Notre Dame Journal of Formal Logic
Volume VIII, Numbers 1 and 2, April 1967

THE CONTRIBUTION OF LEŚNIEWSKI

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The present paper* aims at giving an account of the logical work of Stanislaw Leśniewski. Many other papers, as well as a book, are available, which treat Leśniewski and his work. However, I feel that another paper is called for. None of the articles presently available gives a satisfactory account of what Leśniewski did and why he did it. And the book, *The Logical Systems of Leśniewski,* by E. C. Luschei, which is a complete account of certain aspects of Leśniewski's work, does not make it easy for a person who knows little or nothing about Leśniewski to appreciate Leśniewski's work. The present paper attempts to give a brief, sympathetic, and relatively complete account of Leśniewski's work. What Leśniewski did and his reasons for doing it are both interesting and important—important enough to justify still another paper these many years after his death.

Leśniewski, a Polish logician, died in 1939. He did an enormous amount of research in formal logic, and seems to have had considerable influence in his own country; outside of Poland, little is known about his work. The chief reason for this is that Leśniewski left very little material in print, and what has been published is very abstract and technical. Some explanation of the scarcity of Leśniewski's printed word is afforded by the fact that he was a perfectionist. He would not publish anything that was not just right. He probably intended to present a systematic exposition of his work, but was prevented by his death. And his unpublished results, which were to have been edited by Sobociński, were destroyed during the war.[1]

It is unfortunate that Leśniewski's work is not better known, because it reveals insight and originality. Leśniewski's logical research was motivated and informed by his views about the nature of language and of the world. While these views may not be entirely satisfactory, some of them are very persuasive. The formal systems which Leśniewski devised contain the outlines of a language capable of talking (efficiently) about the

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***Portions of this paper were originally contained in a dissertation submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy at Yale University in 1962.**

Received March 6, 1965

world as Leśniewski saw it. One of the most significant features of these systems is that no provision is made for terms designating (representing, standing for) abstract entities. Should it be the case that there are abstract entities, then it is possible to talk about them using language based on Leśniewski's formal systems. But these systems contain no category, or categories, of expressions which are specifically intended to designate abstract entities.

The relation between Leśniewski's views about the world and his treatment of language which is used to represent the world is very important. This relation, in part, gives Leśniewski's work its continuing interest. The logical systems of Leśniewski are not directly relevant to work being done today. What is relevant is the way that Leśniewski regarded formal systems and the way that he used formal logical systems to express his philosophical views. All too often the logician treats language in isolation from its function, as if language consisted entirely of marks and sounds. It is not possible to understand the relation of one proposition to another or the validity of logical laws, unless one takes account of the meaningful use of language—Leśniewski certainly did this.

WHAT LEŚNIEWSKI THOUGHT HE WAS DOING Leśniewski's view of formal logic (and also mathematics) might be characterized as intuitive formalism[2]—but this does not mean that he is either an intuitionist or a formalist. In his formal systems, Leśniewski considers himself to be formalizing intuition. He writes

> I should not have taken such pains over systematizing and repeatedly checking the rules in my system, if I had not attached to the theorems a certain quite definite, just this and not different, meaning, by virtue of which the axioms of the system and the methods of definition and inference codified in the rules possess for me an intuitive, irresistable validity. I do not see any contradiction in saying that for this reason I practice a rather radical "formalism." I do not know any effectual method of conveying my "logical intuitions" to the reader but the method of formalization of the deductive theory in question, which, however, under the influence of formalization does not cease by any means to consist of clearly meaningful propositions possessing for me an intuitive validity.[3]

Formalization, then, serves to clarify intuition; it neither replaces it nor precedes it.

There may be some question as to what should be understood by 'intuition.' To say that something is intuitively satisfactory is to characterize it very vaguely as being in accord with common sense (perhaps with the "common sense" of the learned). But Jordan states that Leśniewski "believed in the absolute truth of some assumptions and formed his opinions of different systems from this point of view."[4] It would be peculiar if the "absolute truth of some assumptions" consisted in their agreement with common sense. Leśniewski's intuition must be knowledge of some sort (at least he must have thought that it was), but knowledge of what? Perhaps we can say that Leśniewski's intuition is knowledge of the way that the world is put together. For, according to Kotarbiński, Leśniewski's system of Ontology is a "theory of what there is, or general principles of being."[5]

However, this cannot be a satisfactory account of Leśniewski's intuition. Bocheński says that "According to Leśniewski, the phenomenal world is composed exclusively of individuals and contains no properties at all."[6] But Leśniewski's formal systems make provision for terms that can be predicated of more than one object—the intuitions that he formalizes are not simply intuitions about the make-up of the world.

Leśniewski's intuition is best described as knowledge of how language must be if it is to adequately and efficiently represent the world. His intuition is knowledge of the way the world is put together but it is also knowledge of the appropriate way to represent (describe) the world. Leśniewski's intuition is not knowledge of how this or that language might be constructed, although this is how Luschei regards it. For Luschei writes,

> I find it difficult clearly to distinguish.. [Leśniewski's view that he is formalizing intuition].. from the verbally contrasted or "opposed" view that the rules of a formalized system of logic are valid and its theses "true" by convention or definition, being prescribed or asserted in order to establish the grammatical and inferential rules and the categorial and structural framework of any language based on the logical system in question.[7]

In his formal systems, Leśniewski does not think that he is proposing a language or some languages. Instead he regards himself as presenting the outlines for all languages used to talk about the world. In describing the world, physicists will use different terms than chemists, but these terms can be fitted in (introduced) to Leśniewski's formal systems. Leśniewski's formal systems constitute a basis for language used to talk about the world, for these systems enable us to recognize (and describe) those entities which are genuinely constituents of the world without tempting us to admit unreal or fictitious entities.

Leśniewski's reliance on intuition is seen in his treatment of the antinomies which occurred in logic and the foundations of mathematics. He rejects the idea that the contradictions arise from the incompatibility of different intuitions. Leśniewski feels that the antinomies are due to the failure of the contradictory systems to adequately express intuition. "The only realistic method of solving the antinomy is the method of intuitive probing of reasoning or assumptions upon which the contradiction is based."[8] The way to eliminate the antinomies is to understand why they arise. After subjecting the concepts of class and set to an analysis, it is Leśniewski's conclusion that these are not genuine concepts. They result from confusing quite distinct ideas. Leśniewski does not maintain that the customary logical notion of a class is a fiction in the sense that classes might exist, but just happen not to—he claims that the very idea of a class is unintelligible.[9] Leśniewski states that it was after he made this discovery that he "stopped seeing the antinomy as constructed by Mr. Russell."[10]

In attempting to formalize intuition rather than to devise just any sort of system which "works," Leśniewski is choosing to understand rather than simply to invent. His chief concern with formal logic comes from his belief that this must provide a basis for mathematics. Leśniewski rejects the idea that mathematics consists in a group of

> non-contradictory deductive systems which insure.. the possibility of obtaining on their basis a never-ending abundance of new theorems, but which distinguish.. themselves by the lack of any intuitive scientific connection with reality.[11]

A non-intuitive formal system, arbitrarily restricted in order to avoid contradiction, is unsatisfactory from Leśniewski's point of view. Zermelo's set theory is just such a system; Leśniewski says that the question as to whether or not such a theory is consistent is an "indifferent" one.[12] For "non-intuitive mathematics does not have within itself effective remedies for the shortcomings of intuition."[13] Leśniewski contends that formal mathematical systems, although they may contain postulates having a non-logical character, must be based on logical systems. For mathematical systems must contain propositions and proceed by deduction. The language of mathematics is genuinely language, and so its formulas must have meaning. In his work, Leśniewski attempts to provide an adequate and meaningful basis for mathematics, one which does not require the admission of such unsatisfactory entities as sets and classes. In this paper, little will be said about the adequacy of Leśniewski's logical systems as foundations for mathematics. It is sufficient for my purpose that I have indicated that Leśniewski's reason for constructing formal systems was to provide a satisfactory basis for mathematics. Leśniewski feels that mathematical language must be language capable of describing the world. In order to provide a foundation for mathematics, it is necessary to have a general understanding of the way the world is and also to understand what is the most effective way for representing (describing) the world with language. Leśniewski's views on mathematics and the relation of mathematics to logic are not the primary concern of this paper. We shall concentrate on Leśniewski's views about the world, and the expression that these views find in his formal systems.

