STUDIES IN THE AXIOMATIC FOUNDATIONS OF BOOLEAN ALGEBRA

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Section IV

In Section II we made use of the rule for writing propositional definitions in order to define singular inclusion in terms of weak inclusion within the framework of $\mathfrak{S}$ (or $\mathfrak{A}^*$).[11] Our definition had the form of the following expression:

D6. $[a\ b] :.: a \varepsilon b . \equiv : : [\exists c] . \sim (a \subset c) . a \subset b : : [c\ d] .\therefore c \subset a . \supset : a \subset c . \vee . c \subset d$

It is obvious that *D6* can also serve as a definition of singular inclusion within the framework of $\mathfrak{A}$.

One might expect, on purely intuitive grounds, that the familiar proposition

$D_0D1.$ $[a\ b] .\therefore a \subset b . \equiv : [c] : c \varepsilon a . \supset . c \varepsilon b$

could in turn be used as a definition of weak inclusion in terms of singular inclusion, and that the functor of singular inclusion could be employed as a primitive constant term in a system of Boolean Algebra with definitions. Interestingly enough D_0D1 does not seem to be derivable within $\mathfrak{A}$ unless we strengthen

A1. $[a\ b] .\therefore a \subset b . \equiv : [c\ d\ e] : \sim (c \subset d) . c \subset e . c \subset a . \supset . [\exists f\ g] . \sim (f \subset g) . f \subset e . f \subset b$

by subjoining to it

A1.1 $[a\ b] :.: \sim (a \subset b) . \supset : : [\exists c\ d] : : \sim (c \subset d) . c \subset a : : [e\ f] .\therefore e \subset c . \supset : c \subset e . \vee . e \subset f$

Within the framework of $\mathfrak{A}$ proposition *A1.1* appears to be independent of proposition *A.1.* This statement, however, will have to be regarded as a conjecture until an interpretation is found which satisfies *A1* and the rules of $\mathfrak{A}$, including the rule for writing nominal definitions, but fails to satisfy *A1.1.*[12]

Received January 21, 1961

Since we do not know whether *A1.1* can be derived from *A1* within the framework of $\mathfrak{A}$, we cannot regard *A1* as an adequate axiom for the purpose of ontological interpretation. For from the point of view of this interpretation *A1.1*, which in virtue of *D1* and *D6* is inferentially equivalent to the proposition

$$[a] : \mathrm{ex}\,(a) \,.\supset.\, [\exists b] \,.\, b \,\varepsilon\, a$$

raises no intuitive objections, and, consequently, ought to be included among the theses available in the system.

The totality of theses derivable from *A1* and *A1.1* in virtue of the rules R1 – R5 will be described as *Basic Ontology* with nominal definitions. In what follows two systems of Basic Ontology with nominal definitions will be outlined and shown to be inferentially equivalent. The one to be referred to as System $\mathfrak{A}_0$ is based on a single axiom, which takes the form of the following proposition:

A_o1. $[a\, b] \because a \subset b \,.\equiv\therefore [c] \therefore c \subset a :: [d\, e] \therefore d \subset c \,.\supset: c \subset d \,.\vee.\, d \subset e :: \supset .\, c \subset b$

R1 – R5 serve as rules of inference in $\mathfrak{A}_0$. The other system will be referred to as System $\mathfrak{D}_0$. Its single axiom says that

D_o1. $[a\, b] :: a \,\varepsilon\, b \,.\equiv\therefore [\exists c] \,.\, c \,\varepsilon\, a \,.\, c \,\varepsilon\, b \therefore [c\, d] : c \,\varepsilon\, a \,.\, d \,\varepsilon\, a \,.\supset.\, c \,\varepsilon\, d$

Among the rules of $\mathfrak{D}_0$ we have R1 – R4, and instead of R5 we have DR5. This is a rule for writing nominal definitions, which allows us to add to the system new theses of the form

XIV $[a\ldots] : a \,\varepsilon\, x \,.\equiv.\, [\exists b] \,.\, a \,\varepsilon\, b \,.\, \phi\,(b)$

provided '$a \,\varepsilon\, x$' and '$\phi\,(b)$' in these theses satisfy certain conditions analogous to those postulated by R5.[13]

Before we proceed to establish inferential equivalence between $\mathfrak{A}_0$ and $\mathfrak{D}_0$ we have to convince ourselves that A_o1 is inferentially equivalent to *A1* and *A1.1* taken together.

Let us, therefore, assume

$H1 = A1.$

$H2 = A1.1$

and let us proceed with the deductions as follows:

$H3$. $[a\, b\, f\, g\, h] \because [c] \therefore c \subset a :: [d\, e] \therefore d \subset c \,.\supset: c \subset d \,.\vee.\, d \subset e :: \supset .\, c \subset b \because \sim(f \subset g) \,.\, f \subset h \,.\, f \subset a \because \supset .\, [\exists i\, j] \,.\sim(i \subset j) \,.\, i \subset h \,.\, i \subset b$

Proof:

$[a\, b\, f\, g\, h] \because$

(1) $[c] \therefore c \subset a :: [d\, e] \therefore d \subset c \,.\supset: c \subset d \,.\vee.\, d \subset e :: \supset .\, c \subset b \because$

(2) $\sim(f \subset g)\,.$

(3) $f \subset h\,.$

(4) $f \subset a \vdots\vdots \supset :\cdot:$
$[\exists\, i\, j] : :$
(5) $\sim (i \subset j) .$
(6) $i \subset f : :$ [H2, 2]
(7) $[k\, l] \therefore k \subset i . \supset : i \subset k . \vee . k \subset l : :$
(8) $i \subset b .$ [S2, 6, 3]
(9) $i \subset a .$ [S2, 6, 4]
(10) $i \subset b :\cdot:$ [1, 9, 7]
$[\exists\, i\, j] . \sim (i \subset j) . i \subset b . i \subset b$ [5, 8, 10]

H4. $[a\, b] \vdots\vdots [c] :\cdot: c \subset a : : [d\, e] \therefore d \subset c . \supset : c \subset d . \vee . d \subset e : : \supset . c \subset b :\cdot: \supset . a \subset b$

Proof:

