(n)$, $f(0) = 1$.

Proof: I consider numbers N with the following property

$$N = p_o^{e_o} \, p_1^{e_1} \ldots p_n^{e_n} ,$$

where $e_r = \mathbf{e}(N, r) = f(r)$. In other words, we put $e_o = a$ and for $r = 0, 1, \ldots, n-1$ successive

$$e_{r+1} = f(r+1) = \nu(r, f(\lambda_1(r)), \ldots, f(\lambda_m(r))) .$$

Thus N has the property expressed by saying that $\mathbf{F}(N, n) = 0$, where $\mathbf{F}(N, n)$ is the l.el. function

$$\bar{\delta}(\mathbf{e}(N, 0), a) + \sum_{r=o}^{n-1} \bar{\delta}(\mathbf{e}(N, r+1), \nu(r, \mathbf{e}(N, \lambda_1(r)), \ldots, \mathbf{e}(N, \lambda_m(r))) .$$

It is therefore obvious that $y = f(x)$ is equivalent the existence of an N such that

$$\mathbf{F}(N, x) = 0 \ \ \& \ \ y = \mathbf{e}(N, x)$$

or in other words

$$\mathbf{F}(N, x) + \overline{\delta}(\mathbf{e}(N, x), y) = 0 .$$

The integers y for which this is true, that is the y which are values of $f(x)$, are now given by the following equation, where I have put $\mathbf{F}(N, x) + \overline{\delta}(\mathbf{e}(N, x), z) = \mathbf{G}(N, x, z)$,

$$y = z(1 \dot{-} \mathbf{G}(N, x, z)) + a\mathbf{sg}\mathbf{G}(N, x, z) .$$

Here the right hand side is a l.el. function and if we insert

$$N = r_1^{(3)}u , \quad x = r_2^{(3)}u , \quad z = r_3^{(3)}u ,$$

it becomes a l.el. function of u which enumerates the considered numbers y when u runs through all non negative integers.

Remark: If $g(x)$ is l.el. we get of course a l.el. enumeration of the values of $f(g(x))$ by replacing the x in $\mathbf{G}(N, x, z)$ by $g(x)$ and after that taking again N, x, z as $r_r^{(3)}u$, $r = 1, 2, 3$.

Theorem 2. Every recursively enumerable set is l. el. enum.

Proof: As I have explained in a paper published many years ago (see [4]) it is possible to replace the system of equations defining a recursive function by production rules for n-tuples. Indeed the definition of a function of $n-1$ variables, say $x_n = f(x_1, \ldots, x_{n-1})$, is equivalent the generating of a set of n-tuples $(x_1, \ldots, x_n)$ such that every $(n-1)$-tuple $(x_1, \ldots, x_{n-1})$ occurs in just one of the n-tuples. In the defining equation system some earlier defined functions may be present, however we may introduce again the corresponding rules of production of m-tuples for the actual values of m. Then we may assume that in the production rules we have only the successor function left, but so strong a reduction is not always necessary. Every production rule of n-tuples then says that the n-tuples $(b_1, \ldots, b_n)$, $(c_1, \ldots, c_n)$, $\ldots$, $(k_1, \ldots, k_n)$ produce the n-tuple $(a_1, \ldots, a_n)$. Here $a_1, \ldots, a_n$ are some of the $b_1, \ldots, k_n$ or perhaps expressed by some of them by using the successor function a given number of times or perhaps using even some other simple functions which we may assume as l.el. functions. Instead of trying to explain this further in a general way I shall give some examples of the procedure.

That the values of the el. but not l.el. function $x!$ can be l.el. enumerated is seen at once because of Th.1. Indeed this function $f(x)$ is defined by the equations

$$f(0) = 1, \quad f(x + 1) = (x + 1) \cdot f(x)$$

and this is a special case of the recursion treated in Th.1. In the same way it is seen that although the functions 2^x, x^x, $(x!)^x$ and similar ones are not l.el. their corresponding sets of values are l.el.enum.

Let us however look at the function $f(x)$ defined by

$$f(0) = 0, \quad f(x + 1) = 2^{f(x)} .$$

This function is primitive recursive, but probably not elementary. It is here convenient to replace also the function 2^x by a generating of pairs. We then have to deal with 2 kinds of pairs, say (a, b) and $\{a, b\}$. We start with $(0, 0)$ and $\{0, 1\}$ and the production rules are

1) $\{a, b\}$ produces $\{a + 1, 2b\}$
2) $(a, b), \{b, c\}$ produce $(a + 1, c)$.