The quotation from Bocheński found above is a correct statement of Leśniewski's view of the world. Leśniewski holds that there are only objects, or individuals, and not properties or classes. In order to understand his view, it is necessary to consider his formal systems—for these formalize his intuitions.

THE FORMAL SYSTEMS Since Leśniewski's formal logical systems are designed to express his intuition, these systems must be meaningful—and meaningful from the start. He does not begin with a set of marks and rules for their manipulation, and then assign a suitable interpretation. An uninterpreted system is not logical at all, for logic deals with meaningful statements and their relations. The constants in Leśniewski's systems have meaning from the first. Their meaning is not conferred upon them by the axioms, rather it is the meaning they already have that makes the axioms true. Of course, we understand the meaning of these terms best by considering how they are employed in the axioms; this is the reason why

Leśniewski thinks that it is useful to formalize intuition. We understand what these terms mean by considering how they are used—Leśniewski has established his systems to present us the outlines of the language to be used for talking about the world.

Leśniewski's philosophical view is a kind of nominalism. His systems are intended to be compatible with his nominalism, and even to express this nominalism. Leśniewski's three most important systems are Protothetic, Ontology, and Mereology. Of these, only Protothetic and Ontology are, properly speaking, logical systems. The constants introduced by Mereology are terms used for relating objects in the world; they do not affect the grammatical structure of language (in contrast, the fundamental terms of a logical system are terms that bring with them a new grammatical-syntactical structure). Protothetic and Ontology are not dependent upon Leśniewski's nominalism; however, they do not require that any entities other than concrete objects be admitted (they do not require us to recognize abstract entities). Mereology, which deals with the relation of part to whole, is a thoroughgoing expression of Leśniewski's nominalistic position.

To properly understand Leśniewski's formal systems, we must understand how he sets them up. Leśniewski's work is characterized by meticulous formalization, and his systems are constructed according to almost unbelievably elaborate rule-statements, called Terminological Explanations (Terminologische Erklarungen).[14] These Terminological Explanations begin by defining terms that can appear in formulas of the logical systems, and end by describing formulas that can be added to the system as theses at any point in its development.[15] There are not two sets of rules, those for formation and for transformation (i.e., those determining well-formed formulas and those determining theorems), but only one set. The forty-nine Terminological Explanations for Protothetic and the fifty-seven for Ontology define terms necessary for stating the rules of inference. These Terminological Explanations do not contain a definition of a well-formed formula, although this is possible.[16] Leśniewski's systems have this peculiarity, however, that when a well-formed formula is defined, it must be defined relative to the stage of development of the formal system. For definitions are genuine theses of these systems, and statements containing defined terms can be counted as well-formed only following the definitions of these terms. In Leśniewski's systems, then, the formation and transformation rules are inter-dependent. For the rule of definition is a transformation rule (a rule of procedure) of Leśniewski's systems.

Leśniewski's Terminological Explanations (which from now on I shall call T.E.s) accord well with his nominalism. For the T.E.s do not set up a system which is a kind of ideal entity. On Leśniewski's view, there is no such thing as *the* system of Protothetic. There are as many systems of Protothetic as are actually produced (although it is often convenient to speak of Protothetic as if it were just one system, and I shall usually do this). The T.E.s present rules for systems whose expressions are written on paper (or on a blackboard, or whatever). The T.E.s talk about tokens rather than types. Leśniewski's T.E.s are like blueprints—they present the rules for as many systems as one cares to construct. For Leśniewski, a formal system exists in space and time, and it contains just those results which have actually been deduced. This means that there are no unproved theorems of a formal system. However, this claim does not make proofs of completeness and consistency either trivial or senseless. These proofs must be understood as proofs about all possible well-formed formulas or theorems, rather than as proofs about this or that ideal system. Any problems connected with potentiality or possibility are different from those which result from admitting abstract entities. To say that every well-formed formula which can be constructed will either be provable or contradictory is not to say anything which commits us to recognizing *the* formal system containing *the* theorems.

The T.E.s which are used to set up Leśniewski's formal systems are themselves very formal. Each has the form of a definition, defining a technical term used to describe (or designate) expressions which occur in the formal systems. The T.E.s do not refer to the meanings of the terms in the system—these terms are characterized by means of their shapes and positions. The fact that the formal system is set up syntactically does not mean that it is uninterpreted; for its interpretation determines how the system is to be set up. The T.E.s should not be considered as expressions in a formal system, they are statements in the language used by Leśniewski. But the T.E.s make a great deal of use of expressions which do not occur in everyday language. Some of these expressions are defined by the T.E.s, but some of them are taken over from his systems Protothetic, Ontology, and Mereology. Leśniewski uses connectives from Protothetic and Ontology in his T.E.s, for he feels that these terms are genuinely expressions of language, even though he has introduced them into language. The T.E.s are stated in "ordinary" language, although this language is abbreviated symbolically. The symbols used in the T.E.s have the same meaning as they do in the formal system, even though the language of the formal system is not the same as the language we ordinarily speak. For we can use terms with the same meaning both in the formal system and our ordinary language (or, if someone prefers to say it this way: in the object language and the meta-language).

The fact that Leśniewski uses the symbols (and hence the concepts) from Protothetic and Ontology in the T.E.s which formally set up these systems indicates his belief in the fundamental character of the concepts with which these systems deal. Leśniewski also employs the concepts from Mereology, although he does not borrow its symbolism.

It will be helpful to consider in more detail how the T.E.s proceed. The first thing that Leśniewski does is to write down the axioms of the system being constructed. The axioms are the expressions written down, the marks that we can see. Leśniewski refers to the terms occurring in the axioms in order to carry out definitions in the T.E.s. For example, he defines the expression 'left lower corner' with a T.E. roughly equivalent to

> A is a left lower corner if and only if it is equiform with the first word of the axiom of Protothetic.[17]

The first word of the axiom of Protothetic is equiform (i.e., is shaped in the same way as) to: L. After defining general predicates for the different kinds of symbols occurring in the axioms (their kind is determined by their shape), Leśniewski goes on to more elaborate definitions. When he comes to these T.E.s, the fundamental idea of Mereology is very important. T.E. 6 (for Protothetic) is equivalent to

> **A is a word inside B if and only if B is an expression and A is a word in B, neither the first nor the last.**[18]

This T.E. makes use (in the definiens) of the idea of being in something; and this idea, as expressed in Mereology, is capable of being the primitive concept of Mereology. T.E. 7 is equivalent to

> **A is the complex of the b if and only if**
> 1 ***A*** **is an individual expression (a token),**
> 2 **(*C*) (If *C* is a word in *A*, then (∃*D*)(*C* is in *D* and *D* is *b*)),**
> 3 **(*CDE*) (If *C* is *b* and *D* is *b*, and *E* is a word in both *C* and *D*, then *C* is the same individual as *D*),**
> 4 **(*C*) (If *C* is *b*, then *C* is in *A*).**[19]

The complex of the b ('b' is a variable for which a predicate term—adjective or common noun—can be substituted) would be regarded as a class in Mereology. For the class of the b is simply the object composed of all the b and nothing else. A complex is one kind of class, for the mereological requirements are strengthened by condition 1, which requires that A be an expression. In Mereology, any objects at all can compose a class—there is no requirement that they be adjacent to one another, or structured in any way. Condition 2 ensures that A contains no word (symbol) which does not satisfy the defining condition of the complex (i.e., none which does not belong to some b). Condition 3 requires that the b be discrete; this is a condition not imposed upon all mereological classes, for any two pieces of a mereological class are members of the class, whether or not they overlap. And condition 4 is needed so that no b's occur outside of A.