$[a\, b] \vdots\vdots$
(1) $[c] :\cdot: c \subset a : : [d\, e] \therefore d \subset c . \supset : c \subset d . \vee . d \subset e : : \supset . c \subset b :\cdot: \supset \therefore$
(2) $[c\, d\, e] : \sim (c \subset d) . c \subset e . c \subset a . \supset . [\exists\, f\, g] . \sim (f \subset g) . f \subset e . f \subset b \therefore$ [H3, 1]
$a \subset b$ [H1, 2]

$H5 = A_o 1.$ [S2, H4]

Now, let us assume

$J1 = A_o 1.$

From this assumption we derive

J2. $[a] . a \subset a$ [$A_o 1$]

J3. $[a\, b\, c] : a \subset b . c \subset a . \supset . c \subset b$ [$A_o 1$]

J4. $[a\, b\, c\, d\, e] : a \subset b . \sim (c \subset d) . c \subset e . c \subset a . \supset . [\exists\, f\, g] . \sim (f \subset g) . f \subset e . f \subset b$

Proof:

$[a\, b\, c\, d\, e] \therefore$
(1) $a \subset b .$
(2) $\sim (c \subset d) .$
(3) $c \subset e .$
(4) $c \subset a . \supset :$
(5) $c \subset b :$ [J3, 1, 4]
$[\exists\, f\, g] . \sim (f \subset g) . f \subset e . f \subset b$ [2, 3, 5]

J5. $[a\, b\, h] :\cdot: [c\, d\, e] : \sim (c \subset d) . c \subset e . c \subset a . \supset . [\exists\, f\, g] . \sim (f \subset g) . f \subset e . f \subset b \therefore h \subset a : : [c\, d] \therefore c \subset h . \supset : h \subset c . \vee . c \subset d : : \sim (h \subset b) : : \supset . h \subset b$

Proof:

$[a\, b\, h] :\cdot:$
(1) $[c\, d\, e] : \sim (c \subset d) . c \subset e . c \subset a . \supset . [\exists\, f\, g] . \sim (f \subset g) . f \subset e . f \subset b \therefore$
(2) $h \subset a : :$

(3) $[c\ d] .\therefore c \subset b . \supset : b \subset c . \vee . c \subset d ::$
(4) $\sim (b \subset b) :: \supset :$
$[\exists\ f\ g] .$
(5) $\sim (f \subset g) .$
(6) $f \subset b .$ — [1, 4, *J2*, 2]
(7) $f \subset b .$
(8) $b \subset f :$ [3, 6, 5]
$b \subset b$ [*J3*, 7, 8]

J6. $[a\ b] .\therefore [c\ d\ e] : \sim (c \subset d) . c \subset e . c \subset a . \supset . [\exists\ f\ g] . \sim (f \subset g) . f \subset e . f \subset b : \supset . a \subset b$

Proof:

$[a\ b] ::$
(1) $[c\ d\ e] : \sim (c \subset d) . c \subset e . c \subset a . \supset . [\exists\ f\ g] . \sim (f \subset g) . f \subset e . f \subset b .\therefore \supset ::$
(2) $[c] :.: c \subset a :: [d\ e] .\therefore d \subset c . \supset : c \subset d . \vee . d \subset e :: \supset . c \subset b ::$ [*J5*, 1]
$a \subset b$ [*J1*, 2]

J7 = *A1.* [*J4*, *J6*]

J8 = *A2.* [*J1*]

These deductions complete the proof that $A_o 1$ is inferentially equivalent to the set of axioms consisting of *A1* and *A1.1.*

In order to establish inferential equivalence between System $\mathfrak{A}_0$ and System $\mathfrak{D}_0$ we shall have to derive $D_o 1$ and $D_o D1$ within the framework of $\mathfrak{A}_0$. We shall also have to prove that any thesis that could be added to $\mathfrak{D}_0$ by applying DR5, could be obtained in $\mathfrak{A}_0$. Then, assuming $D_o 1$ and $D_o D1$, we shall have to deduce $A_o 1$ and *D6*. In addition we shall have to prove that in $\mathfrak{D}_0$ we could derive any thesis that could be added to $\mathfrak{A}_0$ in virtue of R5.

Since $A_o 1$ implies *A1*, and since the rules of inference in $\mathfrak{A}$ and in $\mathfrak{A}_0$ are the same, we can assume that all the theses derived so far in $\mathfrak{A}$ have been derived in $\mathfrak{A}_0$. Our deductions within the framework of $\mathfrak{A}_0$ continue as follows:

T67. $[a\ b\ c] : a \subset b . c \varepsilon a . \supset . c \varepsilon b$

Proof:

$[a\ b\ c] :.:$
(1) $a \subset b .$
(2) $c \varepsilon a . \supset ::$
(3) $[\exists\ d] . \sim (c \subset d) :$
(4) $c \subset a ::$ — [*D6*, 2]
(5) $[d\ e] .\therefore d \subset c . \supset : c \subset d . \vee . d \subset e ::$
(6) $c \subset b ::$ [$A_o 1$, 1, 4, 5]
$c \varepsilon b$ [*D6*, 3, 6, 5]

T68. $[a\ b\ d] :.: [c] : c \varepsilon a . \supset . c \varepsilon b .\therefore d \subset a :: [e\ f] .\therefore e \subset d . \supset : d \subset e . \vee . e \subset f :: \sim (d \subset b) :: \supset . d \subset b$

Proof:

$[a\ b\ d]\ \because\!\!:$
(1) $[c] : c \mathrel{\varepsilon} a \,.\supset.\, c \mathrel{\varepsilon} b \,\therefore$
(2) $d \subset a ::$
(3) $[e\ f] \therefore e \subset d \,.\supset:\, d \subset e \,.\vee.\, e \subset f ::$
(4) $\sim(d \subset b) ::\supset .$
(5) $d \mathrel{\varepsilon} a .$ $[D6, 4, 2, 3]$
(6) $d \mathrel{\varepsilon} b .$ $[1, 5]$
$d \subset b$ $[D6, 6]$

T69. $[a\ b] \therefore [c] : c \mathrel{\varepsilon} a \,.\supset.\, c \mathrel{\varepsilon} b :\supset .\, a \subset b$

