AN EXTENSION ALGEBRA AND THE MODAL SYSTEM T

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In [4],[1] [5], and [6] J.C.C. McKinsey and A. Tarski proved some far-reaching theorems concerning the modal system S4 and its extensions by using techniques of abstract algebra, and in particular the concept of a closure algebra. In [2], M.A.E. Dummett and the present author applied these results to proving the characteristicity of certain matrices for S4 and some of its extensions. In the present paper, a new kind of algebra is introduced, here called an extension algebra, which is shown to have the same utility in studying the modal system T that closure algebras have in the study of S4. Finally, a particular extension algebra is shown to be characteristic for T; this algebra is very similar to the closure algebra shown to be characteristic for S4 in [2]. Acquaintance with the relevant material in [2], [4], [5], and [6] is presumed in what follows, and proofs which model closely their analogues in these papers are omitted.

I

We define an extension algebra as follows:

Definition 1. $\mathfrak{M} = \langle M, \cup, \cap, -, \mathbf{E} \rangle$ is an *extension algebra* iff M is some set of elements and $\cup, \cap, -, \mathbf{E}$ are operations on these elements such that:

(i) M is a Boolean algebra with respect to $\cup$, $\cap$, and $-$;

(ii) if $x \in M$, then $\mathbf{E}x \in M$;

(iii) if $x \in M$, then $x \subseteq \mathbf{E}x$;

(iv) if $x, y \in M$, then $\mathbf{E}(x \cup y) = \mathbf{E}x \cup \mathbf{E}y$;

(v) $\mathbf{E}\Lambda = \Lambda$.

If we compare this definition with [5] Df. 1.1 we see that, if in addition we stipulate that for $x \in M$ $\mathbf{E}\mathbf{E}x = \mathbf{E}x$, $\mathfrak{M}$ is a closure algebra. Thus our definition is a generalization of that of a closure algebra: all closure algebras are extension algebras, but not conversely.

$\mathbf{E}x$ may be called the *extension* of x, and, in analogy with the interior operator of closure algebras, we may define:

Definition 2. For $x \in M$, $\mathfrak{J}x = -\mathbf{E}-x$.

$\mathfrak{J}x$ is the *intension* of x.

Extension algebras might form the basis of an abstract mathematical study of *growth*. For example, if the elements of M are construed as sets of

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points, then for any $x \mathbf{E} M$ $\mathbf{E}x$ may naturally be construed as the set of points into which x has grown after a certain time-lapse. Conditions (iii) – (v) of Df. 1 then stipulate that the growth of a point-set include the set, that the growth of the union of two point-sets be equal to the union of their separate growths, and that the growth of the null set be the null set. In what follows, however, we are concerned primarily with applications in the field of modal logics.

Following closure-algebraic terminology, we say that $x \mathbf{E} M$ is *closed* iff $x = \mathbf{E}x$ and *open* iff $x = \mathbf{J}x$. Also, we say that $x \mathbf{E} M$ is *extended* iff there is a $z \mathbf{E} M$ such that $x = \mathbf{E}z$ and *intended* iff there is a $z \mathbf{E} M$ such that $x = \mathbf{J}z$. Closed elements are extended and open elements intended, but the converse is not generally true; it is characteristic of closure algebras that the concepts of closed element and extended element and the concept of open element and intended element coincide. Elements which are not extended may be called *atomic*.