By employing the concepts of Mereology, Leśniewski is able to avoid referring to different contexts by means of ellipsis (e.g., '. . . x . . . '). For with the aid of the term 'complex' and the idea of being in something, there is a general means of characterizing any expression which is a component of a larger expression. The T.E.s also avoid recursive definitions; all of them are explicit.

The T.E.s begin by defining general terms for the different symbols in the axioms. After this the T.E.s define quantifiers, variables, functors, functions, and so on. Then the T.E.s needed to handle semantical categories (Leśniewski's version of a theory of types) are introduced. These are very complicated, but once he has established the means for talking about semantical categories, he is able to introduce his rules of procedure. These are the last of the T.E.s, for they are the goals of all the preceding T.E.s. Those T.E.s which present the rules of procedure define a thesis as belonging to the system. Like all the T.E.s these last ones define predicates which apply to individual, actually occurring expressions. Something is a thesis only if it is a visible expression, fulfilling certain conditions.

It was remarked earlier that Leśniewski does not think that there is such a thing as *the* system of Protothetic or *the* system of Ontology. There are potentially an infinite number of these systems.[20] Even if two systems contain the same theses in the same order, they are said to be different systems. Their sameness consists in a similarity of form (their theses may also say the same things). The T.E.s allow one a great deal of freedom in developing the systems whose rules they express. Trivially, all kinds of symbols may be introduced by definition, and unusual marks may also be used as variables (since there are no so-called free variables in any thesis; variables are characterized formally rather than being chosen from a prescribed alphabet). All that is necessary is that different constants and variables should be distinguishable from one another. In a similar fashion, parentheses of different shapes may play analogous roles in two different systems of Protothetic, or Ontology. For differently shaped parentheses are used to indicate different semantical categories—the parentheses are also introduced by definition. In addition to allowing a great deal of choice with regard to the symbolism employed, the T.E.s allow systems whose order of development is quite different. And because definitions are genuine theses of the system, the different order of development can make quite a lot of difference between two systems of Protothetic, or of Ontology. For a definition in one system may contain a defined term in its definiens, which term is defined in a second system with the aid of that symbol it is defining in the first. Clearly, *the* system of Protothetic or Ontology is a fiction, if this is understood as implying the existence of some unique individual.

There are three main systems, or groups of systems, developed by Leśniewski; Mereology, Ontology, and Protothetic. The relations whose expressions are formalized in Mereology are relations which hold between objects in the world; the logical systems (Ontology and Protothetic) present only general linguistic forms. The three systems present the outlines for language used to describe the world, and they serve as a foundation for mathematics. Of the three, Mereology was developed first, then Ontology, then Protothetic. But their logical order is the opposite of this historical one. For the theses of Protothetic are needed in the proofs of ontological theorems, and the theses of both Protothetic and Ontology are used in mereological proofs. The theses of Protothetic are relevant to all reasoning with propositions, and those of Ontology to reasoning concerning individuals.

Mereology apparently was constructed first because Leśniewski came to the study of logic from considering the paradoxes. Once he realized that confusion about sets and classes was the source of the contradiction, he set out to formulate an intuitively satisfactory substitute for the theories dealing with these. In formulating Mereology, the need for a system upon which it can be based became evident—Ontology provides this, but requires, in its turn, a further system on which it can be based—this is Protothetic.

Protothetic is the most fundamental of Leśniewski's formal systems. It can be thought of as a very elaborate system of propositional calculus with quantifiers. Protothetic is not, however, equivalent to propositional calculus with quantifiers in any straightforward sense. For Protothetic contains this as a proper part. Protothetic has functors which take propositions as arguments, and functors which take these functors as arguments, and so on. It is possible to get from any high-level statement of Protothetic to one which is very roughly equivalent and which contains only propositional variables and functors taking these as arguments. But such a reduction may give rise to an incredibly long statement—and because of the possibility of introducing multi-link functors (which are explained below) the "equivalence" is very rough indeed. Perhaps we should say that any proposition of Protothetic can be reduced to a formula of propositional calculus (since propositional calculus with quantifiers can be reduced to the ordinary system), which formula is decidable in that system. But such a reduction (which is really an expansion) is too cumbersome to be of any use.

RULES OF PROCEDURE The rules of procedure in Leśniewski's systems are the end products of the T.E.s. For Protothetic, there are five such rules, or, as Sobociński prefers to say, "one rule of procedure divided into five points."[21] Ontology contains seven rules; it contains the same rules as Protothetic, but needs additional rules to take account of the new propositional (sentential) form which Ontology introduces. The precise form which the rules of procedure take depends upon the axioms that are used. Leśniewski elaborated several systems of Protothetic, having different axioms (having axioms with different forms), and even different primitive terms. He used both implication (material implication) and equivalence as primitive terms for different systems, but he seems to have preferred equivalence, probably because the rule of definition is simpler in a system based on equivalence. The five rules for Protothetic are that of partition of the quantifier, substitution, detachment, definition, and the rule for introducing theses of extensionality. The first rule, that of partition of the quantifier, is needed to carry out detachment, which operates in a customary way (except that in some systems, the principal connective is equivalence and not implication). Partition allows one to distribute the terms in the initial universal quantifier into two quantifiers, one before each argument of the principal connective when this connective is equivalence, (each quantified variable is not enclosed in separate parentheses—all the variables are enclosed by one pair). Partition of the quantifier makes possible the move from

$$(pqr).p \supset (q \supset r) \equiv .p \;\&\; q \supset r$$

to

$$(pqr)(p \supset (q \supset r)) \equiv (pqr)(p \;\&\; q \supset r).$$[22]

The rule of substitution allows substitution only for variables bound by the initial (universal) quantifier, whose scope must be the entire expression. In practice, Leśniewski does not prove theorems by substituting, partitioning the quantifiers, and detaching, although such a procedure is what the T.E.s authorize. He uses instead the method of suppositional proofs. This is a natural deduction technique, where the premises are written first and the conclusion derived in a number of steps. Leśniewski regards this method as an abbreviation for the lengthier process, and so he does not set it up formally—its equivalence with the more tedious method is clear.

The rule of definition is a very important rule in Leśniewski's systems. He agrees with Whitehead that

> The definitions—though in form they remain the mere assignment of names, are at once seen to be the most important part of the subject. The act of assigning names is in fact the act of choosing the various complex ideas which are to be the special object of study. The whole subject depends upon such a choice.[23]

Leśniewski feels that definitions are important enough to be regarded as theses of a system, and not merely as abbreviations. Leśniewski's systems contain definitions, and these definitions serve to extend the systems. The definitions make it possible to prove theorems which could not otherwise be proved. According to one terminology, this means that Leśniewski allows creative definitions, while according to another, Leśniewski's definitions are fruitful rather than creative.[24] I shall say that they are creative. Even though definitions are required for certain proofs, all of the terms introduced by definition are eliminable—any statement containing the defined term can be replaced by one which "means the same thing" but which does not contain the defined term.