Proof:

$[a\ b]\ \vdots\!\cdot\!\vdots$
(1) $[c] : c \mathrel{\varepsilon} a \,.\supset.\, c \mathrel{\varepsilon} b :\supset \vdots\vdots$
(2) $[c] :\cdot: c \subset a :: [d\ e] \therefore d \subset c \,.\supset:\, c \subset d \,.\vee.\, d \subset e ::\supset .\, c \subset b \vdots\vdots$ $[T68, 1]$
$a \subset b$ $[A_o1, 2]$

$T70 = D_oD1.$ $[a\ b] \therefore a \subset b \,.\equiv:\, [c] : c \mathrel{\varepsilon} a \,.\supset.\, c \mathrel{\varepsilon} b$ $[T69, T67]$

T71. $[a] .\, a \subset a$ $[A_o1]$

T72. $[a\ b] : a \mathrel{\varepsilon} b \,.\supset.\, a \mathrel{\varepsilon} a$ $[D6, T71]$

T73. $[a\ b] : a \mathrel{\varepsilon} b \,.\supset.\, [\exists\ c] .\, c \mathrel{\varepsilon} a \,.\, c \mathrel{\varepsilon} b$ $[T72]$

T74. $[a\ b\ c] : a \mathrel{\varepsilon} b \,.\, c \mathrel{\varepsilon} a \,.\supset.\, a \mathrel{\varepsilon} c$

Proof:

$[a\ b\ c]\ \because\!\!:$
(1) $a \mathrel{\varepsilon} b .$
(2) $c \mathrel{\varepsilon} a \,.\supset ::$
(3) $[\exists\ d] .\sim(a \subset d) ::$
(4) $[d\ e] \therefore d \subset a \,.\supset:\, a \subset d \,.\vee.\, d \subset e ::$ } $[D6, 1]$
(5) $[\exists\ d] .\sim(c \subset d) :$
(6) $c \subset a .$ } $[D6, 2]$
(7) $a \subset c :$ $[4, 6, 5]$
$a \mathrel{\varepsilon} c$ $[D6, 3, 7, 4]$

T75. $[a\ b\ c\ d] : a \mathrel{\varepsilon} b \,.\, c \mathrel{\varepsilon} a \,.\, d \mathrel{\varepsilon} a \,.\supset.\, c \mathrel{\varepsilon} d$

Proof:

$[a\ b\ c\ d] :$
(1) $a \mathrel{\varepsilon} b .$
(2) $c \mathrel{\varepsilon} a .$
(3) $d \mathrel{\varepsilon} a \,.\supset .$
(4) $a \mathrel{\varepsilon} d .$ $[T74, 1, 3]$
(5) $a \subset d .$ $[D6, 4]$
$c \mathrel{\varepsilon} d$ $[T67, 5, 2]$

T76. $[a\ e\ f] :: [c\ d] : c \mathrel{\varepsilon} a \,.\, d \mathrel{\varepsilon} a \,.\supset.\, c \mathrel{\varepsilon} d \therefore e \subset a \,.\sim(a \subset e) .\sim(e \subset f) \therefore\supset:\, a \subset e \,.\vee.\, e \subset f$

Proof:

$[a\ e\ f] \mathrel{⁝\cdot⁝}$

(1) $[c\ d] : c \mathrel{\varepsilon} a \,.\, d \mathrel{\varepsilon} a \,.\supset .\, c \mathrel{\varepsilon} d \mathrel{.\cdot.}$

(2) $e \subset a \,.$

(3) $\sim (a \subset e) \,.$

(4) $\sim (e \subset f) \mathrel{.\cdot.} \supset \mathrel{⁝\cdot⁝}$

$\quad [\exists\ g] :\, :$

(5) $\qquad g \subset a :\, :$

(6) $\qquad [h\ i] \mathrel{.\cdot.} h \subset g \,.\supset : g \subset h \,.\vee .\, h \subset i :\, :$ [$A_o 1$, 3]

(7) $\qquad \sim (g \subset e) \,.$

(8) $\qquad g \mathrel{\varepsilon} a \mathrel{⁝⁝}$ [$D6$, 7, 5, 6]

$\qquad [\exists\ h] \mathrel{:\cdot:}$

(9) $\qquad\quad h \subset e :\, :$

(10) $\qquad\quad [i\ j] \mathrel{.\cdot.} i \subset h \,.\supset : h \subset i \,.\vee .\, i \subset j :\, :$ [$A_o 1$, 4]

(11) $\qquad\quad \sim (h \subset f) \,.$

(12) $\qquad\quad h \mathrel{\varepsilon} e \,.$ [$D6$, 11, 9, 10]

(13) $\qquad\quad h \mathrel{\varepsilon} a \,.$ [$T67$, 2, 12]

(14) $\qquad\quad g \mathrel{\varepsilon} h \mathrel{⁝⁝}$ [1, 8, 13]

(15) $\qquad g \mathrel{\varepsilon} e \,.$ [$T67$, 9, 14]

(16) $\qquad g \subset e \mathrel{⁝\cdot⁝}$ [$D6$, 15]

$\quad a \subset e \,.\vee .\, e \subset f$ [7, 16]

T77. $[c\ a] : c \mathrel{\varepsilon} a \,.\supset .\, [\exists\ d] \,.\sim (a \subset d)$

Proof:

$[c\ a] \mathrel{.\cdot.}$

(1) $c \mathrel{\varepsilon} a \,.\supset :$

$\quad [\exists\ d] \,.$

(2) $\qquad \sim (c \subset d) \,.$ [$D6$, 1]

(3) $\qquad \sim (c \mathrel{\varepsilon} d) \,.$ [$D6$, 2]

(4) $\qquad \sim (a \subset d) :$ [$T67$, 1, 3]

$\quad [\exists\ d] \,.\sim (a \subset d)$ [4]

T78. $[a\ b\ c] :\, : c \mathrel{\varepsilon} a \,.\, c \mathrel{\varepsilon} b \mathrel{.\cdot.} [d\ e] : d \mathrel{\varepsilon} a \,.\, e \mathrel{\varepsilon} a \,.\supset .\, d \mathrel{\varepsilon} e \mathrel{.\cdot.} \supset .\, a \mathrel{\varepsilon} b$

