INDEPENDENCE OF TARSKI'S LAW IN HENKIN'S PROPOSITIONAL FRAGMENTS

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The system $\{A1\text{-}3,\ (\varphi)^*\}$ is the system of Henkin's [1], proved by him complete for tautologies in implication and whatever truth-function $\varphi(x_1, \ldots, x_m)$ may be. If $m = 0$, φ is just T or F. The basis of the system is *modus ponens*, the axiom schemata

$$\begin{array}{ll} A1. & A \supset . B \supset A \\ A2. & A \supset B \supset . A \supset (B \supset C) \supset . A \supset C \\ A3. & A \supset C \supset . A \supset B \supset C \supset C \end{array}$$

and a set, $(\varphi)^*$, of 2^m axiom schemata

$$x_1^* \supset . x_2^* \supset . \ldots \supset . x_m^* \supset \varphi'^*$$

in which x_i^* is x_i or $x_i \supset y$ (with y a new variable) in the j-th schema according as x_i is T or F in the j-th valuation (according to some ordering) of $\varphi(x_1, \ldots, x_m)$, and φ^* is $\varphi(x_1, \ldots, x_m) \supset y \supset y$ or $\varphi(x_1, \ldots, x_m) \supset y$ according as $\varphi(x_1, \ldots, x_m)$ is T or F. (Henkin used $x_i \supset y \supset y$ in place of our antecedents x_i, but since $A \supset . B \supset C$ and $A \supset C \supset C \supset . B \supset C$ are equivalent forms in any system containing $A1\text{-}2$, we use the shorter expression.) φ is a function symbol, but we shall usually refrain from indicating its argument places, and this should not cause confusion.

L'Abbé in [2] showed that only the independence of $A3$ is ever in doubt. We here show the general (necessary and sufficient) conditions for $A3$ to be independent[1]), the method of determining this being simple inspection of a truth-table for φ. The term 'Tarskian' in the ensuing theorem is chosen because $A3$ is the often so-called Law of Tarski with commuted antecedents.

Def. $\mathfrak{T}$ For all φ, φ is Tarskian iff there are valuations of φ, say α and β, such that φ is F in α, T in β, and all arguments of φ that are T in β are T in α.

THEOREM. $A3$ is independent in the system $\{A1\text{-}3,\ (\varphi)^*\}$ iff φ is not Tarskian.

[1] We are indebted to Professor Henkin for suggesting this problem.

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Lemma 1. If φ is Tarskian, one of the two following pairs of schemata, *S1*-*S2*, *S3*-*S4* is derivable in the system $\{A1\text{-}2, (\varphi)^*\}$

$$S1.\quad A \supset . \varphi \supset C$$
$$S2.\quad A \supset C \supset . \varphi \supset C \supset C$$

with each i-th argument place ($1 \geqq i \geqq m$) of φ similarly filled in each.

$$S3.\quad A \supset . B \supset C \supset . \varphi \supset C$$
$$S4.\quad A \supset C \supset . B \supset C \supset . \varphi \supset C \supset C$$

with each i-th argument place of φ similarly filled in each.

In proving the Lemma, the only properties of the sub-system $\{A1\text{-}2\}$ which will be used, are the well known ones that $A \supset A$ is provable, provable antecedents can be removed, and all but one of a set of identical antecedents can be removed. Since the rule of commutation is available, differences between schemata owing to the order of their antecedents will be neglected. Schemata corresponding to valuations α and β as in *Def.* $\mathfrak{T}$ will be denoted as α^* and β^*.

By hypothesis, φ is Tarskian, therefore there are α and β as in *Def.* $\mathfrak{T}$, and by the valuation process, β must have fewer T-s than α. Two main cases therefore arise, according as β has no T-s or some.

Case (i) a. All arguments of φ are F in β, all are T in α. In α^* and β^* by taking all arguments as A, and removing all but one of repeated antecedents in each, we get *S1*, *S2*.

Case (i) b. All arguments are F in β, some are T, some F in α. In α^* and β^* take all arguments that are in T in α, F in β, as A; all that are F in both as B ($B \neq A$). Removing superfluous antecedents as before, we get *S3*, *S4*.

Case (ii) Some arguments are T in β. Since φ is Tarskian, these are all T in α. We clear α^* and β^* of all antecedents composed of these arguments by taking each as A $\supset$ A and removing. β^* now has only antecedents containing arguments that are all F in β. The resulting schemata can therefore be treated as in Case (i). The Lemma is proved. We give one example, which should make the working of all cases clear.