Simple examples of extension algebras which are not closure algebras may be constructed as follows. Let M be the set of all subsets of the signed integers $\ldots, -1, 0, 1, \ldots$. Let $\cup, \cap, -$ be the usual set-theoretic operations, and, for any $A \mathbf{E} M$, we put $\mathbf{E}A = \{x : x \mathbf{E} A \text{ or } x+1 \mathbf{E} A \text{ or } x-1 \mathbf{E} A\}$. For example, if $A = \{-1, 3, 4\}$, $\mathbf{E}A = \{-2, -1, 0, 2, 3, 4, 5\}$. Then $M = \langle M, \cup, \cap, -, \rangle$ is an extension algebra with the property that no element except $\wedge$ and $\vee$ is closed and every unit set is atomic. Further, if $B = \{0\}$, then, writing $\underbrace{\mathbf{E} \ldots \mathbf{E}}_{m} A$ as $\mathbf{E}^m A$, we have $\mathbf{E}^m B \neq \mathbf{E}^n B$ for any natural numbers m and n such that $m \neq n$. Again, let M' be the set of all subsets of the natural numbers $0, 1, 2, \ldots$, and let $\cup, \cap, -$ be as before; we put $\mathbf{E}'A = \{x : x \mathbf{E} A \text{ or } x-1 \mathbf{E} A\}$. Then $\mathfrak{M}' = \langle M', \cup, \cap, -, \mathbf{E}' \rangle$ is an extension algebra. In $\mathfrak{M}'$ the set $\{0\}$ is closed, as are the sets $\{0, 1\}$, $\{0, 1, 2\}$, etc., but all other unit sets are again atomic; the set $\{1, 2\}$ is extended but not closed.

Theorem 1.[2] For any extension algebra $\mathfrak{M} = \langle M, \cup, \cap, -, \mathbf{E} \rangle$,

(i) for $x, y \in M$, if $x \subseteq y$ then $\mathbf{E}x \subseteq \mathbf{E}y$ and $\mathbf{J}x \subseteq \mathbf{J}y$;

(ii) $\mathbf{E}\vee = \vee$;

(iii) for $x \in M$, $\mathbf{J}x \subseteq x$;

(iv) for $x, y \in M$, $\mathbf{J}(x \cap y) = \mathbf{J}x \cap \mathbf{J}y$.

(Proofs are immediate by Boolean operations and Dfs. 1 and 2.)

Closed and open elements of extension algebras behave in many respects like their counterparts in closure algebras; the behaviour of extended and intended elements is rather different, however. We have

Theorem 2.[3] In any extended algebra

(i) the complement of an intended (open) element is extended (closed), and the complement of an extended (closed) element is intended (open);

(ii) the sum of any finite number of extended (closed) elements is extended (closed), and the product of any finite number of intended (open) elements is intended (open);

(iii) the product of any number of closed elements is closed, and the sum of any number of open elements is open;

(iv) $\mathbf{E}x$ is extended and $\mathfrak{J}x$ is intended;

(v) $\wedge$ and $\vee$ are both open and closed.

We lack the analogue of (iii) for extended and intended elements. For example, in the extension algebra $\mathfrak{M}'$ given earlier $\{1, 2\}$ and $\{2, 3\}$ are both extended elements, but their product $\{2\}$ is atomic.

We shall need later the

Lemma 1.[4] If x is any open element of an extension algebra and y any element, then $x \cap \mathbf{E}(x \cap y) = x \cap \mathbf{E}y$.

(The proof requires only Boolean operations and Dfs. 1 and 2.)

Also, we shall need appropriate definitions of universal algebras, generalized universal algebras, and extension-algebraic functions (cf, [5] paras. 3 and 4). Thus we stipulate:

Definition 3. Let Π be a class of extension algebras: then $\mathfrak{M}$ is a *universal algebra* for Π iff $\mathfrak{M}$ is an extension algebra and every extension algebra in Π is isomorphic to a subalgebra of $\mathfrak{M}$.

Definition 4. If $\mathfrak{M} = < \mathrm{M}, \cup, \cap, -, \mathbf{E} >$ is an extension algebra and a is an element of $\mathfrak{M} \neq \wedge$, then by $\mathfrak{M}_a$, the *relativized subalgebra of* $\mathfrak{M}$ *with respect to* a, we understand the algebra $<\mathfrak{M}_a, \cup, \cap, -_a, \mathbf{E}_a>$ where, for $x \epsilon M$, $x \epsilon \mathfrak{M}_a$ iff $x \subseteq a$, $-_a\, x = a \cap - x$, and $\mathbf{E}_a x = a \cap \mathbf{E}x$.

It is immediate that $\mathfrak{M}_a$ is itself an extension algebra.