Proof:

$[a\ b\ c] \mathrel{:\cdot:}$

(1) $c \mathrel{\varepsilon} a \,.$

(2) $c \mathrel{\varepsilon} b \mathrel{.\cdot.}$

(3) $[d\ e] : d \mathrel{\varepsilon} a \,.\, e \mathrel{\varepsilon} a \,.\supset .\, d \mathrel{\varepsilon} e \mathrel{.\cdot.} \supset :\, :$

(4) $[\exists\ d] \,.\sim (a \subset d) :\, :$ [$T77$, 1]

(5) $[d\ e] \mathrel{.\cdot.} d \subset a \,.\supset : a \subset d \,.\vee .\, d \subset e :\, :$ [$T76$, 3]

(6) $a \mathrel{\varepsilon} a \,.$ [$D6$, 4, $T71$, 5]

(7) $a \mathrel{\varepsilon} c \,.$ [3, 6, 1]

(8) $c \subset b$ [$D6$, 2]

$\quad a \mathrel{\varepsilon} b$ [$T67$, 8, 7]

T79 $= D_o 1.$ $[a\ b] :\, : a \mathrel{\varepsilon} b \,.\equiv \mathrel{.\cdot.} [\exists\ c] \,.\, c \mathrel{\varepsilon} a \,.\, c \mathrel{\varepsilon} b \mathrel{.\cdot.} [c\ d] : c \mathrel{\varepsilon} a \,.\, d \mathrel{\varepsilon} a \,.$
$\supset .\, c \mathrel{\varepsilon} d$ [$T73$, $T75$, $T78$]

T80. $[a\ b\ \phi] \vdots\vdots\ [b] :: b \subset a . \equiv .\therefore [c\ d] .\therefore \sim (c \subset d) . c \subset b . \supset : [\exists\ e\ f] :\sim (e \subset f) . e \subset c : [\exists\ g] . e \subset g . \phi (g) :\cdot: b\ \varepsilon\ a :\cdot: \supset . [\exists\ c] . b\ \varepsilon\ c . \phi (c)$

Proof:

$[a\ b\ \phi] \vdots\vdots$
(1) $[b] :: b \subset a . \equiv .\therefore [c\ d] .\therefore \sim (c \subset d) . c \subset b . \supset : [\exists\ e\ f] : \sim (e \subset f) . e \subset c : [\exists\ g] . e \subset g . \phi (g) :\cdot:$
(2) $b\ \varepsilon\ a :\cdot: \supset :$
(3) $[\exists\ c] . \sim (b \subset c) :$ [D6, 2]
(4) $b \subset a .$
(5) $b\ \varepsilon\ b :$ [T72, 2]
$[\exists\ c\ d\ e] .$
(6) $\sim (c \subset d) .$
(7) $c \subset b .$ [1, 4, 3, T71]
(8) $c \subset e .$
(9) $\phi (e) .$
(10) $b \subset c .$ [D6, 2, 7, 6]
(11) $b \subset e .$ [S2, 10, 8]
(12) $b\ \varepsilon\ e :$ [T67, 11, 5]
$[\exists\ c] . b\ \varepsilon\ c . \phi (c)$ [12, 9]

T81. $[c\ d\ b\ i\ \phi] .\therefore b\ \varepsilon\ i . \phi (i) . \sim (c \subset d) . c \subset b . \supset : [\exists\ e\ f] : \sim (e \subset f) . e \subset c : [\exists\ g] . e \subset g . \phi (g)$

Proof:

$[c\ d\ b\ i\ \phi] ::$
(1) $b\ \varepsilon\ i .$
(2) $\phi (i) .$
(3) $\sim (c \subset d) .$
(4) $c \subset b . \supset .\therefore$
(5) $b \subset i .$ [D6, 1]
(6) $c \subset i .\therefore$ [S2, 4, 5]
$[\exists\ e\ f] : \sim (e \subset f) . e \subset c : [\exists\ g] . e \subset g . \phi (g)$ [3, T71, 6, 2]

T82. $[a\ b\ \phi] \vdots\vdots\ [b] :: b \subset a . \equiv .\therefore [c\ d] .\therefore \sim (c \subset d) . c \subset b . \supset : [\exists\ e\ f] . \sim (e \subset f) . e \subset c : [\exists\ g] . e \subset g . \phi (g) :\cdot: b\ \varepsilon\ i . \phi (i) :\cdot: \supset . b\ \varepsilon\ a$

Proof:

$[a\ b\ \phi] \vdots\vdots$
(1) $[b] :: b \subset a . \equiv .\therefore [c\ d] .\therefore \sim (c \subset d) . c \subset b . \supset : [\exists\ e\ f] : \sim (e \subset f) . e \subset c : [\exists\ g] . e \subset g . \phi (g) :\cdot:$
(2) $b\ \varepsilon\ i .$
(3) $\phi (i) :\cdot: \supset ::$
(4) $[c\ d] .\therefore \sim (c \subset d) . c \subset b . \supset : [\exists\ e\ f] : \sim (e \subset f) . e \subset c : [\exists\ g] . e \subset g . \phi (g) ::$ [T81, 2, 3]
(5) $b \subset a .$ [1, 4]
(6) $b\ \varepsilon\ b ::$ [T71, 2]
$b\ \varepsilon\ a$ [T67, 5, 6]

T83. $[a\ \phi] \mathbin{\vdots\vdots} [b] :: b \subset a . \equiv .\therefore [c\ d] .\therefore \sim (c \subset d) . c \subset b . \supset : [\exists\ e\ f] : \sim (e \subset f) . e \subset c : [\exists\ g] . e \subset g . \phi(g) :: \supset : [b] : b \ \varepsilon\ a . \equiv . [\exists\ c] . b\ \varepsilon\ c . \phi(c)$ [*T80*, *T82*]

It is evident from *T79* and *T70* that $D_o 1$ and $D_o D1$ are derivable within the framework of $\mathfrak{A}_0$. Moreover, *T83* shows that any thesis that could be obtained in $\mathfrak{D}_0$ by applying DR5 can be obtained in $\mathfrak{A}_0$ with the aid of R5. In order to complete our proof that $\mathfrak{A}_0$ and $\mathfrak{D}_0$ are inferentially equivalent, we assume $D_o 1$, $D_o D1$, and the rules of $\mathfrak{D}_0$, and we proceed to show that from these assumptions we can derive $A_o 1$ and *D6*. In addition we derive a thesis which makes it evident that any thesis obtainable in $\mathfrak{A}_0$ in virtue of R5, can be obtained in $\mathfrak{D}_0$.

