ON THE SINGLE AXIOMS OF PROTOTHETIC

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CHAPTER II*

In this chapter I propose to present the proofs relevant to the preceding discussion of the single axioms of system $\mathfrak{S}_5$ of protothetic. First of all I will prove 1) that thesis A_n mentioned in Chapter I, § 3, and also each of the theses A_o, A_p and A_q, can serve as a single axiom of $\mathfrak{S}_5$[54], and 2) that Leśniewski's metatheorem **L** discussed in § 3,[55] can be replaced by my metatheorem **S**,[56] whose conditions constitute a relatively small fragment of the original prerequisites set out by Leśniewski. In addition a number of questions closely connected with the topics mentioned above will also be discussed in what follows.

It is clear, that if metatheorem **S** is true, then in order to establish that this or that protothetical thesis in which only equivalence occurs as a constant term, can be adopted as a single axiom of $\mathfrak{S}_5$ one must be able to prove that relatively to the rule of procedure of $\mathfrak{S}_5$ the thesis under consideration satisfies the requirements of **S**. And it is evident too, that the truth of metatheorem **S** depends exclusively on whether or not it is possible, by applying the said rule, to deduce metatheorem **L** from the assumptions of **S**. Thus, from the methodological point of view it would appear that thesis A_n should be discussed after metatheorem **S** has been established. There are, however, serious reasons for adopting a reversed order of presentation. For it so happens that the proof concerning A_n, though long and complicated, is much more elementary than the deductions required for the proof of **S**. Hence, it seems to me that in the beginning it is better to show that A_n (and also A_o, A_p and A_q) implies the conditions of **S**. This will enable one who is not familiar with the methods of deduction in protothetic, to understand better subsequent proofs. Moreover, the proof of the main theorem, due to

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which metatheorem **S** holds and due to which almost all the results discussed here hold, is patterned on idea underlying my 1937 investigations of the mutual relations among theses *K1*, *K2*, *K3* and *K4*[57], although, it must be said, these investigations do not involve as complicated deductions as those which establish metatheorem **S**. For this reason, although the results obtained by me in 1937 have been, in a sense, superseded by metarule **S**, I will present them here because they will make the proof of the metarule more understandable.

The detailed plan of this chapter is as follows: In § 4 I shall prove that thesis A_n implies conditions $\mathfrak{a}$ and $\mathfrak{c}$ of metatheorem **L**. Hence, a fortiori it satisfies metatheorem **S** which (assuming the correctness of **S**) proves automatically that A_n can serve as a single axiom of $\mathfrak{S}_5$. In § 5 a proof will be outlined that the same holds for A_o, A_p and A_q. In § 6 I shall show that given the first four points of the rule of $\mathfrak{S}_5$ condition $\mathfrak{c}$ of **L** and a small fragment of theory $\mathfrak{S}$, viz. theses *F1* and *F2*,[58] imply the whole of $\mathfrak{S}$, i.e. condition $\mathfrak{a}$ of Leśniewski's metatheorem **L**. Incidentally, several new metatheorems concerning the completeness of $\mathfrak{S}_5$ will be established in paragraphs 5 and 6. In § 7 a proof will be presented that without the application of the point ϵ (concerning higher extensionalities) of the rule of procedure either thesis *K4* is a consequence of *K1*, *K2* and *K3* or *K3* follows from *K1*, *K2* and *K4*, providing that we have theory $\mathfrak{S}$, condition $\mathfrak{c}$ of **L** and the extensional form of the definition of conjunction.[59] In addition, axiom-system $\bar{S}$, mentioned in § 3,[60] will be discussed in this paragraph. Finally, in § 8 it will be demonstrated with the help of the point ϵ of rule of $\mathfrak{S}_5$ that condition $\mathfrak{b}$ of metatheorem **L** results from condition $\mathfrak{c}$ and theory $\mathfrak{S}$, which means that it follows from condition $\mathfrak{c}$ and theses *F1* and *F2* in virtue of deductions shown in § 6.

Thus, § 6 and § 8 will contain the proof of metatheorem **S**, § 7 will form an explanatory and historical introduction to § 8 and §§ 4-5 will contain deductions concerning theses $A_n - A_o$.

The rule of procedure of $\mathfrak{S}_5$ allows us to add a new thesis to the system on condition that this thesis results from one and only one of the points which constitute the rule.[61] Thus, e.g., if a thesis *A* is a consequence of two theses *B* and *C* (previously proved) in virtue of suitable substitutions in *B* and *C*, the distribution of the quantifier in *B* and the subsequent detachment, then, in fact, 4 new theses have to be inscribed in the roll of the system. I shall follow this prescription only in the first few steps in § 4 in order to show how it works. Later I shall use combined proof lines in which the application of points β, γ and δ of the rule will be indicated jointly. The applications of points α and ϵ will always be pointed out separately as the rule requires. Moreover in the course of deductions several metarules of procedure will be established and put to use. In this way about a thousand uninteresting steps will be omitted without affecting the rigorousness of the deductions.

Concerning the proof lines, mentioned above, it should be noted that: 1) each such line will be closed by parentheses of the form "[" and "]"; 2) An inscription, e.g., "$[B, p \mid q; C, r \mid s]$" indicates that in *B*, which is a thesis of the system, a meaningful formula q is substituted for the variable p; similarly in C, which again is a thesis of the system, s is substituted for r; then the required distributions of the quantifiers are made in B and C, and, finally, the operation of detachment is performed; 3) Whenever a distribution of the quantifier is made as a separate step in the course of deductions, it will be indicated by letter β. E.g., when this point of rule is applied to a result of detachment; 4) The proof line of the theses obtained in virtue of points α and ϵ have the form "[α]" and "[ϵ]"; 5) Symbols: *SI*, *SII*, etc. occurring in the proof lines indicate the application of derivative metarules of procedure, whose validity has been proved already.

Also, it should be noted that symbols "*0*" and "*1*" are not defined constants, but typographical abbreviations of "$[u] \,.\, u$" and "$[u] \,.\, u \,.\, \equiv \,.\, [u] \,.\, u$" respectively.[62] The use of these symbols allow us to present theses and proofs in a shorter, clearer and more understandable manner.

§4. As a single axiom of protothetic we assume:

A_n $\quad [p\ q] : : p \equiv q \,.\, \equiv .\!\cdot\!. [f] .\!\cdot\!. f\,(p\ f\,(p\ [u] \,.\, u)) \,.\, \equiv : [r] : f\,(q\ r) \,.\, \equiv \,.\, q \equiv p$

we then adjust to it the rule of procedure of system $\mathfrak{S}_5$.[63] In virtue of this rule, so adjusted, we deduce from A_n the following theses:

$D1$ $\quad [p] \,.\, p \equiv \mathrm{As}\,(p)$ [In virtue of point α of the rule]

$A1$ $\quad [p] : : p \equiv \mathrm{As}\,(p) \,.\, \equiv .\!\cdot\!. [f] .\!\cdot\!. f\,(p\ f\,(p\ [u] \,.\, u)) \,.\, \equiv : [r] : f\,(\mathrm{As}\,(p)\ r) \,.\, \equiv$
$\,.\, \mathrm{As}\,(p) \equiv p$ [A_n; in virtue of point δ of the rule]