Definition 5. Let Π be a class of extension algebras: then $\mathfrak{M}$ is a *generalized universal algebra* for Π iff $\mathfrak{M}$ is an extension algebra and for each extension algebra $\mathfrak{N} \epsilon \Pi$ there is an open element a of $\mathfrak{M}$ such that $\mathfrak{N}$ is isomorphic to a subalgebra of $\mathfrak{M}_a$.

Similarly, we define *extension-algebraic functions* in exact analogy with the definition of closure-algebraic functions in [5] para. 4. An extension-algebraic function $f(x_1, \ldots, x_n)$ is said to *vanish identically in* $\mathfrak{M}$ iff for all $x_1, \ldots, x_n \epsilon \mathfrak{M} f(x_1, \ldots, x_n) = \wedge$.

II

Using results concerning extension algebras, in the present section we prove certain theorems concerning the system T, in particular (Theorem 8) that T has the *finite model property* in the sense of [3] and so is decidable.[5]

By the system T we understand that modal logic obtained by adding to a classical propositional calculus base formulated with the rules of substitution and detachment the rule (R): if $\vdash_T \alpha$ then $\vdash_T \Box \alpha$, and the two axioms:

(1) $\Box p \rightarrow p$;

(2) $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$.

Let $\mathfrak{A} = \langle A, D, \cup, \cap, -, \mathbf{E} \rangle$ be an algebraic system in which A is a set of elements D is a non-empty proper subset of A, $\cup$ and $\cap$ are binary and $-$ and $\mathbf{E}$ are unary functions defined over A and class-closing on A: then $\mathfrak{A}$ is a *T-matrix* iff A satisfies every provable wff of T when $\cup$ corresponds to &, $\cap$ to v, $-$ to $\neg$, and $\mathbf{E}$ to $\Box$. Also, we stipulate the

Definition 6. $\mathfrak{A} = \langle A, D, \cup, \cap, -, \mathbf{E} \rangle$ is a *regular* matrix iff

(i) A is a Boolean algebra with respect to $\cup$, $\cap$, and $-$;
(ii) if $x \in D$ and $-x \cap y \in D$, then $y \in D$;
(iii) if $x \in D$, then $\mathbf{E}\, x \in D$.

We have at once:

Theorem 3. A regular matrix $\mathfrak{A}$ is a T-matrix iff $\mathfrak{A}$ satisfies (1) and (2).

Theorem 4. $\mathfrak{A} = \langle A, \{d\}, \cup, \cap, -, \mathbf{E} \rangle$ is a regular T-matrix iff $\langle A, \cap, \cup, -, \mathbf{E} \rangle$ is an extension algebra and d is the zero element of this algebra. (The proof, though long, follows closely the lines of [6] Thm. 1.2.)

Corollary. If $\mathfrak{M} = \langle M, \cup, \cap, -, \mathbf{E} \rangle$ is an extension algebra, then $\langle M, \{\wedge\}, \cup, \cap, -, \mathbf{E} \rangle$ is a regular T-matrix.

In view of this corollary, we need not distinguish between an extension algebra and the corresponding matrix in which $\{\wedge\}$ is taken as the designated set: hence we shall speak of extension algebras as themselves T-matrices. Considering extension algebras as T-matrices, we clearly set up a correspondence between any wff α of T and an extension-algebraic function $f^{(\alpha)}$ 7) such that α is satisfied by an extension algebra M iff $f^{(\alpha)}$ vanishes identically in M.

Theorem 5. There is an extension algebra $\mathfrak{T}$ which is a characteristic matrix for T.

Proof. $\mathfrak{T}$ is constructed by Lindenbaum's method, as in [4] Thms. 4 and 11. It is easy to show that $\mathfrak{T}$ is regular and has only one designated element (cf. [6] Thm. 3.6), whence by Theorem 4 $\mathfrak{T}$ is an extension algebra.

If we define a *normal* extension of T to be one which is closed under substitution, detachment, and the rule (R), then Theorem 5 may immediately be generalized to all such normal extensions.