*T83*D$_o$D2.* $[a] : a\ \varepsilon\ \wedge . \equiv . [\exists\ b] . a\ \varepsilon\ b . b\ \varepsilon\ b . \sim (b\ \varepsilon\ b)$ [by applying DR5]

*T83*1.* $[a] . \sim (a\ \varepsilon \wedge)$ [*T83*D$_o$D2*]

*T83*2.* $[a\ b] : a\ \varepsilon\ b . \supset . a\ \varepsilon\ a$ [$D_o 1 = T79$]

*T83*3.* $[a\ b] : a\ \varepsilon\ b . \supset . [\exists\ c] . \sim (a \subset c)$

Proof:

$[a\ b] .\therefore$
(1) $a\ \varepsilon\ b . \supset :$
(2) $a\ \varepsilon\ a .$ [*T83*2*, 1]
(3) $\sim (a \subset \wedge) :$ [$D_o D1 = T70$, 2, *T83* 1*]
$[\exists\ c] . \sim (a \subset c)$ [3]

*T83*4.* $[a\ b\ c] : a\ \varepsilon\ b . c\ \varepsilon\ a . \supset . c\ \varepsilon\ b$

Proof:

$[a\ b\ c] ::$
(1) $a\ \varepsilon\ b .$
(2) $c\ \varepsilon\ a . \supset .\therefore$
(3) $a\ \varepsilon\ a .$ [*T83*2*, 1]
(4) $a\ \varepsilon\ c .\therefore$ [$D_o 1 = T79$, 1, 3, 2]
(5) $[d\ e] : d\ \varepsilon\ c . e\ \varepsilon\ c . \supset . d\ \varepsilon\ e .\therefore$ [$D_o 1 = T79$, 2]
$c\ \varepsilon\ b$ [$D_o 1 = T79$, 4, 1, 5]

*T83*5.* $[a\ b] : a\ \varepsilon\ b . \supset . a \subset b$

Proof:

$[a\ b] ::$
(1) $a\ \varepsilon\ b . \supset .\therefore$
(2) $[c] : c\ \varepsilon\ a . \supset . c\ \varepsilon\ b .\therefore$ [*T83*4*, 1]
$a \subset b$ [$D_o D1 = T70$, 2]

*T83*6.* $[a\ b\ c\ d] .\therefore a\ \varepsilon\ b . c \subset a . \sim (a \subset c) . \sim (c \subset d) . \supset : a \subset c . \vee . c \subset d$

Proof:

$[a\ b\ c\ d] ::$

(1) $a \,\varepsilon\, b$.
(2) $c \subset a$.
(3) $\sim (a \subset c)$.
(4) $\sim (c \subset d) . \supset .\!\cdot\!.$
$[\exists\ e]$:
(5) $e \,\varepsilon\, c$. [$D_o D1{=}T70$, 4]
(6) $e \,\varepsilon\, a$: [$D_o D1{=}T70$, 2, 5]
$[\exists\ f]$.
(7) $f \,\varepsilon\, a$. [$D_o D1{=}T70$, 3]
(8) $\sim (f \,\varepsilon\, c)$.
(9) $f \,\varepsilon\, e$. [$D_o 1{=}T79$, 1, 7, 6]
(10) $f \,\varepsilon\, c \;.\!\cdot\!.$ [$T83{*}4$, 5, 9]
$a \subset c . \vee . c \subset d$ [8, 10]

$T83{*}7$. $[a\ e\ f] :\cdot: [c\ d] .\!\cdot\!.\ c \subset a . \supset : a \subset c . \vee . c \subset d :: e \,\varepsilon\, a . f \,\varepsilon\, a :: \supset . e \,\varepsilon\, f$

Proof:

$[a\ e\ f] :\cdot:$
(1) $[c\ d] .\!\cdot\!.\ c \subset a . \supset : a \subset c . \vee . c \subset d ::$
(2) $e \,\varepsilon\, a$.
(3) $f \,\varepsilon\, a :: \supset :$
(4) $[\exists\ g] . \sim (f \subset g)$: [$T83{*}3$, 3]
(5) $f \subset a$. [$T83{*}5$, 3]
(6) $a \subset f$: [1, 5, 4]
$e \,\varepsilon\, f$ [$D_o D1{=}T70$, 6, 2]

$T83{*}8$. $[a\ b\ e] :\cdot: \sim (a \subset e) . a \subset b :: [c\ d] .\!\cdot\!.\ c \subset a . \supset : a \subset c . \vee . c \subset d :: \supset . a \,\varepsilon\, b$

Proof:

$[a\ b\ e] :\cdot:$
(1) $\sim (a \subset e)$.
(2) $a \subset b$ ::
(3) $[c\ d] .\!\cdot\!.\ c \subset a . \supset : a \subset c . \vee . c \subset d :: \supset .\!\cdot\!.$
(4) $[c\ d] : c \,\varepsilon\, a . d \,\varepsilon\, a . \supset . c \,\varepsilon\, d \;.\!\cdot\!.$ [$T83{*}7$, 3]
$[\exists\ c]$.
(5) $c \,\varepsilon\, a$. [$D_o D1{=}T70$, 1]
(6) $c \,\varepsilon\, b \;.\!\cdot\!.$ [$D_o D1{=}T70$, 2, 5]
$a \,\varepsilon\, b$ [$D_o 1{=}T79$, 5, 6, 4]

$T83{*}9{=}D6$. $[a\ b] :\cdot: a \,\varepsilon\, b . \equiv :: [\exists\ c] . \sim (a \subset c) : a \subset b :: [c\ d] .\!\cdot\!.\ c \subset a . \supset : a \subset c . \vee . c \subset d$ [$T83{*}3$, $T83{*}5$, $T83{*}6$, $T83{*}8$]