Suppose that among the valuations of a quaternary φ there are:

φ	(A	B	D	E)	
F	T	T	T	F	
T	F	F	T	F	, then φ is Tarskian.

The corresponding schemata will be:

$$A \supset . B \supset . D \supset . E \supset C \supset . \varphi(A,B,D,E) \supset C$$
$$A \supset C \supset . B \supset C \supset . D \supset . E \supset C \supset . \varphi(A,B,D,E) \supset C \supset C$$

These are an instance of Case (ii). Putting $A \supset A$ for D, A for B, B for E, removing $A \supset A$ and all but one of repeated antecedents in each schema we get:

$$A \supset . B \supset C \supset . \varphi(A, A, A \supset A, B) \supset C$$
$$A \supset C \supset . B \supset C \supset . \varphi(A, A, A \supset A, B) \supset C \supset C.$$

If there had not been $E \supset C$ in each schema, but say E in the first, $E \supset C$ in the second, we should have put A for E, and obtained $S1$, $S2$.

Lemma 2. If either of the pairs of schemata $S1$-$S2$, $S3$-$S4$ are adjoined to the system $A1$-2 , then $A3$ is provable.

Proof. L'Abbé's [2] shows how to prove $A3$ in the system $\{A1\text{-}2, S1, S2\}$. We need only take his unary φ (negation) as m-ary ($m \geqq 1$), with A or $A \supset A$ in the argument places to obtain a general proof.

Turning to $S3$-$S4$, in any system containing $A1$, $A2$, the deduction theorem is provable and the following primitive or derived rules of inference are available.

$R1.$ $A, A \supset B \vdash B$
$R2.$ $B, A \supset . B \supset C \vdash A \supset C$
$R3.$ $A \supset . B \supset C \vdash B \supset . A \supset C$
$R4.$ $A \supset B \vdash B \supset C \supset . A \supset C$
$R5.$ $A \supset B \supset C \vdash B \supset C$
$R6.$ $A, B, A \supset . B \supset C \vdash C$
$R7.$ $A \supset B, B \supset C \vdash A \supset C$

In the system $\{A1\text{-}2\}$ we prove $A3$ from hypotheses.

1.	$A \supset . B \supset B \supset . D \supset B$	*hyp.*
2.	$A \supset C \supset . B \supset C \supset . D \supset C \supset C$	*hyp.*
3.	$A \supset C$	*hyp.*
4.	$A \supset B \supset C$	*hyp.*
5.	$B \supset B$	[$R6$, $A1$, $A2$]
6.	$A \supset . D \supset B$	[$R2$, 5, 1]
7.	$D \supset . A \supset B$	[$R3$, 6]
8.	$A \supset B \supset C \supset . D \supset C$	[$R4$, 7]
9.	$B \supset C$	[$R5$, 4]
10.	$D \supset C \supset C$	[$R6$, 3,9,2]
11.	$A \supset B \supset C \supset C$	[$R7$, 8, 10]
12.	C	[$R1$, 4, 11]
13.	$A \supset C \supset . A \supset B \supset C \supset C$	[3, 4 $\vdash$ 12]

Hypotheses 3 and 4 have been discharged; $A3$ therefore follows from 1 and 2 in the system $\{A1\text{-}2\}$. If now we take D in 1 and 2 as φ, 1 is $S3$ with C taken as B (and no consequent change in the arguments of φ), 2 is $S4$. Therefore $A3$ is provable in the system $\{A1\text{-}2. S3, S4\}$.

Lemma 3. If φ is Tarskian, $A3$ is provable in the system $\{A1\text{-}2, (\varphi)^*\}$. Proof, from Lemmas 1 and 2.

Lemma 4. If φ is Tarskian, $A3$ is non-independent in the system $\{A1\text{-}3, (\varphi)^*\}$. Proof, from Lemma 3.

We prove the converses of Lemmas 3 and 4 by making use of a system $\{\mathrm{H}\}$ which will now be described.

Def. C $\{\mathrm{S}\} = $ c $\{\mathrm{R}\}$, for, systems $\{\mathrm{S}\}$ and $\{\mathrm{R}\}$ contain all and only the same consequences composed solely of implication and variables.

Def. T $T_i(A_1, \ldots, A_i)$, for, T if $i = 0$, and otherwise for, $A_1 \supset . A_2 \supset \ldots \supset . A_i \supset T$.

Def. H $\{H\}$, for, Heyting's intuitionistic system for implication, alternation, conjunction and negation, with added axioms $T_i(A_1, \ldots, A_i)$ for all i.