Theorem 6.[8] Let α be a wff of T: then $\vdash_T \alpha$ iff $f^{(\alpha)}$ vanishes identically in every extension algebra.

Proof. If $\vdash_T \alpha$, then α is satisfied by every T-matrix, and so by every extension algebra in view of the Corollary to Theorem 4. Conversely, if it is not the case that $\vdash_T \alpha$, then α is not satisfied by the extension algebra $\mathfrak{T}$ of Theorem 5.

Theorem 7. Let $\mathfrak{M} = \langle M, \cup, \cap, -, \mathbf{E} \rangle$ be an extension algebra, and $a_1, \ldots, a_r$ be a finite sequence of elements of **M**. Then there is a finite extension algebra $\mathfrak{M}_1 = \langle M_1, \cup, \cap, -, \mathbf{E}_1 \rangle$ with at most 2^{2^r} elements such that:

(i) for $1 \leq i \leq r$, $a_i \in M_1$;

(ii) If $x \in M_1$ and $\mathbf{E}x \in M_1$, then $\mathbf{E}_1 x = \mathbf{E}x$.

Proof. (Though the proof follows closely those of [4] Thm. 5 and [5] Lemma 2.3 and Thm. 4.14, it may be worthwhile here to give the outlines.) Let M_1 be the set of elements of M obtained from $a_1, \ldots, a_r$ by the Boolean operations $\cup$, $\cap$, $-$. Then M_1 contains at most 2^{2^r} elements, and condition (i) is satisfied. We say that $x \in M_1$ is *covered* by $y \in M_1$ iff $x \subseteq y$ and $\mathbf{E}y \in M_1$. For $x \in M_1$, suppose x to be covered by $x_1, \ldots, x_n$. Then we put $\mathbf{E}_1 x = \mathbf{E}x_1 \cap \ldots \cap \mathbf{E}x_n$. If x is covered by $x_1, \ldots, x_n$, then for $1 \leq i \leq n$ $x \subseteq x_i$, whence by Theorem 1 (i) $\mathbf{E}x \subseteq \mathbf{E}x_i$ and $\mathbf{E}x \subseteq \mathbf{E}_1 x$. If $x \in M_1$ and $\mathbf{E}x \in M_1$, then x is covered by itself, whence by the definition of $\mathbf{E}_1$ $\mathbf{E}x_1 \subset \mathbf{E}x$. Hence condition (ii) is satisfied. It remains to show that $< M_1, \cup, \cap, -, \mathbf{E}_1 >$ is an extension algebra. Of the five conditions in Df. 1, (i) and (ii) are immediate and (v) follows from condition (ii) just established. As to (iii), we have $x \subseteq \mathbf{E}x$ and $\mathbf{E}x \subseteq \mathbf{E}_1 x$ as just shown, whence $x \subseteq \mathbf{E}_1 x$. (iv) is demonstrated exactly as the same condition is demonstrated in [4] Thm. 5.

Corollary 1. If an extension-algebraic function fails to vanish identically in some extension algebra, then there is a finite extension algebra in which it fails to vanish identically.

Corollary 2. If an extension-algebraic function vanishes identically in every finite extension algebra, then it vanishes identically in every extension algebra.

(The proofs of these from Theorem 7 follow the pattern of the proof of [5] Thm. 4.15 from Lemma 4.14.)

Theorem 8. T has the finite model property.

Proof. Let α be some unprovable wff of T. Then α is falsified by the characteristic matrix **T** of Theorem 5, and so fails to vanish identically in some extension algebra. Hence by Corollary 1 to Theorem 7 there is a finite extension algebra – which will itself by Theorem 4 be a T-matrix – which falsifies α.

In [2] Lemma 4, it was proved that any extension of S4 of a certain kind had the infinite model property. The analogous proof for extensions of T breaks down because, as observed in Section I, the product of a finite number of extended elements is not in general itself extended. The same fact seems to explain why there are not propositional calculi standing in a special relation to T and its extension, as there are to S4 and its extensions (namely systems between the intuitionist and the classical calculus).