$T83{*}10$. $[a\ b\ c] : a \subset b . c \subset a . \supset . c \subset b$ [$D_o D1{=}T70$]

$T83{*}11$. $[a\ b\ f] ::\cdot:: [c] :\cdot: c \subset a :: [d\ e] .\!\cdot\!.\ d \subset c . \supset : c \subset d . \vee . d \subset e :: \supset c \subset b ::: f \,\varepsilon\, a ::: \supset . f \,\varepsilon\, b$

Proof:

$[a\ b\ f] ::\cdot::$

(1) $[c] :\cdot: c \subset a :: [d\, e] \therefore d \subset c . \supset : c \subset d . \vee . d \subset e :: \supset . c \subset b ::$

(2) $f \varepsilon a :: \supset ::$

(3) $[\exists\, c] . \sim (f \subset c) :$ ⎫

(4) $f \subset a ::$ ⎬ $[T83^*9, 2]$

(5) $[d\, e] \therefore d \subset f . \supset : f \subset d . \vee . d \subset e ::$ ⎭

(6) $f \subset b ::$ $[1, 4, 5]$

$f \varepsilon b$ $[T83^*9, 3, 6, 5]$

T83 12.* $[a\, b] :: [c] :\cdot: c \subset a :: [d\, e] \therefore d \subset c . \supset : c \subset d . \vee . d \subset e :: \supset . c \subset b :\cdot: \supset . a \subset b$

Proof:

$[a\, b] ::$

(1) $[c] :\cdot: c \subset a :: [d\, e] \therefore d \subset c . \supset : c \subset d . \vee . d \subset e :: \supset . c \subset b :\cdot: \supset \therefore$

(2) $[c] : c \varepsilon a . \supset . c \varepsilon b \therefore$ $[T83^* 11, 1]$

$a \subset b$ $[D_o D1{=}T70, 2]$

T83 13=A_o1.* $[a\, b] :: a \subset b . \equiv :\cdot: [c] :\cdot: c \subset a :: [d\, e] \therefore d \subset c . \supset : c \subset d . \vee . d \subset e :: \supset . c \subset b$ $[T83^* 10, T83^* 12]$

T83 14.* $[a\, d\, e\, f\, \phi] :: [b] : b \varepsilon a . \equiv . [\exists\, c] . b \varepsilon c . c \varepsilon c . \phi(c) \therefore d \subset a . \sim (e \subset f) . e \subset d \therefore \supset . [\exists\, g\, h] . \sim (g \subset h) . g \subset e . \phi(g)$

Proof:

$[a\, d\, e\, f\, \phi] :\cdot:$

(1) $[b] : b \varepsilon a . \equiv . [\exists\, c] . b \varepsilon c . c \varepsilon c . \phi(c) \therefore$

(2) $d \subset a .$

(3) $\sim (e \subset f) .$

(4) $e \subset d \therefore \supset ::$

$[\exists\, g] \therefore$

(5) $g \varepsilon e .$ $[D_o D1{=}T70, 3]$

(6) $g \varepsilon d$ $[D_o D1{=}T70, 4, 5]$

(7) $g \varepsilon a \therefore$ $[D_o D1{=}T70, 2, 6]$

$[\exists\, h] :$

(8) $g \varepsilon h .$ ⎫

(9) $h \varepsilon h .$ ⎬ $[1, 7]$

(10) $\phi(h) .$ ⎭

(11) $h \varepsilon g$ $[T83^* 1, 9, 8]$

(12) $h \varepsilon e :$ $[T83^*4, 5, 11]$

(13) $[\exists\, i] . \sim (h \subset i)$ $[T83^*3, 9]$

(14) $h \subset e ::$ $[T83^*5, 12]$

$[\exists\, g\, h] . \sim (g \subset h) . g \subset e . \phi(g)$ $[13, 14, 10]$

T83 15.* $[a\, b\, c\, d\, e] : a \varepsilon b . c \subset a . d \varepsilon c . e \varepsilon c . \supset . d \varepsilon e$

Proof:

$[a\, b\, c\, d\, e] :$

(1) $a \varepsilon b .$

(2) $c \subset a .$

(3) $d \varepsilon c$.
(4) $e \varepsilon c . \supset .$
(5) $d \varepsilon a$. $[D_0 D1{=}T70, 2, 3]$
(6) $e \varepsilon a$. $[D_0 D1{=}T70, 2, 4]$
$d \varepsilon e$ $[D_0 1{=}T79, 1, 5, 6]$

*T83*16.* $[a\ e\ h\ \phi] :: [b] : b\ \varepsilon\ a . \equiv . [\exists\ c] . b\ \varepsilon\ c . c\ \varepsilon\ c . \phi(c) .\therefore [c\ d] : \sim (c \subset d) . c \subset e . \supset . [\exists\ f\ g] . \sim (f \subset g) . f \subset c . \phi(f) .\therefore h\ \varepsilon\ e .\therefore \supset . h\ \varepsilon\ a$

Proof:

$[a\ e\ h\ \phi] :\cdot:$
(1) $[b] : b\ \varepsilon\ a . \equiv . [\exists\ c] . b\ \varepsilon\ c . c\ \varepsilon\ c . \phi(c) .\therefore$
(2) $[c\ d] : \sim (c \subset d) . c \subset e . \supset . [\exists\ f\ g] . \sim (f \subset g) . f \subset c . \phi(f) .\therefore$
(3) $h\ \varepsilon\ e .\therefore \supset ::$
(4) $h\ \varepsilon\ h :$ $[T83^*2, 3]$
(5) $[\exists\ i] . \sim (h \subset i) :$ $[T83^*9, 3]$
(6) $h \subset e ::$
$[\exists\ f\ g] .\therefore$
(7) $\sim (f \subset g)$.
(8) $f \subset h$. $[2, 5, 6]$
(9) $\phi(f)$.
(10) $h \subset f$ $[T83^*9, 3, 8, 7]$
(11) $h\ \varepsilon\ f .\therefore$ $[D_0 D1{=}T70, 10, 4]$
(12) $[i\ j] : i\ \varepsilon\ f . j\ \varepsilon\ f . \supset . i\ \varepsilon\ j .\therefore$ $[T83^*15, 3, 8]$
(13) $f\ \varepsilon\ f ::$ $[D_0 1{=}T79, 11, 12]$
$h\ \varepsilon\ a$ $[1, 10, 13, 9]$