$A2$ $\quad [p] \,.\, p \equiv \mathrm{As}\,(p) \,.\, \equiv .\!\cdot\!. [p\ f] .\!\cdot\!. f\,(p\ f\,(p\ [u] \,.\, u)) \,.\, \equiv : [r] : f\,(\mathrm{As}\,(p)\ r) \,.\, \equiv$
$\,.\, \mathrm{As}\,(p) \equiv p$ [$A1$; in virtue of point β of the rule]

$A3$ $\quad [p\ f] .\!\cdot\!. f\,(p\ f\,(p\ [u] \,.\, u)) \,.\, \equiv : [r] : f\,(\mathrm{As}\,(p)\ r) \,.\, \equiv \,.\, \mathrm{As}\,(p) \equiv p$
[$A1$; $D1$; in virtue of point γ of the rule]

$D2$ $\quad [p\ q] .\!\cdot\!. p \equiv q \,.\, \equiv \,.\, \mathrm{As}\,(p \equiv q) : \equiv \mathrm{Vr}\,(pq)$ [By α]

The proof of *D1-D2* shows clearly how points α, β, γ and δ of the rule are to be used if we wish to add to the system new theses. In what follows abridged proofs will be given involving steps made in virtue of a joint application of β, γ, and δ. Care will be taken that this does not lead to misunderstandings. It should be noted that *D1* is a definition of "assertium" for one propositional argument and *D2* is a definition of "verum" for two propositional arguments. Obviously, the defined constant functor "As" belongs to the semantical category of proposition–forming functors for one propositional argument. And, *D1* introduces this new category into the system. On the other hand no new category is introduced by *D2* since the defined constant "Vr" belongs to the same category as the variable "*f*" or the constant "≡" in A_n or *A3*, which are proposition–forming functors for two propositional arguments.

$A4$ $\quad [p\ q] \,.\, \mathrm{Vr}\,(p\ q)$ [By β, δ, γ : $D2$; $D1$, $p \mid p \equiv q$]

$A5$ $\quad [p\ r] : \mathrm{Vr}\,(\mathrm{As}\,(p)\ r) \,.\, \equiv \,.\, \mathrm{As}\,(p) \equiv p$ [$A3$, $f \mid \mathrm{Vr}$; $A4$, $q \mid \mathrm{Vr}\,(p\ [u] \,.\, u)$]

$A6$ $\quad [p] \,.\, \mathrm{As}\,(p) \equiv p$ [$A5$; $A4$, $p \mid \mathrm{As}\,(p)$, $q \mid r$]

Now, we are able to prove the following two metarules of procedure:

SI. If in the field of our system we have a thesis of the following structure:

$$[a, b, c, \ldots] : \Phi . \equiv . \Psi$$

(where either the main quantifier does not exist or a, b, c, . . . are the variables which belong to this quantifier and occur as free in the formula "$\Phi . \equiv . \Psi$"), *then we can always add to the system the following thesis*:

$$[a, b, c, \ldots] : \Psi . \equiv . \Phi$$

The proof is evident for anyone, who has understood how *A6* is obtained from *D1*.

Dem.:

$\mathfrak{a}$) $[a, b, c, \ldots] : \Phi . \equiv . \Psi$ [The assumption]

$\mathfrak{b}$) $[a', b', c', \ldots] : \Phi . \equiv . \Psi$ [From the point $\mathfrak{a}$; by means of substitution we change the variables occurring in the main quantifier of $\mathfrak{a}$, so that none of them remain equiform with any variable occurring in any of the quantifier in A_n.]

$\mathfrak{c}$) $[a', b', c', \ldots, f] : : f(\Phi\, f(\Phi\, [u] . u)) . \equiv .\!\cdot\!. [r] .\!\cdot\!. f(\Psi\, r) . \equiv : \Psi . \equiv . \Phi$ $[A_n, p \mid \Phi, q \mid \Psi;\ \mathfrak{b}]$

$\mathfrak{d}$) $[a', b', c', \ldots, r] .\!\cdot\!. \mathrm{Vr}\, (\Psi\, r) . \equiv : \Psi . \equiv . \Phi$ $[\mathfrak{c}, f \mid \mathrm{Vr};\ A4, p \mid \Phi, q \mid \mathrm{Vr}\, (\Phi\, [u] . u)]$

$\mathfrak{e}$) $[a', b', c', \ldots] : \Psi . \equiv . \Phi$ $[\mathfrak{d};\ A4, p \mid \Psi, q \mid r]$

$\mathfrak{f}$) $[a, b, c, \ldots] : \Psi . \equiv . \Phi$ [$\mathfrak{e}$; by the way of substitution we return to the same variables which occur in the main quantifier of $\mathfrak{a}$.]

Q. E. D.

SII. If in the field of the system we have two theses of the following structure:

$$[a, b, c, \ldots] : \Phi . \equiv . \Psi$$

and

$$[a, b, c, \ldots] . \Psi$$

(where a, b, c, . . . are variables which belong to the main quantifiers of these theses and occur either in "Φ" or in "Ψ" or in both), *then we can always add to the system the following thesis*:

$$[a, b, c, \ldots] . \Phi$$

The proof of *SII* is obvious in virtue of *SI* and the points β and γ of the rule. As a scheme of reasoning *SI* says that although we have not yet got the thesis:

$$[p\ q] : p \equiv q \,.\, \equiv .\ q \equiv p^{64}$$

nevertheless, we can always transform any thesis of the form "$[a, b, c, \ldots] : \Phi \,.\, \equiv .\ \Psi$" into a thesis of the form "$[a, b, c, \ldots] : \Psi \,.\, \equiv .\ \Phi$". Evidently, *SII* is a sort of detachment rule working from right to left.

We proceed by proving:

A7 $[p\ f] \mathbin{.\!\cdot\!.} f(\mathrm{As}(p)\, f(\mathrm{As}(p)\, [u] \,.\, u)) \,.\, \equiv : [r] : f(p\ r) \,.\, \equiv .\ p \equiv \mathrm{As}(p)$
[A_n, $p \mid \mathrm{As}(p)$, $q \mid p$; *A6*]

A8 $[p\ r] : \mathrm{Vr}(p\ r) \,.\, \equiv .\ p \equiv \mathrm{As}(p)$
[*A7*, $f \mid \mathrm{Vr}$; *A4*, $p \mid \mathrm{As}(p)$, $q \mid \mathrm{Vr}(\mathrm{As}(p)\, [u] \,.\, u)$]

D3 $[p\ q\ r] \mathbin{.\!\cdot\!.} \mathrm{Vr}(p\ r) \,.\, \equiv .\ p \equiv q : \equiv .\ \Phi\, \alpha \langle p \rangle (q\ r)$ [α[65]]

A9 $[p\ r] \,.\, \Phi\, \alpha \langle p \rangle (\mathrm{As}(p)\ r)$ [*D3*, $q \mid \mathrm{As}(p)$; *A8*]

A10 $[p\ r] : \Phi\, \alpha \langle p \rangle (p\ r) \,.\, \equiv .\ p \equiv \mathrm{As}(p)$
[*A7*, $f \mid \Phi\, \alpha \langle p \rangle$; *A9*, $r \mid \Phi\, \alpha \langle p \rangle (\mathrm{As}(p)\, [u] \,.\, u)$]

A11 $[p\ r] \,.\, \Phi\, \alpha \langle p \rangle (p\ r)$ [*A10*; *D1*; *SII*[66]]

A12 $[p\ r] : \mathrm{Vr}(p\ r) \,.\, \equiv .\ p \equiv p$ [*D3*, $q \mid p$; *A11*; *SII*]

A13 $[p] \,.\, p \equiv p$ [*A12*, $r \mid q$; *A4*]

Thus, the law of identity for equivalence follows from A_n.