One further property of T, which it shares with S4, does not seem to have been noticed before. Inspection of the proof of [5] Thm. 4.12 shows that it applies as well to extension-algebraic functions as to closure-algebraic functions. Thus we have:

Theorem 9. If f and g are extension-algebraic functions (of the same number of variables) and if $\mathbf{E}f \cap \mathbf{E}g$ vanishes identically, then either f or g vanishes identically.

Corollary.[9] If $\vdash_T \Box\alpha \cup \Box\beta$, then either $\vdash_T \alpha$ or $\vdash_T \beta$.

III

We saw in Theorem 5 that a characteristic matrix of a trivial sort exists for T. We proceed in this section to prove the characteristicity of a more interesting matrix. We establish in fact that the matrix in question is a generalized universal algebra for the class of all finite extension algebras. Since it is not obvious that such an algebra is a characteristic matrix for T, we prove this first (Lemma 3).

Lemma 2.[10] If a is an open element of an extension algebra $\mathfrak{M}$, and $x_1, \dots, x_n$ are elements included in a, then for every extension-algebraic function f of n variables $f\mathfrak{M}_a(x_1, \dots, x_n) = a \cap f\mathfrak{M}(x_1, \dots, x_n)$.

Proof. By induction on the order of f, using Lemma 1 for the case where f is of the form $\mathbf{E}g$.

Lemma 3.[11] Let $\mathfrak{M}$ be a generalized universal algebra for the class of all finite extension algebras, and α a wff of T which is not provable in T: then α is falsified by $\mathfrak{M}$.

Proof. By Theorem 8 $f^{(\alpha)}(x_1, \dots, x_n)$ fails to vanish identically in some finite extension algebra, say $\mathfrak{N}$. Since $\mathfrak{M}$ is a generalized universal algebra for all finite extension algebras, there is an open element a of $\mathfrak{M}$ such that $\mathfrak{N}$ is isomorphic to a subalgebra of $\mathbf{M}_a$. Hence under the isomorphism there are elements $a_1, \dots, a_n$ of $\mathfrak{M}_a$ such that $f\mathfrak{M}_a^{(\alpha)}(a_1, \dots, a_n) \neq \Lambda$. Hence by Lemma 2 $a \cap f\mathfrak{M}^{(\alpha)}(a_1, \dots, a_n) \neq \Lambda$, so that $f\mathfrak{M}^{(\alpha)}(a_1, \dots, a_n) = \Lambda$. Thus α is falsified by $\mathfrak{M}$.

As in [2], we shall be concerned with systems and algebras constructed in special ways upon partially ordered sets and quasi-ordered sets. Thus if $\mathfrak{K} = \langle K, \leqq \rangle$ is a quasi-ordered set, by $\mathfrak{K}^e, = \langle K^e, \leqq_e \rangle$ we understand the *partially* ordered set such that K^e is the set of equivalence classes of K under the equivalence relation: $a \leqq b$ and $b \leqq a$, and, for $A, B \in K^e$, $A \leqq_e B$ iff for every $a \in A$, $B \in b$, $a \leqq b$. Further, if $\mathfrak{K} = \langle K, \leqq \rangle$ is a quasi-ordered set, $\mathfrak{K}_1 = \langle K_1, \leqq_1 \rangle$ is defined as follows:

$K_1 = \{\langle a, n \rangle : a \in K$ and n is a natural number$\}$;

$\langle a, m \rangle \leqq_1 \langle b, n \rangle$ iff $a \leqq b$, for all $a, b \in K$ and natural numbers m, n.

A partially ordered set $\mathfrak{K} = \langle K, \leqq \rangle$ is *minimally bounded* iff there is a subset $A \subseteq K$ such that each $a \in A$ is a minimal element of $\mathfrak{K}$ and for every $b \in K$ there is a $c \in A$ such that $c \leqq b$. We understand dually *maximally bounded* partially ordered sets.