The final rule of procedure of Protothetic is that for introducing theses of extensionality. For every semantical category already belonging to the system, it is possible to add a thesis of extensionality.[25]

SEMANTICAL CATEGORIES Expressions in Leśniewski's systems belong to different semantical categories. These are the analogues of the logical types found in other systems.[26] Semantical categories must be distinguished in Protothetic as well as in Ontology. This is because of the occurrence of terms other than propositional variables and constants, singulary and binary connectives. Leśniewski uses the expression 'semantical category' because it suggests the idea of a grammatical distinction. Terms belonging to different categories have different uses, and cannot sensefully be interchanged. Just as in ordinary speech we cannot put verbs in place of nouns, so we cannot substitute a term in one category for that in another (and so we cannot substitute a functor for its argument). Leśniewski states that he would feel obliged to accept (or adopt) his theory of semantical categories even if no antinomies had been turned up.[27]

Every term, with the exception of quantifiers and brackets, and every complete expression belong to a definite semantical category. The axioms of the systems contain terms belonging to primitive categories, and new categories are introduced by defining terms which belong to these categories. Distinct semantical categories are signalled by differently shaped parentheses. For example, all functors having the same number of arguments enclosed in parentheses of the same shape, where the arguments belong to the same (respective) categories, belong to the same category. This adds an aesthetic feature to Leśniewski's systems, but quickly becomes cumbersome.

In Protothetic, the fundamental category is that of propositions. Then there are proposition-forming (sentence-forming) functors which take one propositional (sentential) argument, there are those which take two propositional arguments, and so on. There are also proposition-forming functors which take one proposition-forming functor as argument, those which take two, and so on. We can also introduce proposition-forming functors all of whose arguments do not belong to the same category. Ontology contains the same categories as Protothetic, but introduces the category of names (nouns). Then there are name-forming functors which take name-arguments, proposition-forming functors which take name arguments, and other combinations as well. But even with all these categories, there is another important kind of functor that has not been considered. All functors are not either name-forming or proposition-forming. There are also functors which are used to form other functors.[28]

This last variety of functors is a very important feature of Leśniewski's systems. The functor-forming functors are parts of what are called multi-link functors. An example of a multi-link functor that might be defined is

$$(qp).\mathbf{Eqv}\{q\}(p) \equiv .p \equiv q.$$

Here '**Eqv**' is a functor-forming functor for one propositional argument. '**Eqv**$\{q\}$' is a proposition-forming functor for one propositional argument. Thus multi-link functors are parametric. The multi-link functor '**Eqv**$\{q\}$' belongs to the same semantical category as '$\sim$'—it may be substituted for a variable of this category. These multi-link functors explain the statement made earlier that a translation of a protothetical statement into ordinary propositional calculus may fail to produce a genuinely equivalent statement. There are as many functors of any given category as one cares to introduce; making a translation into ordinary propositional calculus will collapse many distinctions that are important to Protothetic.

Multi-link functors make possible a rule of substitution which is much simpler than a similar rule without them. Leśniewski's rule of substitution permits the replacement only of variables bound by the initial quantifier (whose scope is the entire thesis) of a thesis. In *Principia Mathematica* (which I will call **PM**), the following theorem

$$*13.13 \vdash : \Psi x . x = y . \supset . \Psi y,$$

is used to deduce

$$*14.14 \vdash : a = b . b = (\imath x)(\phi x) . \supset . a = (\imath x)(\phi x).$$

To justify this inference, it is necessary to construe the 'Ψ' of *13.13 as '$= (\imath x)(\phi x)$,' which is an incomplete expression. Leśniewski's rules do not permit such a substitution, but multi-link functors make it possible to achieve the same result. Suppose we have

$$(fpq).f(p) \ \& \ (p \equiv q) \supset f(q),$$

and would like to derive

$$(pq).p \supset 0 \ \& \ (p \equiv q) \supset (q \supset 0).$$[29]

With the following definition,

$$(pq).\mathbf{Imp}\{q\}(p) \equiv .p \supset q,$$

we can get, by substituting in the original formula,

$$(pq).\ \mathbf{Imp}\{0\}(p) \ \& \ (p \equiv q) \supset \mathbf{Imp}\{0\}(q).$$

From this we can derive

$$(pq).p \supset 0 \ \& \ (p \equiv q) \supset (q \supset 0)$$

in a straightforward fashion.

Although the number of categories to which expressions of Leśniewski's systems might belong is enormous (potentially infinite), relatively few categories are actually introduced. The use to which expressions of high-level categories might be put is difficult to determine. In Protothetic, for example, the higher level theorems are of little value, because any reasoning (inferring) which can be represented by, or based on Protothetic is likely to involve only the lowest level theses (those containing singulary and binary proposition-forming functors for propositional arguments--the theses which could be satisfactorily expressed in propositional calculus). It is possible to use high-level statements to serve some metalinguistic purposes, since it is in some respects arbitrary whether we employ a hierarchy of types (categories) or of meta-languages. Protothetic, for example, contains some theses which mean the same as theorems which must be proved in a meta-language with respect to propositional calculus.[30] However, the extensional character of Leśniewski's systems restricts the possibility of talking, in these systems, about the use of an expression of the system (and it is completely out of the question to talk about characteristics of the individual expressions). In Protothetic, high-level statements must ultimately be based on propositions where only truth or falsity is relevant. If two functors produce true statements for exactly the same arguments, these functors must be regarded as having the same meaning—it is not possible to consider different aspects of use. In Ontology, extensionality requires that any expression must be considered only in terms of those existing objects to which it can be related. We can say (in the formal system) of a noun expression, for example, that it is non-empty, or that objects of which it is predicated are also objects of which something else can be predicated, or that it can be predicated of a certain number of objects. But we cannot talk about its use to indicate the presence of a certain feature which can be described independently of our knowing whether or not any objects are characterized by this feature. Thus extensionality limits the adequacy of Leśniewski's formal system, where adequacy might seem to have been guaranteed by the large number of categories (however, being limited is not the same as being defective).

QUANTIFIERS It has already been mentioned that Leśniewski's systems do not contain the so-called free variables. Thus he did not have to resort to such explanations as

> *Any term* is a concept denoting the true variable.... This is just the distinctive point about *any*, that it denotes a term of a class, but in an impartial distributive manner, with no preference for one term over another.[31]

Or as "The notion of ambiguous assertion is very important, and it is vital not to confound an ambiguous assertion with the definite assertion that the same thing holds in *all* cases,"[32] where ambiguous assertion is what we have "when we assert *any* value of a propositional function."[33] It is clear that any system whose theorems contain free variables can be reconstructed in such a way that all variables are bound—and the reconstructed system will contain theorems which mean the same as those in the original system. A senseful formula (one which is, or makes, a statement) which contains free variables can only be understood by construing those variables as bound by a universal quantifier standing at the beginning of the formula. Without such a quantifier, there can be statement forms, but no statements. For variables do not have meaning—they replace expressions which do. Variables are used to make general statements about all statements having a certain form; in order to make such a statement within the formal system, the variables must be bound.

In a system based on Leśniewski's T.E.s, there are no existential quantifiers. However, in his informal exposition, he does use the existential quantifier to abbreviate the universal quantifier with negation on both sides of it. The reason why Leśniewski does not provide for the use of existential quantifiers by means of his T.E.s is that he sees no reason to give priority to the expression 'For at least one...' over 'For at least two...,' 'For at least three...,' etc. He tried to devise a rule that would allow the introduction of an expression for each of these, but he did not succeed in this.[34] It would be a relatively simple matter to expand the T.E.s to include rules for the existential quantifier, but such rules would not be sufficiently general.