A14 $[p\ f] \mathbin{.\!\cdot\!.} f(p\ f(p\ [u] \,.\, u)) \,.\, \equiv : [r] : f(p\ r) \,.\, \equiv .\ p \equiv p$ [A_n, $q \mid p$; *A13*]

Now, we are going to prove two metarules of procedure which make clear that although we have not yet got in the system the thesis:

$$[p\ q] \mathbin{.\!\cdot\!.} p \equiv q \,.\, \equiv : [f] : f(p) \,.\, \equiv .\ f(q)^{67}$$

or any other thesis which, at this stage of the system, could allow us to make the extensional deductions directly, we nevertheless can reason in accordance with the law of extensionality for any expression which is a proposition.

SIII. If two formulas α *and* β *belong to the semantical category of propositions and if in the field of our system we have two theses of the following structure*:

$$[a, b, c, \ldots] : \alpha \,.\, \equiv .\ \beta$$

and

$$[a, b, c, \ldots] \,.\, \Phi(\alpha)$$

(where "Φ" is a constant or a multi-link proposition–forming functor for one propositional argument, and where the variables a, b, c, ... may occur as free variables not only in "α", and "β" but also in "Φ"), *then we can always add to the system the following thesis*:

$$[a, b, c, \ldots] \,.\, \Phi(\beta)$$

Dem.:

$\mathfrak{a}$)	$[a, b, c, \ldots] : \alpha \,.\!\equiv.\, \beta$	[The assumption]
$\mathfrak{b}$)	$[a, b, c, \ldots] \,.\, \Phi(\alpha)$	[The assumption]
$\mathfrak{c}$)	$[a', b', c', \ldots] : \alpha \,.\!\equiv.\, \beta$	[$\mathfrak{a}$; as in the point $\mathfrak{b}$ of the proof of *SI*]
$\mathfrak{d}$)	$[a', b', c', \ldots] \,.\, \Phi(\alpha)$	[$\mathfrak{b}$; as in the point $\mathfrak{b}$ of the proof of *SI*]
$\mathfrak{e}$)	$[a', b', c', \ldots, f] :: f(\alpha\, f(\alpha\,[u]\,.\,u)) \,.\!\equiv\therefore [r] \therefore f(\beta\, r) \,.\!\equiv:\, \beta \,.\!\equiv.\, \alpha$	[A_n, $p \mid \alpha$, $q \mid \beta$; $\mathfrak{c}$]
$\mathfrak{f}$)	$[a', b', c', \ldots, p, q, r] :: \mathrm{Vr}\,(\Phi(p)\, r) \,.\!\equiv:\, \Phi(q) \,.\!\equiv.\, \Phi(p) \therefore\equiv.\, \Psi\alpha \leftarrow q \rightarrow (p\, r)$	[In virtue of point α of the rule. Thus, expression $\mathfrak{f}$ is a scheme of a definition. If any free variables occur in "Φ", "$\Psi\alpha$" is a multi-link functor for these variables. E.g., "$\Psi\alpha$" can have the following form "$\Psi\alpha \leftarrow a', b', c', \ldots \rightarrow$]
$\mathfrak{g}$)	$[a', b', c', \ldots, p, r] \,.\, \Psi\alpha \leftarrow p \rightarrow (p\, r)$	[$\mathfrak{f}$, $q \mid p$; *A12*, $p \mid \Phi(p)$]
$\mathfrak{h}$)	$[a', b', c', \ldots, r] \therefore \Psi\alpha \leftarrow \alpha \rightarrow (\beta\, r) \,.\!\equiv:\, \beta \,.\!\equiv.\, \alpha$	[$\mathfrak{e}$, $f \mid \Psi\alpha \leftarrow \alpha \rightarrow$; $\mathfrak{g}$, $p \mid \alpha$, $r \mid \Psi\alpha \leftarrow \alpha \rightarrow (\alpha\,[u]\,.u)$]
$\mathfrak{i}$)	$[a', b', c', \ldots] : \beta \,.\!\equiv.\, \alpha$	[$\mathfrak{c}$; *SI*]
$\mathfrak{j}$)	$[a', b', c', \ldots, r] \,.\, \Psi\alpha \leftarrow \alpha \rightarrow (\beta\, r)$	[$\mathfrak{h}$; $\mathfrak{i}$; *SII*]
$\mathfrak{k}$)	$[a', b', c', \ldots, r] \therefore \mathrm{Vr}\,(\Phi(\beta)\, r) \,.\!\equiv:\, \Phi(\alpha) \,.\!\equiv.\, \Phi(\beta)$	[$\mathfrak{f}$, $p \mid \beta$, $q \mid \alpha$; $\mathfrak{j}$; *SII*]
$\mathfrak{l}$)	$[a', b', c', \ldots] : \Phi(\alpha) \,.\!\equiv.\, \Phi(\beta)$	[$\mathfrak{k}$; *A4*, $p \mid \Phi(\beta)$, $q \mid r$]
$\mathfrak{m}$)	$[a', b', c', \ldots] \,.\, \Phi(\beta)$	[$\mathfrak{l}$; $\mathfrak{d}$]
$\mathfrak{n}$)	$[a, b, c, \ldots] \,.\, \Phi(\beta)$	[$\mathfrak{m}$; as in point $\mathfrak{f}$ of the proof of *SI*]

Q. E. D.

SIV. If under the same conditions as in SIII in the field of the system we have two theses of the following structures:

$$[a, b, c, \ldots] : \alpha \,.\!\equiv.\, \beta$$

and

$$[a, b, c, \ldots] \,.\, \Phi(\beta)$$

then we can always add to the system the following thesis:

$$[a, b, c, \ldots] \,.\, \Phi(\alpha)$$

The proof of *SIV* follows at once from *SI* and *SIII*.

It has to be noted that whenever we apply *SIII* (or *SIV*) we must introduce into our system two auxiliary definitions in order to perform the required deductions. We need one definition in order to transform the thesis with respect to which we want to apply extensional reasoning into a thesis of the form similar to our assumption $\mathfrak{b}$, i.e. into a thesis which is formed by means of a proposition-forming functor for one propositional argument. The other definition is required in order to perform point $\mathfrak{f}$ in the proof of *SIII*. It is not difficult to construct such definitions for any given case and I shall omit them when using *SIII* or *SIV*. Only, for the purpose of illustration will I give the definitions when *SIII* is used for the first time. Namely:

$A15 \quad [p\ q] \therefore p \equiv q . \equiv . \mathrm{As}\,(p \equiv q) : \equiv . p \equiv p \qquad [A12, r \mid q; D2; SIII]$

In order to give a complete proof of *A15*, we must introduce a definition:

$D\,\mathfrak{A} \quad [p\ q\ r] :: p \equiv q . \equiv . \mathrm{As}\,(p \equiv q) : \equiv r \therefore \equiv . \chi\,\alpha \langle p\ q \rangle\,(r) \qquad [\alpha]$

