Notre Dame Journal of Formal Logic
Volume III, Number 1, January 1962

NOTES ON FOUNDATIONS

G. Y. RAINICH

Note II. On Galois Connections

We deal here with partially ordered sets.[1] In such a set a relation of order is defined which possesses the usual properties except that given two elements it is not necessarily true that one is higher than the other or that they are equally high. As an example we may consider the set of subsets of a given set; given two such subsets it may happen that one includes the other or that each includes the other but, in general neither of these relations is true. We say in a case like this that we have partial order "by inclusion". Although in a sense all cases of partial order may be reduced to this it is convenient to consider other situations independently. Examples of some other cases appear in what follows.

Given two partially ordered sets which are in correspondence it may be that their orderings are related to each other. A classical example occurs in the theory of algebraic equations.[2] As one set we may consider the set of rational functions of the roots of a polynomial—these functions are called natural irrationalities. One such natural irrationality may be called *higher* than another if the second is a rational function of the first but not vice versa. Under this definition the natural irrationalities constitute a partially ordered set. On the other hand, we may consider the groups of permutations of the roots of our polynomial. If in connection with every natural irrationality we consider the group of permutations of the roots under which the natural irrationality is invariant we have established a correspondence between the two partially ordered sets, the set of natural irrationalities and the set of groups of permutations ordered by inclusion. It is true that when a natural irrationality α is a rational function of β the permutations that do not affect β would not affect α so that when α is higher than β its group includes the group of β. There is then a connection between the partial orders of the two sets. This forms the basis of the Lagrange-Galois theory.

If there is a correspondence between two partially ordered sets such that whenever an element α of the first set is higher than an element β it is true that the corresponding elements of the second set are in the same order relation we say that the partially ordered sets are *isotaxic*; the term "Galois connected" is used with about the same meaning. The two sets considered above in connection with an algebraic equation are isotaxic.

Received December 19, 1961

A very simple example of two isotaxic sets is given by a monotone increasing sequence a_n of real numbers. The set of indices n and the set of numbers of the sequence are both ordered (by magnitude; order is a special case of partial order) and the fact that $a_m > a_n$ when $m > n$ means that the sets are isotaxic.

Another classical example of isotaxic sets is given by the so called Erlanger Program. One set here is the set of geometries (Euclidean, Projective, Affine, Equiaffine, Equiform, Inversive). A geometry may be called higher than another if every proposition of the first is a significant and true proposition of the second (it is enough to ascertain that every axiom of the second is provable in the first). On the other hand to every geometry (whose proposition or axioms form a categorical set) may be assigned a group, and it is true that (possibly after some adjustment of terminology) a higher geometry has a higher group. The set of geometries is then isotaxic with the set of groups.

As the last example we shall consider a definition of a Riemann integral (slightly generalized). Here we begin with an interval $I = (a, b)$ of real numbers. We consider the divisions of I into subintervals, such a division being brought about by inserting between a and b a (finite) number of dividing points. Of two such divisions α and β we consider β higher (or finer) if it is obtained from α by inserting additional dividing points. We consider next a real valued function f defined on I. Given a division of I we consider in connection with every subinterval i of I what we shall call a vertical interval, namely the interval whose endpoints are the lower and upper bounds of the set of values that f takes on i. We then form a sum Σ of products whose second factors are the lengths of the subintervals of the division and whose first factors are some numbers belonging to the corresponding vertical subinterval. There are many such sums corresponding to a given division because we have freedom to choose as the first factors any numbers in the vertical intervals.[3] We'll speak of the sum-sets corresponding to a division of I (we consider a fixed function f which we do not change during the whole discussion). We want to show now that if we have two divisions A and B and if B is finer than A then the sum-set corresponding to B is part of the sum-set corresponding to A.

To do this it is enough to show that every new sum corresponding to a division obtained by inserting *one* additional point (we consider the same function f throughout) will be equal to one of the old sums. Denoting the new dividing point by y and the next smaller and the next larger points of the old division by x and z, so that $x < y < z$ we have to compare the term $p(z - x)$ of the old sum with the sum of *two* terms

$$p_1 (y - x) + p_2 (z - y)$$

of the new sum. We must prove that given p_1 and p_2 we can find p so that the two expressions are equal. Obviously we must choose as the number p the number

$$p = p_1 \frac{y - x}{z - x} + p_2 \frac{z - y}{z - x}$$

We must prove that this number belongs to the vertical interval corresponding to the interval (x, z) if p_1 and p_2 belong to the vertical intervals corresponding to (x, y) and (y, z) respectively. If $p_1 \leq p_2$ we have

$$p_1 = p_1 \frac{y - x}{z - x} + p_1 \frac{z - y}{z - x} \leq p_1 \frac{y - x}{z - x} + p_2 \frac{z - y}{z - x}$$

$$\leq p_2 \frac{y - x}{z - x} + p_2 \frac{z - y}{z - x} = p_2$$

So that p is between p_1 and p_2 and we obtain the same result in a similar fashion when $p_1 \geq p_2$. Since p_1 and p_2 are in the vertical intervals corresponding to (x, y) and (y, z) the number p must be in the vertical interval corresponding to (x, z) because the endpoints of that interval are the smaller of the two lower bounds and the larger of the two upper bounds of the other two vertical intervals. It follows that every sum corresponding to a finer division is equal to one of the sums corresponding to the original division and this proves the isotaxy of the lattices.

Of course we may consider special cases and generalizations. In the direction of specialization, if all the sum-sets corresponding to all possible divisions have only one point in common we call this point or number the integral of f over I. For example, if f is continuous in I it follows without effort that the integral exists.

On the other hand, we may consider instead of an interval of real numbers a more general domain such as an arc or a surface (in the second case we will have dividing contours instead of dividing points), instead of numerical function—a function whose values are vectors or tensors, and instead of products of two numbers for instance products of two vectors, one of which is connected with the dividing contour (corresponding to the difference of the end-points of the subinterval) and the other is the value of a vector valued function defined on the domain. —We have thus as a special case a surface integral.

REFERENCES

[1] *Cf.* Garret Birkhoff, *Lattice Theory*, Amer. Math. Soc. 1948.

[2] *Ibidem.*, p. 56.

[3] This point of view according to which the sums Σ is a many valued function of the division seems to have been first indicated by D. Kryschanowsky, *J D M V*, vol. 22, 1913.

University of Notre Dame
Notre Dame, Indiana