In Leśniewski's systems, vacuous occurrences of terms are not permitted in the quantifiers—an expression containing such a vacuous occurrence is not a proposition (sentence, well-formed formula). And there is not a separate quantifier for each bound variable; the quantifier is a "box" containing expressions equiform to all the bound variables. The terms in a quantifier are not considered to be "ranging" over objects in the world. Leśniewski does not read the universal and existential quantifiers '(x)' and '$(\exists x)$' as "'everything x is such that' and 'something x is such that.'"[35] In Protothetic, the temptation to regard the quantified variables as talking about objects in the world is perhaps not so great as it is in Ontology, which formalizes talk about individuals. But there are surely those who would regard the terms in a quantifier in a protothetical thesis as ranging over propositions which are supposed to be ideal entities, not ingredient in space and time. This is not Leśniewski's view; we read that

> in times when I did not know how to operate by means of quantifiers, but in the colloquial language which I used needed something to correspond to expressions of types "(∃a). f(a)," "(∃X, a). f(X,a)," etc., which are expressions of the symbolic language, I used corresponding expressions of the type "For some significant word "a," f(a)," "For some significant words "X" and "a," f(X,a)."[36]

The universal quantifier '(x)' would, according to this scheme, be read "For every significant (senseful) expression 'x,'" or "For any senseful expression 'x.'" The existential quantifier can be read with 'some' or it might be read "There is," but it must not be construed as asserting that some non-linguistic object exists—it simply means that an expression can be formed.

There appear to be difficulties with Leśniewski's view of quantifiers, for it seems to require that we regard quantified formulas as statements about statements (i.e., as meta-linguistic statements). It is true that, putting it in English we would read

$(x).(x$ is a man$) \supset (x$ is mortal$)$

as

For every significant expression 'x,' if x is a man, then x is mortal.

or, perhaps,

for every expression 'x' which makes sense of it, the following will be (or make) a true statement: if x is a man, then x is mortal.

But the English statement should not cause us to misinterpret the quantified statement. The quantified formula of the formal system is truly a general statement of that system. From

$(x).(x$ is a man$) \supset (x$ is mortal$)$

and

John Smith is a man.

we can infer

John Smith is mortal.

We have not moved from a meta-language to an object language, or vice versa. The fact that we cannot say certain things in English in the manner that they can be expressed in formal systems constitutes one of the advantages of formal systems.

A variable is not a word, it takes the place of words (or longer expressions). Variables are used in forming the statements in a formal system; they do this by replacing certain expressions in order to call attention to the general forms of other (and longer) expressions. Variables must be viewed as gaps—we put letters in place of the gaps to indicate that certain gaps are to be filled by equiform expressions (the "same" expressions). One can say that

$$(x + y)^2 = x^2 + 2xy + y^2$$

is true for any numbers x and y, but to get from the preceding formula to

$$(3 + 5)^2 = 3^2 + 2\cdot 3\cdot 5 + 5^2,$$

we must substitute numerals, not numbers. Quantifiers are needed to form statements in situations where variables are used. They do this by indicating to what extent statements of a certain form will be true (for any senseful expression replacing the variables or for at least one).

There is a view which holds that a bound variable "is rather like a pronoun,"[37] and this view is incompatible with the one outlined above. The pronoun view maintains that "The quantifiers '$(\exists x)$' and '(x)' mean 'there is some entity x such that' and 'each entity x is such that.!"[38] Quine claims that "In general, *an entity is assumed by a theory if and only if it must be counted among the values of the variables in order that the statements affirmed in the theory be true*."[39] Now it is unclear what is the proper way of understanding 'values of variable.' Quine surely thinks that these values are the objects which the variables "denote." But Leśniewski's interpretation of quantifiers sees the quantifiers as making statements about expressions which can be substituted for the variables. These expressions might be called the values of the variables. But this does not require us to hold that these expressions are ideal entities. Quantifiers enable us to make general statements from which particular statements can be inferred. Quine is nearer to Leśniewski's view when he discusses schematic letters. He says, "There is no need to view statements as names, nor to view 'p', 'q,' etc. as variables which take entities named by statements as values."[40] The reason for this is that

> 'p,' 'q,' etc. are not used as bound variables subject to quantifiers. We can view 'p,' 'q,' etc. as schematic letters... and we can view '$[(p \supset q).\sim q] \supset \sim p$,'... not as a sentence but as a schema or diagram such that all actual statements of the depicted form are true.[41]

If we use theorems of formal systems as diagrams "such that all actual statements of the depicted form are true" we are using them to make statements about statements. If we are using them to make statements about all statements having a certain form, then we might as well have bound the variables with an initial universal quantifier. Quantifiers do not require that we hypostatize abstract entities, or that we recognize any already existing entities which make the quantified statement true (if the variables in question are name variables, then the truth of some statements which contain these will depend indirectly upon the existence of certain objects). The quantifier, on Leśniewski's view, is a device which enables us to make statements from expressions which contain gaps.

ONTOLOGY Ontology, or the calculus of names, is added to Protothetic. The theses of Protothetic are used in the proofs of many of the theorems of Ontology. There are a separate set of T.E.s for Ontology, but many of them are the same as those for Protothetic. Those which are different are different in virtue of the new propositional (sentential) structure involving terms belonging to the category of names joined by the primitive term 'ε'. All of the rules of procedure for Protothetic are rules for Ontology as well. In addition, Ontology has two rules which are peculiar to this system: a rule of definition and a rule for adding theses of extensionality.

The 'ε' of Ontology is not the customary sign of class membership. If someone chooses, with Sobociński, to talk of a distributive class, he can speak of class membership; but

> If one takes the term "class" in this sense, the formula "$A\varepsilon \mathrm{Kl}(a)$" signifies the same thing as "*A* is an element of the extension of objects *a*," which is to say, more briefly: "*A* is *a*."[42]

The so-called distributive class is not an object at all, hence there is no reason for talking about it. The 'ε' is to be read "is" in the sense of "Socrates is white," "The current president of the United States is a Democrat," and "James is James." This sense of 'is' requires that the subject of the sentence be the name of an existent individual if the sentence is a true affirmative one. It does not matter whether the term following the 'ε' is a general term (e.g., 'red') or an individual name (as above, 'James'). The existential import of the 'ε' is what prompted Leśniewski to call this system Ontology.

In Ontology, the distinction between adjectives and nouns is collapsed. For the sentences 'James is tall.' and 'James is a man.' are considered to be exemplifying the form

$$A\varepsilon b.$$

In this respect Ontology is more similar to languages like Latin or Polish than to English. For the former two have no articles—though it is possible to distinguish nouns from adjectives.

The axiom of Ontology introduces two semantic categories, the category of names and the category of proposition-forming functors for two name arguments ('ε' belongs to this category). It is an important feature of Ontology that individual and general names (proper and common nouns) belong to the same category. This means that a general name can significantly precede the 'ε,' although such a statement will be false. It should be noted that Leśniewski uses capital Latin letters as variables for individual names, and small letters for general names. But this is an informal, heuristic device, which could be abandoned without changing the system. The original axiom of Ontology is

$$(Aa).(A\varepsilon a) \equiv .(\exists B)(B\varepsilon A)\&(BC)((B\varepsilon A)\&(C\varepsilon A) \supset (B\varepsilon C))\&(B)((B\varepsilon A) \supset (B\varepsilon a)).$$

Sobociński devised the following shorter axiom

$$(Aa).(A\varepsilon a) \equiv (\exists B)((A\varepsilon B)\&(B\varepsilon a)),$$

but I will discuss the original axiom, because it is easier to understand the meaning of 'ε' in this axiom. From the first conjunct in the coimplicate (the right side) of the axiom, we can see that 'ε' has existential import. If we say that A is something, then there must be an A. The second conjunct indicates that if A is anything, then there is at most one A. And the third conjunct says that if A is something, then anything that is A is also this something. This last makes it possible to prove the principle of extensionality for names (nouns)—the rule for introducing theses of extensionality does not include the category of names. If A and B are the same object, then whatever can be said about A can be said about B.

The propositions of Ontology which contain 'ε' are existential statements (or their negations): the "existential" quantifier is not sufficient to allow us to make an existential statement. The existential quantifier does not indicate that objects exist, it only indicates that an expression can be formed which can be used in a certain way. The 'ε', however, is not used to talk about symbolism—it is the term we use when we talk about objects. In Ontology, it is not possible to prove that even one object exists. There is no thesis of the form

$$(\exists A)(A \varepsilon \phi).$$ [44]

In systems containing both free and bound variables, such a formula is usually provable, because of an axiom of substitution for bound variables. But this cannot be done in Ontology, which is surely a virtue of Ontology.