From $D\,\mathfrak{A}$ and *D2* we get:

$I \quad [p\ q] . \chi\,\alpha \langle p\ q \rangle\,(\mathrm{Vr}\,(p\ q)) \qquad [D\,\mathfrak{A}, r \mid \mathrm{Vr}\,(p\ q); D2]$

Thesis I corresponds to our assumption $\mathfrak{b}$ and thesis $A12\,(r \mid q)$ to $-\ \mathfrak{a}$. Then, we can easily prove the theses corresponding to $\mathfrak{c} - \mathfrak{e}$. Next we introduce the second definition corresponding to point $\mathfrak{f}$:

$D\,\mathfrak{B} \quad [p\ q\ r\ s\ t] :: \mathrm{Vr}\,(\chi\,\alpha \langle s\ t \rangle\,(p)\ r) . \equiv : \chi\,\alpha \langle s\ t \rangle\,(p) . \equiv . \chi\,\alpha \langle s\ t \rangle$
$(q) \therefore \equiv . \chi\,\beta \langle\!\langle s\ t \rangle\!\rangle \langle q \rangle\,(p\ r) \qquad [\alpha]$

and subsequently we get the theses corresponding to points $\mathfrak{g} - \mathfrak{n}$, i.e. we finally obtain a thesis:

$II \quad [p\ q] . \chi\,\alpha \langle p\ q \rangle\,(p \equiv p)$

which in virtue of $D\,\mathfrak{A}\,(r \mid p \equiv p)$ and *SII* gives *A15*.

$A16 \quad [p\ q] \therefore p \equiv q . \equiv . p \equiv q : \equiv . p \equiv p \qquad [A15; A6, p \mid p \equiv q; SIII]$

$A17 \quad [p\ q] \therefore p \equiv p . \equiv : p \equiv q . \equiv . p \equiv q \qquad [A16; SI]$

$D4 \quad [p\ q] \therefore p \equiv p . \equiv . q \equiv q : \equiv . \mathrm{Vr}_1\,(q\ p) \qquad [\alpha$ [68]$]$

Now, in order to make the formulas clearer and easier to read I introduce the following two abbreviations: "*0*" will be used for "$[u] . u$" and "*1*" for "$[u] . u . \equiv . [u] . u$". We should always remember that these symbols ("*0*" and "*1*") are used here only as typographical abbreviations; they are not introduced into the system by means of definitions although definitions:[69]

$$DI \quad [u] . u . \equiv . 0$$

and

$$DII \quad [u] . u . \equiv . [u] . u : \equiv . 1$$

could be added to the system in accordance with the rule of procedure. Hence, e.g., the real form of *A18*, to be proved next, is:

$$\mathrm{Vr}_1\,([u] . u, [u] . u . \equiv . [u] . u)$$

$A18 \quad \mathrm{Vr}_1\,(0\ 1)$
$[D4, p \mid [u] . u . \equiv . [u] . u, q \mid [u] . u; A16, p \mid [u] . u, q \mid [u] . u]$

$A19$ $1 . \equiv . \mathrm{Vr}_1 (0\ 0)$ [$A17$, $p \mid 0$, $q \mid 0$; $D4$, $p \mid 0$, $q \mid 0$; $SIII$]

$A20$ $\mathrm{Vr}_1 (0\ \mathrm{Vr}_1 (0\ 0))$ [$A18$; $A19$; $SIII$]

$A21$ $[r] : \mathrm{Vr}_1 (0\ r) . \equiv . 0 \equiv 0$ [$A14$, $p \mid 0$, $f \mid \mathrm{Vr}_1$; $A20$]

$A22$ $[r] . \mathrm{Vr}_1 (0\ r)$ [$A21$; $A13$; $p \mid 0$; SII]

$A23$ $[p] : p \equiv p . \equiv . 1$ [$D4$, $q \mid 0$; $A22$, $r \mid p$; SII]

$A24$ $[p] : 1 . \equiv . p \equiv p$ [$A23$; SI]

$A25$ $1 \equiv 1 . \equiv . 1 \equiv 0 : \equiv : 1 \equiv . 1 \equiv 0$
[$A13$, $p \mid 1 \equiv . 1 \equiv 0$; $A23$, $p \mid 1$; SIV]

$A26$ $[p] :: 0 \equiv . 0 \equiv 1 : \equiv .\therefore 0 \equiv : 0 \equiv . p \equiv p$
[$A13$, $p \mid 0 \equiv . 0 \equiv 1$; $A24$; $SIII$]

$A27$ $[p] :\cdot: 0 \equiv . 0 \equiv 1 : \equiv 1 .\therefore \equiv :: 0 \equiv : 0 \equiv . p \equiv p .\therefore \equiv 1$
[$A13$, $p \mid 0 \equiv . 0 \equiv 1 : \equiv 1$; $A26$; $SIII$]

$D5$ $[p\ q] :: p \equiv : p \equiv . q \equiv q .\therefore \equiv . \mathrm{Vr}_2 (p\ q)$ [α]

$A28$ $[p] .\therefore 0 \equiv . 0 \equiv 1 : \equiv \mathrm{Vr}_2 (0\ p)$ [$A26$; $D5$, $p \mid 0$, $q \mid p$; SIV]

$A29$ $[r] :: 0 \equiv . 0 \equiv 1 : \equiv 1 .\therefore \equiv : \mathrm{Vr}_2 (0\ r) . \equiv . 1$
[$A27$, $p \mid r$; $D5$, $p \mid 0$, $q \mid r$; $SIII$]

$A30$ $0 \equiv . 0 \equiv 1 : \equiv 1 .\therefore \equiv : [r] : \mathrm{Vr}_2 (0\ r) . \equiv . 1$ [$A29$; β]

The proof of $A30$ deserves our attention, because $A30$ follows from $A29$ by the distribution of the quantifier. Since we do not know any other way of obtaining $A30$ in this stage and since it is a necessary step to get $A32$ and consequently the important $A37$, it is clear that the point β of the rule plays an essential role in the deductions presented in this paragraph.

$A31$ $0 \equiv . 0 \equiv 1 : \equiv : [r] : \mathrm{Vr}_2 (0\ r) . \equiv . 1$
[$A14$, $p \mid 0$, $f \mid \mathrm{Vr}_2$; $A28$, $p \mid \mathrm{Vr}_2 (0\ 0)$; SIV]

$A32$ $0 \equiv . 0 \equiv 1 : \equiv .\therefore 0 \equiv . 0 \equiv 1 : \equiv 1$ [$A31$; $A30$; SIV]

$D6$ $[p\ q] .\therefore p \equiv . p \equiv q : \equiv . \Phi\beta (q\ p)$ [α]

$A33$ $\Phi\beta (1\ 0) . \equiv : \Phi\beta (1\ 0) . \equiv 1$ [$A32$; $D6$, $p \mid 0$, $q \mid 1$; $SIII$]

$A34$ $\Phi\beta (1\ \Phi\beta (1\ 0))$ [$D6$, $p \mid \Phi\beta (1\ 0)$, $q \mid 1$; $A33$]

$A35$ $[r] : \Phi\beta (1\ r) . \equiv . 1 \equiv 1$ [$A14$, $p \mid 1$, $f \mid \Phi\beta$; $A34$]