Remark: We could also remove the function $2n$ so that only the successor function is used inside the pairs, but that would require the treatment of 3 kinds of pairs and that is not necessary because the function $2n$ is l.el.

We now introduce an enumerating function φ for the pairs letting $\varphi(2n)$ enumerate the pairs (a, b) and $\varphi(2n + 1)$ the pairs $\{a, b\}$. This can be performed so: We put $\varphi 0 = \mathfrak{p}_2(0, 0) = 0$, $\varphi 1 = \mathfrak{p}_2(0, 1) = 1$ and

1) as often as we have put $\varphi(2n + 1) = \mathfrak{p}_2(a, b)$ we put $\varphi(2n + 3) = \mathfrak{p}_2(a + 1, 2b)$,
2) as often as we have put $\varphi(2m) = \mathfrak{p}_2(a, b)$ and $\varphi(2n + 1) = \mathfrak{p}_2(b, c)$ we put $\varphi(2\, \mathfrak{p}_2(m, n + 1)) = \mathfrak{p}_2(a + 1, c)$,
3) similarly we write $\varphi(2\, \mathfrak{p}_2(m, n + 1)) = 0$ when $\varphi(2m) = \mathfrak{p}_2(a, b)$ and $\varphi(2n + 1) = \mathfrak{p}_2(c, d)$, $b \neq c$.

Remark: One may notice that $2\, \mathfrak{p}_2(m, n + 1)$ is always $>$ max $(2m, 2n + 1)$ and takes different values for different pairs m, n. Therefore these argument values are always available without confusion.

The definition of φ can be written more concisely thus:

$$\varphi 0 = 0, \quad \varphi 1 = 1 \quad \text{and for } n > 0$$

$$\varphi(n + 1) = \mathfrak{p}_2(\tau_1^{(2)} \varphi(n - 1) + 1,\ 2\,\tau_2^{(2)} \varphi(n - 1))\ \mathbf{rm}(n + 1, 2)$$
$$+\ \mathfrak{p}_2\left(\tau_1^{(2)} \varphi\left(2\tau_1^{(2)} \left[\frac{n + 1}{2}\right]\right) + 1,\ \tau_2^{(2)} \varphi\left(2\tau_2^{(2)} \left[\frac{n + 1}{2}\right] - 1\right)\right)$$
$$\text{times } \delta\left(\tau_2^{(2)} \varphi\left(2\tau_1^{(2)} \left[\frac{n + 1}{2}\right]\right),\ \tau_1^{(2)} \varphi\left(2\tau_2^{(2)} \left[\frac{n + 1}{2}\right] - 1\right)\right)\ \mathbf{rm}(n, 2)$$

It is obvious that this recursive definition is just of the form dealt with in Th.1. Now $y = f(x)$ means that for some n we have $\varphi(2n) = \mathfrak{p}_2(x, y)$, whence $y = \tau_2^{(2)} \varphi(2n)$. However according to the remark to Th.1 the set of values of $\tau_2^{(2)} \varphi(2n)$ is l.el.enum.

Let us take as a further example the function of R. Péter which is not primitive recursive and is defined thus:

$$\psi(0, n) = n + 1$$
$$\psi(m + 1, 0) = \psi(m, 1)$$
$$\psi(m + 1, n + 1) = \psi(m, \psi(m + 1, n))\ .$$

The translation of this into a generation of triples is so: We consider a set S of triples generated by following rules:

1) Every triple $(0, n, n+1)$ belongs to S
2) As often as we have got $(a, 1, b)$ in S we put $(a+1, 0, b)$ in S
3) As often we have already $(a+1, b, c)$ and (a, c, d) in S we put $(a+1, b+1, d)$ in S

Clearly this generation of triples is just what the computation of the values of ψ amounts to, the third element in any triple being the value of the ψ-function of the first and second element. Here we may define a function φ so:

$$\varphi(3n+1) = \mathfrak{p}_3(0, n, n+1), \; \varphi(3n+2) = \mathfrak{p}_3(a+1, 0, b) \text{ when } \varphi n = \mathfrak{p}_3(a, 1, b),$$
$$\text{otherwise } \varphi(3n+2) = 1,$$

$$\varphi(3n) = \mathfrak{p}_3(a+1, b+1, d) \text{ when } \varphi\tau_1^{(2)} n = \mathfrak{p}_3(a+1, b, c) \text{ and } \varphi\tau_2^{(2)} n = \mathfrak{p}_3(a, c, d),$$
$$\text{otherwise } \varphi(3n) = 1.$$

