A NOTE CONCERNING THE MANY-VALUED PROPOSITIONAL CALCULI

BOLESŁAW SOBOCIŃSKI

For the purpose of investigating many-valued propositional calculi the following simple theorem is often useful:

THEOREM: *Any implicational thesis of the bi-valued propositional calculus in which only one variable occurs is a thesis in a many-valued system* $\mathfrak{S}$ *provided the following conditions are satisfied:* (i) *implication is primitive or defined in* $\mathfrak{S}$; (ii) *the ordinary operations of substitution and detachment in respect of implication are valid in* $\mathfrak{S}$; (iii) *any implicational thesis of* $\mathfrak{S}$ *is a thesis of the bi-valued calculus, and* (iv) *the following theses hold in* $\mathfrak{S}$: *A1 Cpp; A2 CpCCqrCCpqr or A2* CCqrCpCCpqr; A3 CCrpCCqpCrCqp or A3* CCqpCCrpCrCqp; A4 CCprCCqpCCrpCqp or A4* CCqpCCprCCrpCqp; A5 CCprCCqpCqCCrpp or A5* CCqpCCprCqCCrpp.*

PROOF: Assume that **F** is an arbitrary implicational bi-valued thesis in which there occurs only one variable, say "p". Then the last consequent of **F** is equiform with "p" and this variable is preceded by n (for: $1 \leq n < \infty$) antecedents. Hence, **F** is either **TI**: $C\alpha_1 C\alpha_2 \ldots C\alpha_{n-1} C\alpha_n p$ or **TII**: $C\alpha p$. If **F** is **TI**, then there must exist such α_k (for: $1 \leq k \leq n$) that formula **G**: $C\alpha_k p$ which is shorter than **F** and is **TII** itself, is a thesis of the classical propositional calculus. Otherwise, **F** could not be a thesis of $\mathfrak{S}$. Consider now a formula α_j which is a part of **F** and is such that $j \neq k$ and $1 \leq j \leq n$. If α_j is false, then $C\alpha_j p$ is a thesis of the bi-valued propositional calculus, and together with **G** and *A3* (or *A3**) implies theses: *B1* $C\alpha_j C\alpha_k p$ and *B2* $C\alpha_k C\alpha_j p$. On the other hand, if α_j is a thesis, then its form is: $C\beta_1 C\beta_2 \ldots C\beta_{m-1} C\beta_m p$, for: $1 \leq m < \infty$. Since this is an implicational thesis, the following formula: $CpC\beta_1 C\beta_2 \ldots C\beta_{m-1}\beta_m$, to be referred to as *C1*, is of the same length, and is also a thesis of the bi-valued propositional calculus. In such a case *C1*, **G**, *A4* (or *A4**) and *A5* (or *A5**) imply *B1* and *B2* again. Therefore, elementary inductive reasoning shows that if **F** is **TI**, then it can always be obtained from *A3*, *A4*, *A5* (or *A3**, *A4**, *A5**), and shorter theses in which only one variable occurs. Consequently, any thesis of type **TI** can be reduced to shorter theses of type **TII**. But, any thesis of this type is either *A1* or has the following form: *D1* $CC\gamma_1 C\gamma_2 \ldots C\gamma_{r-1} C\gamma_r pp$, for: $1 \leq r < \infty$. Since *D1* is a thesis, then any of its components γ_j, for: $1 \leq j \leq r$, must also be a thesis of the bi-valued calculus. Hence *D1* follows from *A1*, *A2* (or *A2**) and the theses $\gamma_1, \gamma_2, \ldots \gamma_r$ each of which is shorter than *D1*. Since any thesis γ_j is of type **TI** or **TII**, an elementary proof by induction

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shows that our arbitrary thesis **F** can be obtained from *A1 - A5* (or *A1 - A5**). Thus, the theorem is proved.

I omit here the easy proof that theses *A1 - A5* (or *A1 - A5**) can be deduced from the following four mutually independent theses: *M1 CCqrCCpqCpr* (or *CCpqCCqrCpr*); *M2 CpCCpqq*; *M3 Cpp*; *M4 CpCpp*. This being the case the theorem holds for any system in which the positive logic is included. On the other hand there are systems which do not contain the positive logic and for which the theorem is valid, e.g. Łukasiewicz's many-valued systems (including the infinite-valued one) and a system which I have investigated.[1] The following matrix in which *1* and *3* are the designated values:

	C	0	1	2	3
	0	1	1	1	1
*	1	0	1	0	0
	2	0	1	3	0
*	3	0	1	2	3

shows that our theorem does not hold for Church's weak positive implicational calculus.[2] For the matrix verifies Church's axioms: *N1 CCpCpqCpq*; *N2 CCqrCCpqCpr*; *N3 CCpCqrCqCpr*; *N4 Cpp* but falsifies *M4* for $p = 2$ and *A3* (*A3**) for $p = q = r = 2$. Thus, Church's system does not contain all implicational theses in which only one variable occurs.

NOTES

1. *Cf.* B. Sobociński: Axiomatization of a partial system of three-valued calculus of propositions. *The Journal of Computing Systems*, vol. I (1952), pp. 23-55. For that system I proved a slightly stronger theorem, *cf.* Theorem II, pp. 37-38. The theorem given in the present paper holds not only for that my system but also for its implicative part investigated by Alan Rose. *Cf.* A. Rose: Le degré de saturation du calcul propositional implicatif à trois valeurs de Sobociński. *Comptes Rendus hebdomadaires des séances de l'Academie des Sciences* (Paris), vol. 235 (1952), pp. 1000-1002; A. Rose: A formalization of Sobociński's three-valued implicational propositional calculus. *The Journal of Computing Systems*, vol. I (1953), pp. 150-154; A. Rose: An alternative formalization of Sobociński's three-valued implicational propositional calculus. *Zeitschrift für mathematische Logic und Grundlagen der Mathematik* (Berlin), vol. 2 (1956), pp. 166-172.
2. *Cf.* A. Church: The weak positive implicational propositional calculus. *The Journal of Symbolic Logic*, vol. 16 (1951), p. 238, and A. Church: *The weak theory of implication*. In "Kontrolliertes Denken, Untersuchungen zum Logikkalkül und zur Logik der Einzelwissenschaften". Munich, 1951, pp. 22-37. Also *cf.* my remark concerning Church's system in *op. cit.*, p. 54.

University of Notre Dame
Notre Dame, Indiana