$A36$ $[r] . \Phi\beta (1\ r)$ [$A35$; $A13$, $p \mid 1$; SII]

$A37$ $[p] : p \equiv . p \equiv 1$ [$D6$, $q \mid 1$; $A36$, $r \mid p$; SII]

$A38$ $[p] : p \equiv 1 . \equiv p$ [$A37$; SI]

$A39$ $[p\ q] .\therefore p \equiv : p \equiv . q \equiv q$ [$A37$; $A24$, $p \mid q$; $SIII$]

$A40$ $[p\ q] .\therefore p \equiv q . \equiv : p \equiv 1 . \equiv q$ [$A13$, $p \mid p \equiv q$; $A37$; $SIII$]

$A41$ $[p\ q] .\therefore p \equiv q . \equiv : p \equiv 1 . \equiv . q \equiv 1$ [$A40$; $A37$, $p \mid q$; $SIII$]

A42 $[p] \therefore p \equiv . p \equiv 1 : \equiv 1$ [*A23*; *A37*; *SIII*]

A43 $[p] \therefore 1 \equiv : p \equiv . p \equiv 1$ [*A42*; *SI*]

A44 $[p] \therefore 1 \equiv . 1 \equiv 1 : \equiv : p \equiv . p \equiv 1$ [*A43*, $p \mid 1$; *A43*; *SIII*]

Now, although we have not yet got the thesis:

$[f] :: f([u] . u) . \equiv \therefore f([u] . u . \equiv . [u] . u) . \equiv : [p] : f([u] . u) . \equiv . f(p)$[70]

or any other thesis which, at this stage of the system, could allow us to make the deductions of generalization directly, we are in a position to prove a scheme of reasoning which says that in the field of the system we can always reason in accordance with the principle of bivalence for propositions. Viz: we prove the following metarule of procedure:

SV. If in the field of the system a formula

$$[a, b, c, \ldots, p] . \Phi(p)$$

possesses a sense, i.e. if it is a well formed formula, in which "p" belongs to the semantical category of propositions, and if the variables a, b, c . . . are free in "Φ" which is a simple or a multi-link functor belonging to the category of proposition-forming functors for one propositional argument, *and, finally, if the following two formulas*

$$[a, b, c, \ldots] . \Phi([u] . u)$$

and

$$[a, b, c, \ldots] . \Phi([u] . u . \equiv . [u] . u)$$

are already proved in the system, then we can always add to it as a new thesis the formula

$$[a, b, c, \ldots, p] . \Phi(p)$$

Dem.:

𝔞) $[a, b, c, \ldots] . \Phi(0)$ [The assumption]

𝔟) $[a, b, c, \ldots] . \Phi(1)$ [The assumption]

𝔠) $[a', b', c', \ldots] . \Phi(0)$ [𝔞, as in point 𝔟 of the proof of *SI*]

𝔡) $[a', b', c', \ldots] . \Phi(1)$ [𝔟, as in point 𝔟 of the proof of *SI*]

𝔢) $[a', b', c', \ldots, p] : \Phi(0) . \equiv . p \equiv p$ [*A39*, $p \mid \Phi(0)$, $q \mid p$; 𝔠]

𝔣) $[a', b', c', \ldots, p] : \Phi(1) . \equiv . p \equiv p$ [*A39*, $p \mid \Phi(1)$, $q \mid p$; 𝔡]

𝔤) $[a', b', c', \ldots, p, q] \therefore \Phi(q) . \equiv . p \equiv p : \equiv . \Psi \beta(p\ q)$
[In virtue of point α of the rule, as in point 𝔣 of the proof of *SIII*]

𝔥) $[a', b', c', \ldots, p] . \Psi \beta(p\ 0)$ [𝔤, $q \mid 0$; 𝔢]

𝔦) $[a', b', c', \ldots, p] . \Psi \beta(p\ 1)$ [𝔤, $q \mid 1$; 𝔣]

$\mathfrak{j})$ $[a', b', c', \ldots, p] : \Psi\, \beta\, (p\ 0) \,.\equiv.\, 1$ $[A37, p \mid \Psi\, \beta\, (p\ 0);\ \mathfrak{h}]$

$\mathfrak{k})$ $[a', b', c', \ldots, p] \,.\, \Psi\, \beta\, (p\ \Psi\, \beta\, (p\ [u] \,.\, u))$ $[\mathfrak{i};\ \mathfrak{j};\ SIV]$

$\mathfrak{l})$ $[a', b', c', \ldots, p, r] : \Psi\, \beta\, (p\ r) \,.\equiv.\, p \equiv p$ $[A14, f \mid \Psi\, \beta;\ \mathfrak{k}]$

$\mathfrak{m})$ $[a', b', c', \ldots, p, r] \,.\, \Psi\, \beta\, (p\ r)$ $[\mathfrak{l};\ A13;\ SII]$

$\mathfrak{n})$ $[a', b', c', \ldots, p] : \Phi\, (p) \,.\equiv.\, p \equiv p$ $[\mathfrak{g}, q \mid p; \mathfrak{m}, r \mid p;\ SII]$

$\mathfrak{o})$ $[a', b', c', \ldots, p] \,.\, \Phi\, (p)$ $[\mathfrak{n};\ A13;\ SII]$

$\mathfrak{p})$ $[a, b, c, \ldots, p] \,.\, \Phi\, (p)$ [$\mathfrak{o}$; as in point $\mathfrak{f}$ of the proof of *SI*]

Q.E.D.

The proof of *SV* shows that A_n satisfies the condition $\mathfrak{c}$ of the metatheorems **L** and **S**. Just as in the case of applying *SIII* (or *SIV*) whenever we use *SV*, we must introduce into our system two auxiliary definitions. Since the construction of such definitions is not difficult, I shall omit them using *SV*. Only for purposes of illustration will I give the definitions, when *SV* is used for the first time. Viz.:

A45 $[p] \,.\cdot.\, 1 \equiv p \,.\equiv:\, 1 \equiv 0 \,.\equiv.\, p \equiv 0$
$[A17, p \mid 1, q \mid 0;\ A39, p \mid 1 \equiv 0, q \mid 0;\ SV]$

In order to have a complete proof of *A45*, we must introduce the following definition:

$D\, \mathfrak{C}$ $[p] :: 1 \equiv p \,.\equiv:\, 1 \equiv 0 \,.\equiv.\, p \equiv 0 \,.\cdot.\equiv.\, X\, \gamma\, (p)$ $[\alpha^{71}]$

From D $\mathfrak{C}$ and *A39* we get:

III $X\, \gamma\, (0)$ $[D\, \mathfrak{C}, p \mid 0;\ A39, p \mid 1 \equiv 0, q \mid 0]$

and from $D\, \mathfrak{C}$ and *A17*:

IV $X\, \gamma\, (1)$ $[D\, \mathfrak{C}, p \mid 1;\ A17, p \mid 1, q \mid 0]$

Theses *III* and *IV* correspond to our assumptions $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Then an easy proof gives theses corresponding to points $\mathfrak{c}$ - $\mathfrak{f}$. Next, we introduce a second definition according to point $\mathfrak{g}$:

$D\, \mathfrak{D}$ $[p\ q] \,.\cdot.\, X\, \gamma\, (q) \,.\equiv.\, p \equiv p :\equiv.\, X\ \delta\, (p\ q)$ $[\alpha]$

and, subsequently, we get the theses corresponding to points $\mathfrak{h}$ - $\mathfrak{p}$, i.e., finally, we obtain

V $[p] \,.\, X\, \gamma\, (p)$

which in virtue of $D\, \mathfrak{C}$ and *SII* gives *A45*.