Then the values of φ are just $\mathfrak{p}_3$ of all the generated triples. The value $1 = \mathfrak{p}_3(0, 0, 1)$ is taken by φ infinitely often, the other values once each. The recursive definition of φ can be written in concise form thus: $\varphi\, 0 = 1$ and

$$\begin{aligned}\varphi(n+1) = {} & \mathfrak{p}_3\left(0, \left[\frac{n}{3}\right], \left[\frac{n}{3}\right] + 1\right) \delta\,(\mathbf{rm}\,(n+1, 3), 1) + \mathfrak{p}_3\left(\tau_1^{(3)}\varphi\left(\left[\frac{n}{3}\right]\right)\right. \\ & \left. + 1, 0, \tau_3^{(3)}\varphi\left(\left[\frac{n}{3}\right]\right)\right) \delta\left(\tau_2^{(3)}\varphi\left(\left[\frac{n}{3}\right]\right), 1\right) + \bar{\delta}\left(\tau_2^{(3)}\varphi\left(\left[\frac{n}{3}\right]\right), 1\right)\Bigg) \\ & \delta\,(\mathbf{rm}\,(n+1, 3), 2) + \left(\mathfrak{p}_3\left(\tau_1^{(3)}\varphi\tau_1^{(2)}\left(\left[\frac{n+1}{3}\right]\right), \tau_2^{(3)}\varphi\,\tau_1^{(2)}\left(\left[\frac{n+1}{3}\right]\right)\right.\right. \\ & \left.\left. + 1, \tau_3^{(3)}\varphi\,\tau_2^{(2)}\left(\left[\frac{n+1}{3}\right]\right)\right)\delta\left(\tau_3^{(3)}\varphi\tau_1^{(2)}\left(\left[\frac{n+1}{3}\right]\right), \tau_2^{(3)}\varphi\tau_2^{(2)}\left(\left[\frac{n+1}{3}\right]\right)\right)\right. \\ & \left. + \bar{\delta}\left(\tau_3^{(3)}\varphi\tau_1^{(2)}\left(\left[\frac{n+1}{3}\right]\right), \tau_2^{(3)}\varphi\tau_2^{(2)}\left(\left[\frac{n+1}{3}\right]\right)\right)\right) \delta\,(\mathbf{rm}(n+1, 3), 0)\end{aligned}$$

This is a recursion of the kind treated in Th.1 according to which we can find a l.el. function, say $f(n)$, which takes the same set of values as φn. Now the values of Péter's function constitute the set of values of $\tau_3^{(3)}\varphi n$ which again coincides with the set of values of $\tau_3^{(3)} f(n)$ and this is a l.el. function.

By the way there is another method of proof of which I shall give a hint. E. L. Post has developed a theory on sets of strings of letters. In particular he has shown that the recursive sets can be conceived as the so-called canonical sets in his normal systems (see [5], p. 170). In a normal system one string of symbols 1 and b is given as axiom. Further there are say m rules of production for strings of symbols 1 and b, say $\sigma_{1,r}\, \alpha \to \alpha\, \sigma_{2,r}$, $r = 1, \ldots, m$, where the σ's are given strings, α arbitrary. Using the prime number decomposition of positive integers we can represent the strings by numbers, letting the exponent of each prime be the number of letters in a corresponding sequence of equal symbols in the string. Then it turns out that the number corresponding to $\alpha\,\sigma_{2,r}$ becomes a l.el. function, say $l_r(x)$, of the number x corresponding to $\sigma_{1,r}\alpha$. If a represents the axiom, we thus get generated the numbers a, $l_r(a)$, $l_s l_r(a)$, and so on. These numbers are the values of the function φ defined thus:

$$\varphi(0) = a,\ \varphi(n+1) = \sum_{r=1}^{m} l_r \varphi\left[\frac{n}{m}\right] \delta(\mathbf{rm}(n+1,\, m),\, r)\ .$$

Since this recursive definition is of the kind considered in Th.1 we have according to this theorem a l.el. function $f(n)$ whose set of values is the same as the set of values of φn.