In [2], we defined a constant closure operator on subsets of quasi-ordered sets. Similarly, we now define an extension operator. First, let $\mathfrak{K} = \langle K, \leqq \rangle$ be a quasi-ordered set: then for $a \in K$ by *e(a)* we understand the equivalence class in $\mathfrak{K}^e$ to which a belongs. Now for $a,b \in K$ we say that a *extends*

b iff either $e(a) = e(b)$ or $e(a)$ covers $e(b)$ in $\mathfrak{K}^e$. (We note that if $\mathfrak{K}$ is a partially ordered set then for a, b in $\mathfrak{K}$ a extends b iff either $a = b$ or a covers b in $\mathfrak{K}$, by the isomorphism between $\mathfrak{K}^e$ and $\mathfrak{K}$.) Finally, by $\mathfrak{K}$ we understand the algebra $< K^{\#}, \cup, \cap, -, \mathbf{E} >$, where $K^{\#}$ is the set of all subsets of K, $\cup$, $\cap$, and $-$ are the usual set-theoretic operations, and $\mathbf{E}$ is an operator on $K^{\#}$such that, for $A \in K^{\#}$, $\mathbf{E}A = \{a :$ for some b, $b \in A$ and a extends $b\}$. It is easily verified that $\mathfrak{K}^{\#}$ is an extension algebra, which we call the *order extension algebra* on $\mathfrak{K}$. 12)

Now we prove a succession of lemmas, corresponding to lemmas in [2], which we shall need later.

Lemma 4. (Cf. [2] Lemma 1). Any finite extension algebra is isomorphic to the order extension algebra $\mathfrak{K}^{\#}$ on some finite quasi-ordered set $\mathfrak{K}$.

Proof. Let $\mathfrak{A} = < A, \cup, \cap, -, \mathbf{E} >$ be a finite extension algebra. Then $\langle A, \cup, \cap, - >$ is a finite Boolean algebra, and hence isomorphic to the field of all subsets of some finite set A'. Where ϕ is the isomorphism, for any $B \subseteq A'$ we define $\mathbf{E}'B = \phi\mathbf{E}\phi^{-1}B$. For any $a, b \in A'$ we define: a extends b iff $a \in \mathbf{E}'\{ b \}$; Then we say that $a \leqq b$ iff there are elements $c_1, \ldots, c_m \in A' (m \geqq 0)$ such that b extends c_1, c_1 extends c_2, $\ldots$, c_m extends a. Then it is readily seen that $\mathfrak{K} = < A', \leqq >$ is a quasi-ordered set and that the extension operator in $\mathfrak{K}^{\#}$ coincides with $\mathbf{E}'$, so that $\mathfrak{A}$ is isomorphic to $\mathfrak{K}^{\#}$.

Lemma 5. For any finite partially ordered set $\mathfrak{K}' = < K', \leqq >$, there is a finite partially ordered set $\mathfrak{K}' = < K', \leqq' >$ with the property that in it no element is covered by more than one element, such that $\mathfrak{K}^{\#}$ is isomorphic to a subalgebra of $\mathfrak{K}'^{\#}$.

Proof identical with the proof of [2] Lemma 3, altering $\mathfrak{K}^+$ and $\mathfrak{K}'^+$ to $\mathfrak{K}^{\#}$ and $\mathfrak{K}'^{\#}$ respectively.

Lemma 6. Let $\mathfrak{K} = < K, \leqq >$ be a countable (i.e. finite or denumerable) quasi-ordered set and $\mathfrak{L} = < L, \leqq >$ be a partially ordered set, such that $\mathfrak{K}^{e\#}$ is isomorphic to a subalgebra of $\mathfrak{L}^{\#}$. Then $\mathfrak{K}^{\#}$ is isomorphic to a subalgebra of $\mathfrak{L}_1{}^{\#}$.

Proof identical with the proof of [2] Lemma 5, *mutatis mutandis (*in particular, we note that members of equivalence classes in quasi-ordered sets are *collectively* either members or not members of extended sets, by the definition of $\mathfrak{K}^{\#}$ from $\mathfrak{K}$).