Our deductions proceed as follows:

A46 $[p] :: 1 \equiv .\, 1 \equiv p :\equiv.\cdot.\, 1 \equiv 0 \,.\equiv:\, 1 \equiv 0 \,.\equiv p$
$[A45, p \mid 1 \equiv 0;\ A44, p \mid 1 \equiv 0;\ SV]$

A47 $1 \equiv :\, 1 \equiv .\, 1 \equiv 0 \,.\cdot.\equiv.\, 1 \equiv 0$
$[A46, p \mid 1 \equiv 0;\ A39, p \mid 1 \equiv 0, q \mid 1 \equiv 0;\ SIV]$

A48 $[p] \,.\cdot.\, 1 \equiv p \,.\equiv.\, 1 \equiv 0 :\equiv:\, 1 \equiv .\, p \equiv 0$ $[A25;\ A16, p \mid 1, q \mid 0;\ SV]$

A49 $1 \equiv \therefore 1 \equiv . 1 \equiv 0 : \equiv 0$ [$A48, p \mid 1 \equiv . 1 \equiv 0; A47$]

A50 $1 \equiv . 1 \equiv 0 : \equiv 0$ [$A49; A13, p \mid 0$]

A51 $[q] \therefore 1 \equiv . 1 \equiv q : \equiv q$ [$A50; A42, p \mid 1; SV$]

A52 $[q] \therefore 0 \equiv . 0 \equiv q : \equiv q$ [$A38, p \mid 0; A42, p \mid 0; SV$]

A53 $[p\ q] \therefore p \equiv . p \equiv q : \equiv q$ [$A52; A51; SV^{72}$]

A54 $[p\ q] \therefore q \equiv : p \equiv . p \equiv q$ [$A53; SI$]

A55 $[p\ q] \therefore q \equiv 1 . \equiv : p \equiv . p \equiv q$ [$A54; A37, p \mid q; SIII$]

A56 $[q] \therefore 0 \equiv q . \equiv : 1 \equiv . q \equiv 0$ [$A55, p \mid 1, q \mid 0; A37, p \mid 0 \equiv 0; SV$]

A57 $[p\ q] \therefore p \equiv q . \equiv : p \equiv 0 . \equiv . q \equiv 0$ [$A56; A45, p \mid q; SV$]

A58 $[p\ q\ r] \therefore p \equiv q . \equiv : p \equiv r . \equiv . q \equiv r$ [$A57; A41; SV$]

A59 $[p\ q\ r] : : p \equiv q . \equiv \therefore p \equiv . q \equiv r : \equiv r$
[$A58, r \mid q \equiv r; A53, p \mid q, q \mid r; SIII$]

A60 $[p\ q] : : p \equiv \therefore p \equiv . 1 \equiv q : \equiv q$ [$A59, q \mid 1, r \mid q; A38; SIII$]

A61 $[p\ q] : : p \equiv . 1 \equiv q : \equiv q \therefore \equiv p$ [$A60; SI$]

A62 $[p\ q] :\cdot: p \equiv : : p \equiv . 1 \equiv q : \equiv 1 \therefore \equiv q$ [$A60; A37, p \mid p \equiv . 1 \equiv q; SIII$]

A63 $[p\ q] \vdots\vdots\ p \equiv :\cdot: p \equiv . 1 \equiv q : \equiv 1 \therefore \equiv 1 : : \equiv q$
[$A60; A37, p \mid p \equiv . 1 \equiv q : \equiv 1; SIII$]

A64 $[p] :\cdot: p \equiv . 1 \equiv p : \equiv :\cdot: p \equiv . 1 \equiv p : \equiv 1 \therefore \equiv 1 : : \equiv p \vdots\vdots \equiv :\cdot: p \equiv . 1 \equiv p : \equiv 1 \therefore \equiv 1 : : \equiv p$ [$A61, q \mid p; A63, q \mid p; SIII$]

D7 $[p\ q] :\cdot: 0 \equiv . p \equiv 0 : \equiv . q \equiv 0 \therefore \equiv . q \equiv 0 : : \equiv . \Phi\, \gamma\, (p\ q)$ [α]

A65 $0 \equiv . 1 \equiv 0 : \equiv . \Phi\, \gamma\, (1\ 0) \equiv 0 \therefore \equiv . \Phi\, \gamma\, (1\ 0) \equiv 0$
[$A64, p \mid 0; D7, p \mid 1, q \mid 0; SIII$]

A66 $\Phi\, \gamma\, (1\ \Phi\, \gamma\, (1\ 0))$ [$D7, p \mid 1, q \mid \Phi\, \gamma\, (1\ 0); A65$]

A67 $[r] : \Phi\, \gamma\, (1\ r) . \equiv . 1 \equiv 1$ [$A14, p \mid 1, f \mid \Phi\, \gamma; A66$]

A68 $[q] . \Phi\, \gamma\, (1\ q)$ [$A67, r \mid q; A13, p \mid 1; SII$]

A69 $[q] : : 0 \equiv . 1 \equiv 0 : \equiv . q \equiv 0 \therefore \equiv . q \equiv 0$ [$D7, p \mid 1; A68; SII$]

A70 $0 \equiv . 1 \equiv 0 : \equiv . 0 \equiv 0$ [$A69, q \mid 0; A13; p \mid 0; SII$]

A71 $0 \equiv . 1 \equiv 0$ [$A70; A13, p \mid 0; SII$]

A71 **constitutes a crucial point in this proof, since it gives:**

A72 $[p] : p \equiv . 1 \equiv p$ [$A71; A37, p \mid 1; SV$]

A73 $[p] : 1 \equiv p . \equiv . p \equiv 1$ [$A37; A72; SIII$]

A74 $[p] : p \equiv 1 . \equiv . 1 \equiv p$ [$A73; SI$]

A75 $[p] : p \equiv 0 . \equiv . 0 \equiv p$ [$A13, p \mid 0 \equiv 0; A73, p \mid 0; SV$]

$A76 \quad [p\ q] : p \equiv q . \equiv . q \equiv p$ [$A75$; $A74$; SV]

Thus, we have proved that equivalence is symmetrical, and that as from now we can dispense with SI and SII.

$A77 \quad [p\ q\ r] \therefore p \equiv q . \equiv : q \equiv r . \equiv . p \equiv r$
[$A58$; $A76$, $p \mid p \equiv r$, $q \mid q \equiv r$; $SIII$]

$A78 \quad [p\ q\ r] \therefore p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$ [$A77$; $A76$, $p \mid q$, $q \mid r$; $SIII$]

Since $A78$ is Łukasiewicz's axiom $Ł1$ of the classical equivalential propositional calculus and since the rules of procedure of that theory are, obviously, included into the rule of $\mathfrak{S}_5$, I have proved that from A_n system $\mathfrak{S}$ can be obtained, which satisfies condition $\mathfrak{a}$ of metatheorem $\mathbf{L}$. Condition $\mathfrak{c}$ of that metatheorem is also satisfied by A_n as can be seen from metarule SV. Since condition $\mathfrak{b}$ of $\mathbf{L}$ is superfluous as will be shown in § 8 of this chapter, we have a proof that A_n can serve as a single axiom of system $\mathfrak{S}_5$ of protothetic. It is worth noting that although we did not use point ϵ of the rule of procedure so far, we were able to obtain very strong deductive results. On the other hand I am unable to prove that A_n (or A_o or A_p or A_q) is a single axiom of $\mathfrak{S}_5$ without the application of reasonings, which will be discussed in § 8 and in which point ϵ plays an essential role.