Theorem 3. For any recursive relation

$$\rho(x_1, \ldots, x_n) = 0$$

there is an equivalent parametric representation

$$x_1 = f_1(t), \ldots, x_n = f_n(t)\ ,$$

where the f_r are l.el. functions, provided that there is at least one n-tuple $(a_1, \ldots, a_n)$ satisfying the relation. Indeed there is a l.el. function $f(t)$ such that

$$x_r = r_r^{(n)} f(t),\ r = 1, \ldots, n\ .$$

Proof: Putting $y = \mathfrak{p}_n(x_1, \ldots, x_n)$ and $a = \mathfrak{p}_n(a_1, \ldots, a_n)$ we have $x_r = r_r^{(n)}\, y$ so that if we write

$$\rho(r_1^{(n)}\, y, \ldots, r_n^{(n)}\, y) = \sigma y\ ,$$

the n-tuples $(x_1, \ldots, x_n)$ satisfying $\rho(x_1, \ldots, x_n) = 0$ are just given by $x_r = r_r^{(n)}\, y$, $r = 1, \ldots, n$, y satisfying $\sigma y = 0$. These numbers y are all given by the formula

$$y = z(1 \dot{-} \sigma z) + a \cdot \mathbf{sg}\, \sigma z = \chi(z)\ .$$

Now $\chi(z)$ is usually not a l.el. function, but according to Th.2 there is a l.el. function $f(t)$ taking the same set of values as $\chi(z)$. Therefore every n-tuple $(x_1, \ldots, x_n)$ such that $\rho(x_1, \ldots, x_n) = 0$ and only these are given by

$$x_1 = r_1^{(n)} f(t), \ldots, x_n = r_n^{(n)} f(t)$$

letting t here run through all non negative integers.

I shall give some concluding remarks.

If we possess l.el. functions $f_1, \ldots, f_n$ enumerating respectively the sets $M_1, \ldots, M_n$, it is trivial to find a l.el. function enumerating the union of these sets. The same must be said for the intersection, provided of course that it is not empty. Let however $M_0, M_1, \ldots$ be a l.el. enumerated infinite set of sets that is we have a l.el. function $f(x, y)$ such that for any given x the set M_x is enumerated by putting $y = 0, 1, \ldots$ in $f(x, y)$ for this x. Then it may be noticed that the elements of the union of $M_0, M_1, \ldots, M_{x-1}$ will for arbitrary x be enumerated by the l.el. function

$$g(x, z) = \sum_{r=0}^{x-1} f\left(r, \left[\frac{z}{x}\right]\right) \delta(\mathbf{rm}(z, x), r) .$$

Let us further assume that 0 belongs to all M_r, $r = 0, 1, \ldots, x$. Then the intersection of these sets consists of the numbers z for which for some u

$$\sum_{r=0}^{x} \overline{\delta}(z, f(r, e(u, r))) = 0 .$$

Indeed if z is in the intersection, then there are numbers $y_0, y_1, \ldots, y_x$ such that

$$z = f(0, y_0) = f(1, y_1) = \ldots = f(x, y_x)$$

and putting

$$u = p_0^{y_0} \ldots p_x^{y_x}$$

we obtain for $r = 0, 1, \ldots, x$

$$z = f(r, e(u, r)) ,$$

whence

$$\sum_{r=0}^{x} \overline{\delta}(z, f(r, e(u, r))) = 0 .$$

Let on the other hand the last equation be valid. Then the preceding one is valid for $r = 0, 1, \ldots, x$ which means that z is in the intersection. Now the z for which the last equation is valid for some u are enumerated by the l.el. function

$$g(x, v) = r_1^{(2)} v \cdot (1 \dot{-} \psi(x, r_1^{(2)}(v), r_2^{(2)}(v)),$$

where

$$\psi(x, z, u) = \sum_{r=0}^{x} \overline{\delta}(z, f(r, e(u, r))) .$$

Finally the union of the infinitely many sets $M_0, M_1, \ldots$, where M_x is enumerated by $f(x, y)$, $y = 0, 1, \ldots$, $f(x, y)$ being a l.el. function of x and y, is enumerated very simply by the l.el. function $f(r_1^{(2)} z, r_2^{(2)} z)$.

It has been asked whether the recursively enumerable sets are all of them diophantine sets (see [6], Chapter 7). A diophantine set of numbers x is the set of x's such that numbers $y_1, \ldots, y_n$ can be found such that a given diophantine equation in $x, y_1, \ldots, y_n$ is satisfied. I regret not having had the opportunity yet to study this question seriously.

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University of Oslo
Oslo, Norway