Lemma 7. Let $\mathfrak{K} = < K, \leqq >$ be a countable quasi-ordered set. Then there is a denumerable partially ordered set $\mathfrak{K}^q = < K^q, \leqq_q >$ such that $\mathfrak{K}^{\#}$ is isomorphic to a subalgebra of $\mathfrak{K}^{q\#}$. If $\mathfrak{K}^e$ is maximally bounded, then so is $\mathfrak{K}^q$.

Proof. (Compare [2] Lemma 7; as the construction there differs in certain respects from that required here, we give the proof in full.) We define $\mathfrak{K}^q$ from $\mathfrak{K}^e$ as follows:

$\mathfrak{K}^q \doteq \{< A, m, n >: A \in K^e$ and m, n are natural numbers$\}$;

$< A, m, n > \leqq_q < B, m', n' >$ iff $A \leqq_e B$ and either $n > n'$ or both $n = n'$ and $m = m'$.

It is easily shown that $\mathfrak{K}^q = < K^q, \leq_q >$ is a partially ordered set, and that if $\mathfrak{K}$ is finite or denumerable then $\mathfrak{K}^q$ is denumerable. Also, $\mathfrak{K}^q$ is maximally bounded if $\mathfrak{K}^e$ is. Next, we define a mapping ϕ of elements of K into denumerable subsets of K^q, using the equivalence classes of $\mathfrak{K}$. If $A \epsilon K^e$ contains finitely many numbers of K, $a_0, \ldots, a_n$ $(n \geq 0)$, then for $i < n$ we put $\phi(a_i) = \{< A, i, k > : k$ is a natural number$\}$ and $\phi(a_n) = \{< A, i, k >; i \leq n$ and k is a natural number$\}$. If $A \epsilon K^e$ contains denumerably many members of K, a_i for $i \geq 0$, then we put $\phi(a_i) = \{< A, i, k > : k$ is a natural number$\}$. Finally we map every subset $B \subseteq K$ into a subset $\Theta(B)$ of K^q, namely $\Theta(B) = \bigcup_{b \epsilon B} \phi(b)$. Clearly ϕ maps every element of K into unique, mutually exclusive, and collectively exhaustive subsets of K^q, so that Θ is an isomorphism with respect to $\cup$, $\cap$, and $-$. That Θ is also such an isomorphism with respect to $\mathbf{E}$ is evident when we observe that, for $a \epsilon K$, $\mathbf{E}\phi(a)$ contains, for every b that extends a, every element $< B, m, n > \epsilon K^q$ such that B is the equivalence class containing b and m, n are natural numbers.

To construct the matrix characteristic for T, we use the sequence $\{\mathfrak{H}_i\}$ of systems $< H_i, \leq_i >$ defined in [2][13]: namely, for each i, H_i is the set of ordered couples $< n, m >$ of natural numbers such that $m \leq i$ and $m! \,.\, n \leq i!$; $\leq_i$ *is* the result of confining to H_i the relation $\leq$ such that $< n, m > \leq < i, k >$ iff $m \leq k$ and $k! \,.\, i \leq m! \,.\, n < k! \,(f+1)$. Also we use the partially ordered set $\mathfrak{H}$ such that, for each i, $\mathfrak{H}_i$ is a subsystem of $\mathfrak{H}$: namely, H is the set of all ordered couples of natural numbers, $\leq$ is defined as above, and $\mathfrak{H} = < H, \leq >$. In [2] it is proved that the order closure algebra on $\mathfrak{H}_1$, $\mathfrak{H}_1{}^+$, is a characteristic matrix for S4. We now prove that the order *extension* algebra on $\mathfrak{H}_1$, $\mathfrak{H}_1{}^\#$, is a characteristic matrix for T.

Lemma 8. Let $\mathfrak{K} = < K, \leq_k >$ be a finite partially ordered set in which no element is covered by more than one element. Then if $\mathfrak{K}$ has a greatest element, $\mathfrak{K}^\#$ is isomorphic to a subalgebra of $\mathfrak{H}_i{}^\#$ for some i; if $\mathfrak{K}$ has no greatest element, $\mathfrak{K}^\#$ is isomorphic to a subalgebra of $\mathfrak{H}_i{}'^\#$ for some i, where $\mathfrak{H}_i{}' = < H_i{}', \leq_i >$ is the partially ordered set obtained from $\mathfrak{H}_i$ by subtracting from H_i its greatest element $< o, i >$.