Evidently $F1$ and $F2$ (appearing in the metarule $\mathbf{S}$) can be derived from $A78$ alone by substitution and detachment. $F1$ has been proved already as it results by substitution from $A13$ ($p \mid [u] . u$) and $F2$ is obtainable easily as follows:

$A79 \quad [p\ q] \therefore p \equiv : q \equiv p . \equiv q$ [$A54$, $p \mid q$, $q \mid p$; $A76$, $p \mid q$, $q \mid q \equiv p$; $SIII$]

§ 5. A simple inspection of the deductions presented in the preceding paragraph will convince us that each of the theses:

$A_o \quad [p\ q] :: p \equiv q . \equiv \therefore [f] \therefore f(q\ f(q\ [u] . u)) . \equiv : [r] : f(p\ r) . \equiv . q \equiv p$

$A_p \quad [p\ q] :: p \equiv q . \equiv \therefore [f] \therefore f(p\ f(q\ [u] . u)) . \equiv : [r] : f(q\ r) . \equiv . q \equiv p$

$A_q \quad [p\ q] :: p \equiv q . \equiv \therefore [f] \therefore f(q\ f(p\ [u] . u)) . \equiv : [r] : f(p\ r) . \equiv . q \equiv p$

can serve as a single axiom of system $\mathfrak{S}_5$. It is evident that the theses $A1$ - $A14$ and the metarules SI and SII can be obtained from each of the discussed theses in exactly the same way as they were obtained from A_n. An entirely analogous proof of $SIII$ exist for A_p. In the case of A_o and A_q a little modification has to be introduced in order to get this metarule. Viz., a scheme of a definition used in point $\mathfrak{f}$ has to have now the following form:

$\mathfrak{f}$ *) $[a', b', c', \ldots, p, q, r] :: \mathrm{Vr}(\Phi(q)\ r) . \equiv : \Phi(p) . \equiv . \Phi(q) \therefore \equiv .$
$\Psi^* \ \alpha \leftarrow q \rightarrow (p\ r)$

from which and points $\mathfrak{a}$ - $\mathfrak{e}$ the analogues of points $\mathfrak{g}$ - $\mathfrak{n}$ can be obtained without any difficulty. Now, the metarules SIV and SV and the theses $A15$ - $A78$ are provable in each of these cases in exactly the same way as in § 4, since no direct application of axiom is used in order to obtain them. Hence each of the discussed theses can serve as a single axiom of system $\mathfrak{S}_5$.

A more penetrating analysis of the demonstrations given in § 4 allows us to establish the following metatheorem **S***:

METATHEOREM **S***: An axiom-system of protothetic having the rules of procedure inferentially equivalent to the rule of $\mathfrak{S}_5$ yields a complete system, if in its field the following conditions are satisfied:

I. Thesis *A14* is provable.
II. Metarules *SI* and *SIII* are provable.

In fact in virtue of the rule of $\mathfrak{S}_5$ alone we have *D1*, *D2*, *A4*; and *SII* and *SIV* follow from *SI* and *SIII*, hence, as an inspection of § 4 shows, one can obtain *A12*, *A13*, *SV* and *A15* - *A78* using only the conditions of **S***. Therefore, having *A78* and *SV*, we get the required conditions of **S**. It seems to me that without the application of point ϵ (concerning higher extensionalities) it is impossible to prove A_n (or A_o, A_p, A_q) from the conditions of **S***. But it is easy to deduce axiom A_l mentioned in § 3.[73]

Dem.: Obviously, in virtue of **S*** we have *A77* and *A78*. And due to Łukasiewicz's proof, the following two theses can be obtained from *A78*.

M1 $[p\ q\ r\ s\ t]$:·: $s \equiv . t \equiv t : \equiv : : s \equiv .\!\cdot\!.\ p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$

M2 $[p\ q]$.·. $p \equiv : p \equiv q . \equiv q$

Therefore, we have:

M3 $[f\ p\ q]$:·: $[r] : f(p\ r) . \equiv . p \equiv p : \equiv : : [r] : : f(p\ r) . \equiv .\!\cdot\!.\ p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$ [$M1$, $s \mid f(p\ r)$, $t \mid p$; and the application of point β of the rule in respect to r]

M4 $[f\ p\ t] : : [r] : f(p\ r) . \equiv . p \equiv p : \equiv t .\!\cdot\!. \equiv : f(p\ f(p\ [u] . u)) . \equiv t$
[$A77$, $p \mid f(p\ f(p\ [u] . u))$, $q \mid [r] : f(p\ r) . \equiv . p \equiv p$, $r \mid t$; β; $A14$, in virtue of **S***]

M5 $[f\ p\ q]$:·: $f(p\ f(\ p\ [u] . u)) . \equiv : : [r] : : f(p\ r) . \equiv .\!\cdot\!.\ p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$
[$M4$, $t \mid [r] : : f(p\ r) . \equiv .\!\cdot\!.\ p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$; β; $M3$]

A_l $[f\ p\ q]$:·: $f(p\ f(p\ [u] . u)) . \equiv : : [r] : : f(p \equiv q . \equiv q\ r) . \equiv .\!\cdot\!.\ p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$ [$M5$; $M2$; $SIII$, in virtue of **S***]

Q.E.D.

On the other hand there is no simple way of obtaining the conditions of **S*** from A_l alone. Indeed, using elementary deductions we can get *A78* from it at once as, in virtue of the rule, we have at our disposal *D1*, *D2* and *A4*. Therefore, we have also *SI* (by *A78*) and we can prove easily:

N1 $[f\ p\ q] : f(p\ f(p\ [u] . u))$.·$\equiv . [r] . f(p \equiv q . \equiv q\ r)$

But, *A14* and *SIII* cannot be deduced directly from *A78* and *N1*, as the last thesis gives only an extremely narrow possibility of extensional deductions. In order to obtain the discussed thesis *A14* and metarule *SIII* from those assumptions we have to use metatheorem **S**, i.e. we have to prove previously that its condition ¢ (*SV*) follows from *A78* and *N1*. And this can be done only by an application of point ϵ of the rule and the reasonings which *mutatis mutandis* are similar to those to be presented in § 8 of this chapter. I will return to this question in chapter III of the present paper.

To conclude these considerations concerning the structure of A_n I would like to make the following historical remark. The methods of deductions due to which *SI* and *SV* were obtained in the preceding paragraph had been established by Leśniewski and applied to axioms A_a - A_m. On the other hand the proofs by which *A13*, *A37*, *A71* and metarule *SIII* are obtained from A_n in § 4 were previously unknown. The "decompositions" of A_a - A_m are much more simple than the decomposition of A_n and do not involve so complicated deductions.