A statement of the form

$$A \varepsilon A,$$

affirms the existence of the object A. This statement is the definiens of the definition of '$\vee$':

$$(A).(A \varepsilon \vee) \equiv (A \varepsilon A).$$

'$A \varepsilon \vee$' is read "A is an object." The possibility of statements of the form '$A \varepsilon A$' minimizes the difference between 'ε' and '$\subset$,' which is defined

$$(ab).(a \subset b) \equiv (A).(A \varepsilon a) \supset (A \varepsilon b).$$

For in Ontology we have

$$(Ab).(A \varepsilon b) \supset (A \subset b)$$

and

$$(Ab).(A \subset b) \& (A \varepsilon \vee) \supset (A \varepsilon b)$$

as theses. It is still possible to distinguish between

John is mortal.

and

All men are mortal.

but the distinction is not so great as in systems where 'ε' signifies class membership.

In Protothetic, it is possible to define constant propositions (these are '0' and '1') and functors related to propositions. In Ontology, it is also possible to define names and functors related to names. There are restrictions on definitions which may be introduced, which restrictions eliminate the possibility of defining terms which could be used to derive a contradiction. These restrictions are actually conditions for the senseful use of language (in the formal system). The only restriction which I shall discuss here is that which prevents one from defining a term which would produce a form of Russell's paradox. In all name-definitions in Ontology, the defined term must occur as the predicate of the definiendum (if a name-forming functor is being defined, it must occur *in* the predicate of the definiendum). This makes it impossible to posit the existence of a named individual in the definition. The subject of the definiendum must be a variable, and it must appear as the subject of at least one conjunct in the definiens. Otherwise it would be possible to introduce the following

$$(A).(A\,\varepsilon *) \equiv \sim(A\,\varepsilon A)$$

as a definition in Ontology. From this it would be possible to deduce

$$(*\varepsilon *) \equiv \sim(*\varepsilon *).$$

Since, in Ontology, names perform roughly the same function as is performed by class terms in **PM**, this would be the closest thing to Russell's paradox that Ontology could produce. The restriction which makes the above statement inadmissible as a definition is not introduced merely to keep out the contradiction. The real trouble with the proposed definition is that the definiendum is stronger than the definiens. For the definiendum says both that A is an individual and that A is *. The definiens either denies that A is an individual, or, if A is an individual, denies that A is A. In general, a statement of the form

$$\sim(b\,\varepsilon c),$$

is true if there is not exactly one b, or if the sole b is not c. The definition of a predicate which can be predicated of individuals is a satisfactory definition only if the definiens guarantees that the names which are the subjects of the sentences whose predicates are the defined terms are actually names of individuals. This requirement that the subject of the definiendum also occur as a subject in the definiens is similar to the criterion of being an element that is used by Quine in *Mathematical Logic*. His axiom scheme ***202**

If β is not α nor free in Φ,

$$\vdash \ulcorner(\exists\beta)(\alpha)(\alpha\,\varepsilon\,\beta\,.\equiv.\,\alpha\,\varepsilon\,\mathrm{V}\,.\,\Phi)\urcorner,$$ [45]

contains '$\alpha\,\varepsilon\,\mathrm{V}$' to rule out the same kind of contradictions that are eliminated by the condition in Ontology. Indeed, from Leśniewski's standpoint, ***202** is a pseudo-definition (or definition schema).

I shall discuss some of the definitions of Ontology, and the theses which can be stated with their help. The first definition to be considered is that for '$\wedge$.' This is the analogue in Ontology of expressions for the null class in ordinary theories—it means approximately the same as 'nothing.' The definition is

$$(A).(A\,\varepsilon\wedge)\equiv.(A\,\varepsilon A)\,\&\sim(A\,\varepsilon A).$$

If the first conjunct of the definiens were eliminated, we would have the contradictory definition described above. From the definition of '$\wedge$,' it is a trivial matter to prove

$$(A).\sim(A\,\varepsilon\wedge).$$

Hence, even

$$\sim(\wedge\varepsilon\wedge)$$

may be added as a thesis. The formula above states that there is no object called $\wedge$. But it is the case that the following existentially quantified statement

$$(\exists a)(A).\sim(A\,\varepsilon a)$$

is verified by

$$(A).\sim(A\,\varepsilon\wedge).$$

This is because the existentially quantified statement does not assert that some object exists, but only that an expression can be found whose use satisfies certain conditions.

An example of a creative definition, or one which extends the system by making it possible to prove what would be otherwise unprovable is

$$(A\phi).(A\,\varepsilon\,\mathbf{stsf}<\phi>)\equiv.(A\,\varepsilon A)\,\&\,\phi\{A\}.$$

This definition makes it possible to prove the following statement of extensionality,

$$(AB\phi).(A=B)\,\&\,\phi\{A\}\supset\phi\{B\}.$$ [46]

The proof is

	$(AB\phi)$	
1	$A=B$	
2	$\phi\{A\}.\supset.$	
3	$A\,\varepsilon\,B$	1, def of '='
4	$B\,\varepsilon A$	1, def of '='
5	$A\,\varepsilon A$	3, previous theorem
6	$A\,\varepsilon\,\mathbf{stsf}<\phi>$	5, 2, def of '**stsf**'
7	$B\,\varepsilon\,\mathbf{stsf}<\phi>$	4, 6, axiom
8	$\phi\{B\}$	7, def of '**stsf**'

The definition of '**stsf**' makes it possible to express a functional statement by means of a statement containing names. If we could not make a name from a proposition-forming functor for one name argument, then the theorem could not be proved. Creative definitions make it possible to replace one statement by an equivalent statement which contains terms belonging to different semantical categories than the terms in the first.

Ontology contains the Aristotelian syllogistic. This is expressed with the terms introduced by the following definitions.

$$(ab).(a \sqsubset b) \equiv .(\exists A)(A\, \varepsilon a)\&(a \subset b);$$

this is existential inclusion—"every a is b, and there is an a."

$$(ab).(a \Delta\, b) \equiv (\exists A).(A\, \varepsilon a)\&(A\, \varepsilon b);$$

"some a is b (some a's are b's)."

$$(ab).(a \nabla b) \equiv (\ A).(A\, \varepsilon a) \supset \sim (A\, \varepsilon b);$$

"no a is b." The functor '∇' will not give the law of subalternation, because it does not indicate that either of its arguments can be predicated of some existing individuals. A new functor would have to be defined if it were desired to include a law of subalternation whose antecedent is a universal negative statement.

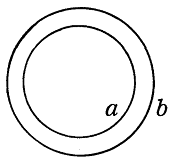
$$(ab).(a \sqsupset b) \equiv (\exists A).(A\, \varepsilon a)\& \sim (A\, \varepsilon b);$$

"some a is not b." With the terms that have been defined, it is possible to add all the valid forms of syllogism as theses of Ontology. The so-called algebra of logic (Boolean algebra) is also incorporated in Ontology. Besides the definition of inclusion, this requires definitions of '$a \cup b$,' '$a \cap b$,' 'a-b' (a and not b), '$a \div b$' (either a or b, but not both). However, these definitions are similar to the familiar ones.

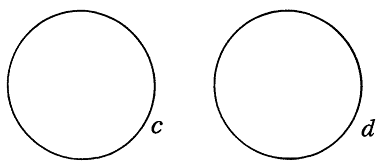
Ontology is equivalent to **PM** as a foundation for mathematics. This fact indicates that classes are not needed to accomplish the work for which Whitehead and Russell introduced them. Abstract entities can be dispensed with. Because it does not contain terms for abstract entities, Ontology seems (to me anyway) a more natural vehicle for the founding of mathematics than is the system of **PM**. Unfortunately, Ontology is equally as cumbersome as the system of **PM**. Mereology was intended to carry some of the weight in providing a foundation for mathematics, but Mereology proved to be unequal to this task.