Proof follows exactly the proof of [2] Lemma 10 as far as the construction of the isomorphism Θ is concerned. That Θ is an isomorphism with respect to $\mathbf{E}$ follows from considerations analogous to those given there in connexion with the closure operation.

Theorem 10. $\mathfrak{H}_1{}^\#$ is a characteristic matrix for T.

Proof. If $\mathfrak{A}$ is a finite extension algebra, then by Lemma 4 $\mathfrak{A}$ is isomorphic to a finite order extension algebra $\mathfrak{K}^\#$ for some quasi-ordered set $\mathfrak{K}$. Since $\mathfrak{K}^e$ will be finite and partially ordered, $\mathfrak{K}^{e\#}$ is isomorphic to a subalgebra of $\mathfrak{K}^{e'\#}$, where $\mathfrak{K}^{e\,\prime}$ is the partially ordered set in which no element is covered by more than one element, whose existence is assured by Lemma 5. $\mathfrak{K}^{e\,\prime}\#$ is in turn isomorphic to a subalgebra of either $\mathfrak{H}_i\#$ or $\mathfrak{H}_i'\#$ for some i, by Lemma 8. Hence $\mathfrak{K}^e\#$ is isomorphic to a subalgebra of either $\mathfrak{H}_i\#$ or $\mathfrak{H}_i'\#$. Now using Lemma 6, we see that $\mathfrak{K}\#$, and so $\mathfrak{A}$, is isomorphic to a subalgebra of either $\mathfrak{H}_{i_1}\#$ or $\mathfrak{H}_i'{}_1\#$. Clearly H_{i_1} and $H_i'{}_1$ are both always open elements of $\mathfrak{H}_1\#$, since their complements are not only extended but also closed. Hence both $\mathfrak{H}_{i_1}\#$ and $\mathfrak{H}_i'{}_1\#$ are relativised subalgebras of $\mathfrak{H}_1\#$ with respect to open elements $\mathfrak{H}_{i_1}$ and $\mathfrak{H}_i'{}_1$ of $\mathfrak{H}_1\#$. It follows that $\mathfrak{H}_1\#$ is a generalized universal algebra for the class of all finite extension algebras, whence by Lemma 3 we have the theorem.

By Lemma 7, since $\mathfrak{H}_1$ is denumerable, we can find a denumerable *partially* ordered set $\mathfrak{H}_1{}^q$ such that $\mathfrak{H}_1\#$ is isomorphic to a subalgebra of $\mathfrak{H}_1{}^q\#$. $\mathfrak{H}_1{}^q\#$ will obviously also be a characteristic matrix for T; unlike $\mathfrak{H}$, which is minimally but not maximally bounded, $\mathfrak{H}_1{}^q$ is neither minimally nor maximally bounded, as its construction from $\mathfrak{H}_1$ reveals.

NOTES

1. Numerals in square brackets refer to items listed in the bibliography at the end of the paper

2. Cf. [5] Corollaries 1.2 and 1.4

3. Cf. [5] Corollary 1.7

4. Cf. [5] Corollary 1.8 (i)

5. On the system T, see Sobociński [7]. T is equivalent to von Wright's M (in [8]), as Sobociński shows. The result that T is decidable is not, of course, new: it is proved by Anderson, using quite different techniques, in [1]. But that T, like S2, S4, and S5, has the finite model property seems to be a new result.

6. As in [2], the correspondence is the *dual* of that in e.g. [4].

7. Cf. [6] pp. 4-5

8. Cf. [6] Thm. 1.4.

9. Cf. [6] Thm. 2.2.

10. Cf. [5] Thm. 4.8.

11. Cf. [5] Thm. 5.7

12. Cf. the definition of order closure algebra in [2].

13. Cf. [2] Lemma 9.

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