NOTES

54. *Cf.* [35], p. 67.

55. *Cf.* [35], p. 63.

56. *Cf.* [35], pp. 67-68.

57. *Cf.* [35], pp. 65-66.

58. Concerning this rule *cf.* [35], § 2, pp. 56-63. Also, *cf.* [5], pp. 59-78.

59. It is well known, *cf.*, e.g., [31], pp. 1-8 and [20], pp. 97-100, that Tarski has established that it is possible to define conjunction in terms of equivalence provided one is allowed to use quantifiers and variable functors. I.e., such definitions can be established in a system of the calculus of propositions in which variables of higher semantical categories are allowed, and in which quantifiers are used with appropriate rules to bind variables of any category. It is also known that definitions suggested by Tarski are of two different types.

The following propositions can serve as example of the one type of definition:

1 $[p\ q] :.: [f] : : p \equiv .\therefore [r] : p \equiv . f(r) : \equiv : [r] : q \equiv . f(r) : : \equiv . p . q$

2 $[p\ q] :.: [f] : : q \equiv .\therefore [r] : p \equiv . f(r) : \equiv : [r] : q \equiv . f(r) : : \equiv . p . q$

In this type of definitions the arguments of definiendum ("p" and "q" in the examples given above) do not occur as arguments of the variable functors in the definiens (we have "$f(r)$" in the definiens but not "$f(p)$" and "$f(q)$").

The other type of definitions can be examplified with the aid of the following propositions:

3 $[p\ q] : : [f] .\therefore p \equiv : f(p) . \equiv . f(q) .\therefore \equiv . p . q$

4 $[p\ q] : : [f] .\therefore q \equiv : f(p) . \equiv . f(q) .\therefore \equiv . p . q$

In *3* and *4* the arguments of the definiendum (i.e. "p" and "q" in *3* and *4*) occur as arguments of the variable functor in the definiens (in the formula *3* and *4* we have "$f(p)$" and "$f(q)$").

Only in the field of a theory of propositions which is enriched by the addition of the law of extensionality for propositions we are able to give a proof that both types of these definitions are inferentially equivalent. *Cf.* [28], [29], [31], pp. 1-8 and [20], pp. 97-100. It is quite obvious that in the field of full systems of protothetic, such as $\mathfrak{S}_1$-$\mathfrak{S}_5$, this equivalence holds.

Definitions of both types admit of modifications. Thus, for instance, we have Tarski's *thesis:*

5 $[p\ q] :\cdot: [f] :: p \equiv .\therefore [r] . f(r) . \equiv : [r] : q \equiv . f(r) :: \equiv . p . q$

cf. [6], p. 27, which belongs to the first type. On the other hand, the formulas established by Leśniewski:

6 $[p\ q] :: [f] .\therefore p \equiv : f(q) . \equiv . f(1) .\therefore \equiv . p . q$

7 $[p\ q] :: [f] .\therefore q \equiv : f(p) . \equiv . f(1) .\therefore \equiv . p . q$

cf. [6], p. 24, and the definitions, suggested by myself:

8 $[p\ q] .\therefore [f] : f(p\ q) . \equiv . f(1 1) : \equiv . p . q$

9 $[p\ q] .\therefore [f] : f(p\ 1) . \equiv . f(1\ q) : \equiv . p . q$

10 $[p\ q] .\therefore [f] : f(1\ p) . \equiv . f(q\ 1) : \equiv . p . q$

cf. [22] and [23], belong to the second class. In § 8 of this paper I shall use a definition of conjunction by equivalence of the second type but much more complicated than the formulas *3* or *4*. The results discussed in § 7 hold for such definitions of the second kind which possess strictly "extensional" form, as the theses *3* and *4* or their suitable modifications. Namely, in order to obtain the deductions presented in § 7 we can adopt an arbitrary definition of conjunction by equivalence, but this definition must satisfy the following additional condition: Both arguments of the definiendum must occur as the *n* - th arguments (for *n*: *1*, *2*, *3*, . . .) of the variable functor in the definiens. This additional condition is satisfied by the formulas *3* and *4*, but not by the theses *6*-*10*.

The meaning of symbol "*1*" which is used in the formulas *6*-*10* is explained at the end of this introduction.

60. *Cf.* [35], pp. 65-66.

61. *Cf.* [35], p. 57, [5], p. 76 and [7]. Leśniewski observed this requirement consequently in [7].

62. *Cf.* [35], p. 63 and p. 70, note 39.

63. A formulation of the rule of procedure of $\mathfrak{S}_5$ requires to adjust this rule to each particular axiom–system of $\mathfrak{S}_5$. There exists a method using which we can make such adjustment automatically. *Cf.* [5], pp. 59-76, especially p. 63, *T. E. I* - *T. E. IV*.

64. This theorem will be proved later, viz. as the thesis *A76*.

65. In this and the subsequent definitions I am using the symbols "$\Phi\,\alpha$", "$\Phi\,\beta$", . . . "$\Psi\,\alpha$". "$\Psi\,\beta$" a. s. o. when the defined terms possess no important logical meanings. I.e., when the suitable definitions have purely auxiliary character.

Obviously, an expression "$\Phi\,\alpha \,\langle p \rangle$" occurring in the definiendum of *D3* belongs to the same semantical category as "*f*" in *A7*, and, therefore, we can substitute it for this variable in order to obtain *A10*.

66. Evidently, instead of *SII* we can use *SI*, but in such a case the deductions would be longer.

67. *Cf.* [35], p. 61, formula *Z14*.

68. In this and the subsequent definitions I am using symbols "Vr_1", "Vr_2" etc. in order to indicate that the defined terms are really the same as "Vr", introduced by *D2*. At this stage we are not yet able to show that these terms are the same. This will be possible at later stage, when the system is sufficiently developed. *Cf.* the use of such symbols in [7], e.g. *D9*, p. 126 and *D10*, p. 130.

69. The formulas *DI* and *DII* belong to the type of definitions which in Leśniewski's system are called the absolute protothetical definitions (in protothetic). *Cf.* [35], p. 69, note 22.

70. In connection with this thesis *cf.* [35], p. 66, thesis *SI*. Obviously, if we have at our disposal *SI*, then the rule of $\mathfrak{S}_5$ allows us to prove that these theses are inferentially equivalent.

71. Obviously, the symbols "*0*" and "*1*" which occur in the definiens of this definition are only typographical abbreviations. *Cf.* the introduction to the chapter II of this paper.

72. The application of *SV* in this case requires, obviously, an auxiliary definition which must be multi-link, viz.:

$$[p\ q] : : p \equiv . \ p \equiv q : \equiv q \ .\because \equiv . \ \chi\,\delta \,\langle q \rangle\, (p)$$

Similarly, in the proof of *A57* a. s. o.

73. *Cf.* [35], p. 66 (where due to a typographical error the axioms A_k and A_l and, also, the formula *Z18* are numbered A_f, A_1 and *Z15* respectively).

BIBLIOGRAPHY

Bibliography [1] - [34] is given at the end of the first part of this paper. See *Notre Dame Journal of Formal Logic*, vol. I (1960), pp. 71-73. It can now be supplemented by:

[35] Bolesław Sobociński: On the single axioms of protothetic, I. *Notre Dame Journal of Formal Logic,* vol. I (1960), pp. 52-73.

To be continued.

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