*T83*17.* $[a\ e\ \phi] :: [b] : b\ \varepsilon\ a . \equiv . [\exists\ c] . b\ \varepsilon\ c . c\ \varepsilon\ c . \phi(c) .\therefore [c\ d] : \sim (c \subset d) . c \subset e . \supset . [\exists\ f\ g] . \sim (f \subset g) . f \subset c . \phi(f) .\therefore \supset . e \subset a$
$[T83^*16, D_0 D1{=}T70]$

*T83*18.* $[a\ \phi] :: [b] : b\ \varepsilon\ a . \equiv . [\exists\ c] . b\ \varepsilon\ c . c\ \varepsilon\ c . \phi(c) : \supset .\therefore [b] .\therefore b \subset a . \equiv : [c\ d] : \sim (c \subset d) . c \subset b . \supset . [\exists\ e\ f] . \sim (e \subset f) . e \subset c . \phi(e)$
$[T83^*14, T83^*17]$

By deriving *T83*13*, *T83*9*, and *T83*18* we have shown that $A_0 1$, *D6*, and any thesis introduced into System $\mathfrak{A}_0$ in virtue of R5 are all obtainable within the framework of System $\mathfrak{D}_0$, which completes the proof that the two systems are inferentially equivalent.

When we compare $\mathfrak{A}_0$ with $\mathfrak{D}_0$, we can hardly fail to notice that while the axioms of the two systems are of equal complexity, the rule for writing nominal definitions in $\mathfrak{D}_0$ is much simpler than the corresponding rule in $\mathfrak{A}_0$. In fact, DR5 appears to coincide with our intuitions about definitions in ordinary discourse. For in ordinary discourse we often define an object as being *so and so* if and only if it is something that *satisfies such and such conditions*. Now, this way of talking finds its expression in the form postulated for nominal definitions by DR5. It is the intuitiveness of DR5 that provides justification for R5, which although a little more difficult to grasp is nevertheless inferentially equivalent to DR5.

A more formal justification of the two rules for writing nominal definitions amounts to proving consistency of $\mathfrak{D}_0$ or $\mathfrak{A}_0$, and involves a re-interpretation of 'ε' or '$\subset$' which makes $\mathfrak{D}_0$ or $\mathfrak{A}_0$ part of a system known to be consistent.

A very simple re-interpretation of 'ε' for the purpose of proving consistency of Leśniewski's Ontology has been suggested by the late Mr. Kruszewski.[14] Following his idea we re-interpret 'ε' as the functor of conjunction for propositional arguments, and we re-interpret the nominal variables of $\mathfrak{D}_0$ as propositional variables. On this re-interpretation D_o1 becomes a thesis of Protothetic, which is Leśniewski's system of the logic of propositions, while the rules of inference available in $\mathfrak{D}_0$ turn into valid protothetical rules. The case of D_o1 and R1 - R4 appears to be obvious. Now, any proposition of the form

$$[p \ldots] : p \,.\, x \,.\, \equiv \,.\, [\exists\, q] \,.\, p \,.\, q \,.\, \phi\,(q)$$

which satisfies DR5 on the latter's re-interpretation, is easily derivable from the corresponding proposition of the form

$$[\ldots] : x \,.\, \equiv \,.\, [\exists\, q] \,.\, q \,.\, \phi\,(q)$$

which, considering the stipulations of DR5, can be regarded as a protothetical definition.

Consistency of $\mathfrak{A}_0$ follows from the fact that $\mathfrak{A}_0$ is inferentially equivalent to $\mathfrak{D}_0$. It can, however, be established independently by re-interpreting '$\subset$' as the functor of implication.

It is obvious that the proposition

$$[p\; q] \mathrel{.\cdot.} p \supset q \,.\, \equiv : [r] : r \supset p \,.\, \supset \,.\, r \supset q$$

holds in Protothetic. Since the proposition

$$[p\; q\; r] : p \supset q \,.\, \vee \,.\, q \supset r$$

also holds, A_o1 will hold on re-interpreting '$\subset$' as '$\supset$'.

Moreover, a proposition of the form

K. $[p \ldots] \mathrel{.\cdot.} p \supset x \,.\, \equiv : [q\; r] : \sim (q \supset r) \,.\, q \supset p \,.\, \supset \,.\, [\exists\, s\; t] \,.\, \sim (s \supset t) \,.\, s \supset q \,.\, \phi\,(s)$

which satisfies R5 under our re-interpretation, can be derived from the corresponding proposition of the form

K1. $[\ldots] \mathrel{.\cdot.} x \equiv : [q\; r] : \sim (q \supset r) \,.\, \supset \,.\, [\exists\, s\; t] \,.\, \sim (s \supset t) \,.\, s \supset q \,.\, \phi\,(s)$

which, in view of the stipulations of R5, can be regarded as a protothetical definition. The following simple deductions show that *K1* implies *K*.

K2. $[p\; q\; r \ldots] : p \supset x \,.\, \sim (q \supset r) \,.\, q \supset p \,.\, \supset \,.\, [\exists\, s\; t] \,.\, \sim (s \supset t) \,.\, s \supset q \,.\, \phi\,(s)$

Proof:

$[p\; q\; r \ldots] \mathrel{.\cdot.}$

(1) $p \supset x \,.$

(2) $\sim (q \supset r)$.
(3) $q \supset p . \supset :$
(4) q . [2]
(5) p . [3, 4]
(6) x : [1, 5]
$[\exists\ s\ t] . \sim (s \supset t) . s \supset q . \phi(s)$ [*K1*, 6, 2]

K3. $[p\ u\ v \ldots] :: [q\ r] : \sim (q \supset r) . q \supset p . \supset . [\exists\ s\ t] . \sim (s \supset t) . s \supset q . \phi(s) .\because p . \sim (u \supset v) .\because \supset . [\exists\ s\ t] . \sim (s \supset t) . s \supset u . \phi(s)$

Proof:

$[p\ u\ v \ldots] ::$
(1) $[q\ r] : \sim (q \supset r) . q \supset p . \supset . [\exists\ s\ t] . \sim (s \supset t) . s \supset q . \phi(s) .\because$
(2) p .
(3) $\sim (u \supset v) .\because \supset :$
(4) $u \supset p$: [2]
$[\exists\ s\ t] . \sim (s \supset t) . s \supset u . \phi(s)$ [1, 3, 4]

K4. $[p \ldots] :: [q\ r] : \sim (q \supset r) . q \supset p . \supset . [\exists\ s\ t] . \sim (s \supset t) . s \supset q . \phi(s) .\because p .\because \supset x$

Proof:

$[p \ldots] ::$
(1) $[q\ r] : \sim (q \supset r) . q \supset p . \supset . [\exists\ s\ t] . \sim (s \supset t) . s \supset q . \phi(s) .\because$
(2) $p .\because \supset .\because$
(3) $[q\ r] : \sim (q \supset r) . \supset . [\exists\ s\ t] . \sim (s \supset t) . s \supset q . \phi(s) .\because$ [*K3*, 1, 2]
x [*K1*, 3]

Now, *K2* and *K4* imply *K*.

Our justification of R5 having been concluded it remains to consider the relationship between Boolean Algebra with definitions and Leśniewski's Ontology. It is evident from the preceding discussion that Boolean Algebra with definitions is part of Basic Ontology with nominal definitions. For by Basic Ontology with nominal definitions we understand the totality of theses derivable from *A1* and *A1.1* in virtue of the rules R1 - R5 whereas Boolean Algebra with definitions has been described above as the totality of theses so derivable from *A1* alone. In what follows I will briefly indicate a series of steps which lead from Basic Ontology with nominal definitions to Ontology proper. They all concern the rules of inference.

(i) Since Ontology is conceived as containing a system of the logic of propositions, the condition of R4 and R5 which bars semantical categories obtainable in the logic of propositions from appearing in the definienda of either propositional or nominal definitions has to be dropped.

(ii) We have to generalise R4 and R5 in order to accommodate many-link functions. In definitions which satisfy the stipulations of R4 or R5 in their present form new functors are always constant terms. The rules of definition in Ontology proper allow for definitions in which new functors are functional expressions. This generalisation seems to be entirely in keeping with certain tendencies in ordinary usage, and it increases the flexibility of the ontological syntax enormously.[15]

(iii) Moreover, the rule for writing nominal definitions has to be strengthened to allow for definitions of the following form:

XV $\quad [a \ldots] : a \varepsilon x . \equiv . \phi(a)$

The stipulations of the new rule can be outlined as follows. On the assumption that a thesis T is the last thesis in the system, an expression E of type XV can be added to the system as a new thesis provided the following conditions are fulfilled: 'x' in E is a constant name which does not occur in T or in any thesis preceding T in the system, or it is a nominal function; if the latter is the case then the functor of the first link (in a simple nominal function there is only one link) is a constant term which does not occur in T or in any thesis preceding T in the system while the arguments in the links are all variables; none of the variables in '$a \varepsilon x$' occurs in that expression more than once; '$\phi(a)$' in E is of the form '$a \varepsilon y$' or it is a conjunction with at least one conjunct of the form '$a \varepsilon y$'; '$\phi(a)$' in E is, with respect to T, a meaningful propositional expression, *i.e.*, every constant in '$\phi(a)$' occurs in T or in a thesis preceding T in the system, and every variable occurring in '$\phi(a)$' belongs to a semantical category (logical type) already available in the system; every variable occurring in '$a \varepsilon x$' occurs as a free variable in '$\phi(a)$' and every free variable in '$\phi(a)$' occurs in '$a \varepsilon x$'; there are no free variables in E.

In this version the rule for writing nominal definitions enables us to derive the proposition

$$[a\, b] : a = b . \phi(a) . \supset . \phi(b)$$

from the propositions

$$[a\, b\, c] : a \varepsilon b . b \varepsilon c . \supset . a \varepsilon c$$

and

$$[a\, b] : a \varepsilon b . \supset . a \varepsilon a$$

without employing any law of extensionality. It also enables us to replace $D_o 1$ as the axiom of Ontology by the following simple thesis

$$[a\, b] : a \varepsilon b . \equiv . [\exists\, c] . a \varepsilon c . c \varepsilon b$$ [16]

(iv) Finally, two rules of extensionality have to be adopted: the rule of *propositional* extensionality and the rule of *nominal* extensionality.

For a detailed and authoritative statement of the rules of inference in Ontology the reader will be well advised to consult S. Leśniewski, 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', *Fundamenta Mathematicae* 14 (1929) and 'Über die Grundlagen der Ontologie', *Comptes rendus des séances de la Societé de Sciences et des Lettres de Varsovie*, Classe III, XXIII Année, Warszawa 1930.[17]

NOTES

11. Introduction, Section I, and Section II of my 'Studies in the axiomatic foundations of Boolean Algebra' have been published in Vol. I No. 1 of *Notre Dame Journal of Formal Logic*, pp. 23-47; Section III of the essay has appeared in Vol. I No. 3 of the Journal, pp. 91-106.

12. *A1* and *A1.1* can be shown to be mutually independent in a system with R1 - R4 as its only rules of inference.

13. For historical details concerning $D_o 1$ and $A_o 1$ see my 'On Leśniewski's Ontology', *Ratio* 1 (1957-58), pp. 150-176.

14. See J. Słupecki, 'S. Leśniewski's Calculus of Names', *Studia Logica*, 3 (1955), p. 66.[61]

15. The idea of many-link functions in Protothetic is explained informally by B. Sobociński in his 'On the single axioms of protothetic', *Notre Dame Journal of Formal Logic*, vol. I (1960), pp. 52-73.

16. See B. Sobociński, 'O kolejnych uproszczeniach aksjomatyki "ontologji" prof. St. Leśniewskiego' (On Successive Simplifications of the Axiom-system of Leśniewski's 'Ontology'), *Księga Pamiątkowa Fragmenty Filozoficzne*, Warszawa 1934.

17. In preparing the present essay for publication I have been helped by generous advice and illuminating criticism from Professor Sobociński.

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