MEREOLOGY. Mereology, the last of Leśniewski's three major formal systems, is not a logical theory, but deals with a relation which holds between objects in the world. Any collection of one or more objects is treated as an individual, no matter how distant these objects are from one another. Such objects are heaps, but not heaps that are heaped together. This is the same sense of 'individual' that Goodman intends when he says, "An individual is simply a segment of the world or of experience, and its boundaries may be complex to any degree."[47] The individuals involved need not be heaps of physical objects—whatever other objects there are can also be lumped together, whether they are events or minds.

Mereology was originally designed as an intuitively acceptable substitute for set theory. Leśniewski did not give up his claim that it is intuitively acceptable, but he did realize that it cannot replace set theory. Many theorems of set theory have analogues in Mereology, but most of these require, in their mereological form, an added conjunct in the antecedent, to guarantee that the members of the class or set in question are discrete. For it is possible that one individual will overlap with another, or be contained in another, and when this happens, the number of different collections (sets) containing these individuals is reduced. As an example of this, consider the circular discs in the following illustration.



There are two discs, *a* and *b*. The smaller disc, *a*, is contained in *b* as a proper part. Because of this, there are only two individuals which the discs constitute. There is the disc *a* and the disc *b*. The individual composed from *a* and *b* is no different from *b*. In the following,



there are also two discs, but they do not overlap at all. Consequently, three distinct individuals can be composed from these discs. There is the disc *c* and the disc *d*, and the individual composed from the two of them. It is because individuals which are parts of other individuals cannot be kept distinct from these others that Mereology is weaker than set theory. Most of the important theorems of set theory cannot be proved within Mereology—Mereology cannot be used as a basis for number theory.

I shall briefly discuss Leśniewski's presentation of Mereology, because this will make it clear how he came to realize that Mereology is too weak to play the role he had in mind. Leśniewski's first axiomatization was clumsy, for he included defined (defined within Mereology itself) terms in the axioms. The following are his original axioms and definitions.

A1 $(PQ).(P\varepsilon\mathbf{Prt}[Q]) \supset (Q\varepsilon{\sim}[\mathbf{Prt}[P]])$[48]

This states that if P is part of Q, then Q is an individual which is not part of P.

A2 $(PQR).(P\varepsilon\mathbf{Prt}[Q]) \,\&\, (Q\varepsilon\mathbf{Prt}[R]) \supset (P\varepsilon\mathbf{Prt}[R])$

D1 $(PQ).(P\varepsilon\mathbf{Ing}[Q]) \equiv .(P=Q) \vee (P\varepsilon\mathbf{Prt}[Q])$

D2 $(aP).(P\varepsilon\mathbf{Cla}[a]) \equiv .(P\varepsilon\mathrm{V}) \,\&\, (a \sqsubset \mathbf{Ing}[P]) \,\&$
$(Q).(Q\varepsilon\mathbf{Ing}[P]) \supset (\exists A).(A\varepsilon a) \,\&\, (\mathbf{Ing}[Q]\,\Delta\,\mathbf{Ing}[A])$

These definitions define an ingredient of an object, and the class of *a*'s. The definition of the class of *a*'s requires that the class be an individual, that there be at least one a, and that every ingredient of the class either contain some a's or parts of a's, or be contained in an a. Mereology is similar to set theory in that a class (or a set) is an individual, and its name belongs to the semantic category of names. The type hierarchy does not intrude.

A3 $(aPQ).(P\varepsilon \mathbf{Cla}[a])\&(Q\varepsilon \mathbf{Cla}[a]) \supset (P\varepsilon Q)$
A4 $(a).(\mathsf{V}\Delta a) \supset (\bar{\mathsf{V}}\Delta \mathbf{Cla}[a])$

A4 says that if there are some a's, then there is a class of them.

Leśniewski later formulated an equivalent axiom system which does not contain defined terms, and Sobociński and Lejewski have each developed single axioms.[49]

Leśniewski originally added the following definition to Mereology,

$(PQ).(P\varepsilon \mathbf{Ele}[Q]) \equiv (\exists a).(Q\varepsilon \mathbf{Cla}[a])\&(P\varepsilon a)$

However, the functor '**Ele**' (element of) was discovered to be equivalent to '**Ing**.' Thus the definition of an element is useless if an ingredient has already been defined. It is not possible in Mereology to treat a class as is done in other theories, for there is no privileged way of taking a mereological class apart.

HOW LEŚNIEWSKI'S FORMAL SYSTEMS ORIGINATED Leśniewski began his work in logic because of his concern about the paradoxes. Apparently it was Russell's paradox that really caught his interest.[50] His study of the matter led him to conclude that classes, as normally treated by logicians and mathematicians, are fictitious entities. Because there is no such thing as a class, there is no class which is or is not a member of itself. Sobociński claims that Leśniewski recognized two distinct ideas "hidden" in the customary logical understanding of the term 'class,' which ideas are that of a distributive class and that of a collective class.[51] However, in his original article on Mereology, Leśniewski did not mention such a confusion. In this article, Leśniewski formalizes what he considers to be the natural theory of classes, and shows how it avoids the paradox. He also remarks that he finds unintelligible the customary theories dealing with classes.

In arguing against classes and sets, Leśniewski states that these notions, in their logical form, do not make sense. Of a class which is the extension of an idea, Leśniewski writes,

> The concept of Frege, which treats classes as extensions of ideas, I cannot subject to a well-deserved analysis, because I have so far been unable, despite sincere efforts, to understand what various authors are really talking about when they use the expression 'the extension of an idea.'[52]

In his article "O podstawach matematyki I," Leśniewski deals with statements made by Cantor, Frege, Felix Hausdorff, Sierpinski, and Fraenkel. He objects to the remarks of all of these except Cantor, because he does not agree "with many mathematicians that one can invent something at will which does not exist, even something which is unthinkable."[53] Leśniewski does not agree with Cantor either, he simply declines to discuss Cantor's view at any length. What Leśniewski is objecting to is the abstract entity status of classes. His point of attack is the admission of a null class. Clearly, a class which, like those Leśniewski recognizes, is no different from its members must contain members; a class without members is nothing.

Leśniewski feels that Mereology represents the proper way of dealing with complex objects (i.e., those composed of parts), which is what he takes classes and sets to be. From his view of classes, he answers the following objection raised by Frege. Frege argues that an individual must be different from a class whose only member is that individual. For suppose there is a class *P* of several individuals. Let *Q* be a class whose only member is *P*. If *Q* is the same object as *P*, then its members are the members of *P*, contrary to the assumption that *P* is the only member of *Q*.[54] Leśniewski denies that there is a class having only one member, unless that member is simple (i.e., without parts)—in such a case, the simple object *is* the class whose only member is itself). Hence Frege's objection misses its mark.