The following six properties of the system $\{H\}$ are assumed as either well known or easily verifiable. We use $\triangle$ to denote possibly empty sets of formulae.

(*H1*) $\{H\} =_c \{A1\text{-}2\}$, and $A3$ is not provable in $\{H\}$.

(*H2*) $\triangle \vdash_H T_i(A_1, \ldots, A_i) \supset C \supset C$

(*H3*) $\triangle \vdash_H \sim T_i(A_1, \ldots, A_i) \supset C$

(*H4*) If $\psi(A_1, \ldots, A_i, B_1, \ldots, B_j)$ is $A_1 \supset . A_2 \supset \ldots . A_i \supset . B_1 \vee B_2 \vee \ldots \vee B_j$, then $A_1, \ldots, A_i, B_1 \supset C, \ldots, B_j \supset C \vdash_H \psi(A_1, \ldots, A_i, B_1, \ldots, B_j) \supset C$.

(*H5*) If $\psi(A_1, \ldots, A_i, B_1, \ldots, B_j)$ is as in (*H4*) and X is one of $B_1, \ldots, B_j$, then $X, \triangle \vdash_H \psi(A_1, \ldots, A_i, B_1, \ldots, B_j)$.

(*H6*) If $\triangle \vdash_H B_1, \ldots, \triangle \vdash_H B_j$, then $\triangle \vdash_H (B_1 \& B_2 \& \ldots \& B_j) \supset C \supset C$.

Lemma 5. If φ is not Tarskian, then schemata $(\varphi)^*$ are interpretable as valid schemata in $\{H\}$.

Proof. By the valuation procedure, three cases arise, according as φ is always T, always F, or sometimes T and sometimes F.

Case 1. φ is always T. If φ_i is interpreted as T_i, then all schemata $(\varphi_i)^*$ are valid in $\{H\}$, by (*H2*).

Case 2. φ is always F. If φ_i is interpreted as $\sim T_i$, then all schemata $(\varphi_i)^*$ are valid in $\{H\}$, by (*H3*).

Case 3. φ is sometimes T, sometimes F. Let α_r be the r-th of the m valuations in which φ is F, and in α_r let $A_1^r, \ldots, A_i^r$ be the arguments of φ which are T, $B_1^r, \ldots, B_j^r$ those which are F. Let $\psi_r(A_1^r, \ldots, A_i^r, B_1^r, \ldots, B_j^r)$ be $\psi(A_1^r, \ldots, A_i^r, B_1^r, \ldots, B_j^r)$ as in (*H4*). We interpret ϕ as the conjunction of all ψ_r from 1 to m. Then for each r, the schema $\alpha_r{}^*$ is valid in $\{H\}$, by (*H4*).

By the hypothesis of the case, φ is T in some valuation, and by the hypothesis of the lemma, viz. that φ is not Tarskian, for all valuations α in which φ is F, and for all valuations β in which φ is T, there are some arguments T in β which are F in α. Let β_p be the p-th of the n valuations in which φ is T, and let $X_1^p, \ldots, X_s^p$ be the arguments valued T in β_p. Then for each α_r, some of $X_1^p, \ldots, X_s^p$ are F in α_r, i.e. are among the $B_1^r, \ldots, B_j^r$ of α_r, and so, by (*H5*):

$X_1^p, \ldots, X_s^p \vdash_H \psi_r(A_1^r, \ldots, A_i^r, B_1^r, \ldots, B_j^r)$. Hence, by (*H6*), $X_1^p, \ldots, X_s^p, \triangle \vdash_H \varphi \supset C \supset C$. The case and the lemma are proved.

Lemma 6. If φ is not Tarskian, $\{A1\text{-}2, (\varphi)^*\} =_c \{\mathrm{H}\}$.

Proof, from Lemma 5 and (*H1*).

Lemma 7. If φ is not Tarskian, A3 is independent in $\{A1\text{-}3, (\varphi)^*\}$.

Proof, from Lemma 6 and (*H1*).

Lemmas 4 and 7 prove the THEOREM.

REFERENCES

[1] Henkin, Leon, *Fragments of the Propositional Calculus,* The Journal of Symbolic Logic, vol. 14 (1949), pp. 42-48.

[2] L'Abbé, Maurice, *On the Independence of Henkin's Axioms for Fragments of the Propositional Calculus,* The Journal of Symbolic Logic, vol 16 (1951), pp. 43-45.

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