To the spurious concept of a set in ordinary set theory, Leśniewski opposes the concept of a class as formulated in Mereology. This

> concept is entirely in agreement with the method of using the terms 'class' and 'collection' ['set'] established in the colloquial speech of people who have never had to deal with any theory of classes or theory of sets.[55]

What Leśniewski is trying to express is the ordinary concept of a group, or a collection, or a bunch. The objects of such concepts occur in our experience, and are not at all abstract. When the concept that Leśniewski has formalized in Mereology is compared with its set-theoretical counterpart, there is little that they have in common. Leśniewski feels that he is capturing the genuine concept which gave rise to sets and classes, though he surely realizes how different are his uses of the terms 'class' and 'set' ('collection') from those of the normal logician or mathematician. Unlike set theory, which cut itself free from its origins, Mereology produces no contradictions, and is, indeed, provably consistent (relative to an interpretation in the real number system using decimal expansions[56]). The remedies proposed to eliminate the contradictions from set theory serve, in Leśniewski's eyes, only to carry it farther from its intuitive origins, making it even less capable of being used to describe the real world.[57]

Leśniewski's nominalistic outlook will not allow him to recognize abstract entities. It is not clear whether his nominalism resulted from his study of the paradoxes, or whether it preceded this study. Whichever is the case, it is clear that his rejection of abstract entities enables him to eliminate Russell's paradox. Of course, there are other formal procedures which lead to contradiction. And Leśniewski eliminates these in the customary ways. His hierarchy of semantical categories is very much like the theory of types. The substitution of an expression belonging to one category for that belonging to another produces nonsense—just like an improper substitution in a system based on the theory of types. Thus it is not possible to define terms by using self-reference. Leśniewski asserts that the semantic category structure is essential for senseful talk, and would be required even if no paradoxes had been encountered.[58] For the different semantical categories signal different linguistic functions—they are grammatically distinct.[59]

The most puzzling aspect of Russell's paradox is that it seems to be concerned with objects in the world, rather than with linguistic considerations. And the theory of types makes it look as if the universe contains objects of different levels, or types. But Leśniewski feels that all objects belong to the same category (in a non-linguistic sense of 'category'), though surely individual objects differ greatly from one another. The paradoxes appear to make trouble for the real world; they are eliminated when we realize that certain kinds of things (classes, for example) are not found in the real world. Other contradictions are eliminated by distinguishing different linguistic functions.

Ontology is the system which Leśniewski has devised to replace class theories (such as that of **PM**—in contrast to set theory). Sobociński states that Ontology deals with distributive classes.[60] However, the distributive class is not an object at all, and so the term is misleading. It is more appropriate to speak of distributive predication. In Ontology, there are general and individual names (nouns); and variables functioning like them. But the general names (common nouns) are not to be construed as names of entities, as red, or redness, humanity, etc. There are no terms for abstract entities in Ontology. Both general and individual names—including "empty" names—belong to the same semantical category. For Leśniewski feels that their grammatical role is the same. General names (common nouns) in Ontology are the closest thing to classes in other systems. But particular names (proper nouns) are only a special case of general names. Names do not indicate the existence of corresponding objects—the question of which names do so is an extra-logical one. Leśniewski feels that the real world contains only concrete objects; Ontology is a system which formalizes language used to talk about such objects—it is a system which does not add new objects to the world. Ontology is equivalent to the system of **PM**, but lacks the commitment to abstract entities that characterizes **PM**. Instead of defining inclusion, intersection, union, etc., for classes, Ontology defines these for names.

Ontology is designed to talk about objects in the world. Leśniewski believes that Ontology is structurally adequate to this task. All ontological statements are based on talk about concrete individuals (based on statements containing places for names of such individuals—in Ontology these places are filled by variables). High-level statements (containing expressions of high semantical categories) are dependent on statements containing names (name variables). For expressions of higher categories can at first be introduced only by definitions containing names in their definiens. It is true that high-level functors may be regarded as making possible statements about expressions (analogous to meta-linguistic statements). It was remarked earlier that the distinction between a hierarchy of meta-languages and a hierarchy of semantical categories (logical types) is to some extent arbitrary. We can regard '$\rightleftarrows \{f\}$' in Ontology[61] as saying "'f' is a one-one functor," but the ordinary-language rendering blurs the grammatical distinctions of Ontology. For 'f' in '$\rightleftarrows\{f\}$' does not occupy a name role, although our "translation" gives it the appearance of doing so. The only objects which Ontology requires are concrete ones.

Mereology is the formal system devised to talk about complex objects. Mereology, Ontology, and Protothetic together form a satisfactory (formal) basis for talking about the world—according to Leśniewski. Since Ontology is a structurally adequate system for discussing objects, Mereology contains no structural features peculiar to itself (requires no new semantical categories). Parts and wholes are objects of the same (non-linguistic) category. Thus the addition of Mereology raises no new problems in connection with abstract entities.

Leśniewski's formal systems are clearly compatible with his nominalistic position; indeed, they are designed to express this position. Even the directives for setting up these systems (the T.E.s) are stated in a nominalistic manner. The T.E.s talk about tokens instead of types, and make no reference to merely potential objects (such as unproved theorems). Of course, if one agrees with Quine that "To be assumed as an entity is, purely and simply, to be reckoned as the value of a [bound] variable,"[62] then Leśniewski's systems are anything but nominalistic. For expressions of every semantical category must be understood as representing entities. Since there are many more semantical categories than that of names, most of the signified entities are abstract (indeed, even common names will signify abstract entities). However, there is no reason to accept Quine's view.

It is Leśniewski's position that only certain expressions designate objects. But this does not mean that the non-naming expressions are senseless; language requires more than names. Names, however, are absolutely essential to language—other types of expressions depend on names. While Leśniewski does not associate an object with every expression, neither does he go to the opposite extreme and deny that language as representing the non-linguistic is important to the logical analysis of language. Leśniewski is concerned to formulate systems which provide an adequate basis for talking about the world. Syntactical structure is thus dependent upon the symbolic function (i.e., the representation of the non-linguistic) of language. It is not a logical concern to determine what objects are to be found in the world—but it is the business of logic to analyze language used to talk about objects in the world. Leśniewski rejects certain ontological commitments of ordinary language and of ordinary logical systems. His formal systems embody what he regards as the correct ontological commitments. In his view, the world consists simply of objects. Concrete entities are the only ones that he recognizes; they are the only ones for which he provides a category of expressions. Only words occupying the name role can indicate existing objects—and proper (individual) names are the only *names* that do so.

AN EVALUATION OF LEŚNIEWSKI'S WORK I feel that some evaluation of Leśniewski's work is in order here, although this must be brief, for the chief purpose of the present paper has already been accomplished.

I whole-heartedly agree with Leśniewski's rejection of abstract entities. There is no real ground for recognizing properties, classes, sets, propositions (when interpreted as abstract entities), and the rest. The question of whether or not such entities exist is not a logical matter (although it may be one that interests the logician). While I agree that there are no such entities, the issue need not and cannot be settled within logic. It is clear, however, that there is no need to recognize such entities in order to do logic. Leśniewski effectively illustrated this in his own work. His formal systems do not contain terms, or categories of terms, intended to represent abstract entities. If there are such entities, the terms in Leśniewski's logical systems will be adequate for discussing them; but his systems in no way require that abstract entities exist. Leśniewski's logical systems of Protothetic and Ontology constitute entirely adequate systems of classical logic. But these systems do not commit us to talking about classes, sets, or properties.

It is clear that within Leśniewski's systems there is no need to admit abstract entities. In this connection, it is important to understand his interpretation of quantifiers. It is equally clear that we do not need abstract entities to set up or to discuss his systems. On Leśniewski's view, there are no ideal theorems or systems. There is not any such thing as *the* system of Protothetic, or Ontology, or Mereology. There are as many such systems as have been constructed, and more of them can be added in the future. Formal systems are physical objects as much as houses are physical objects. Leśniewski's very careful Terminological Explanations show conclusively that there is no more reason to accept abstract entities in dealing with the system(s) from without than there is to accept them when we are working within the system.

Leśniewski feels that a formal system, if it is a logical system, is one whose formulas mean something. In this, I also agree with Leśniewski. Logic is concerned with language (primarily with inferences made in, or with, language). If a formula does not mean anything, then it has no interest for logic. Leśniewski did not begin with an uninterpreted formal system, and then give it an interpretation. The interpretation is there from the beginning, for it guides him in constructing the formal system.

In Leśniewski's view, not all formal systems are properly considered logical. But then, why should they be? Logic is concerned with language; there are other reasons for constructing formal systems than the study of logical matters. Of course, any formal system that begins with formulas and rules, and proceeds to derive other formulas, has some relation to logic. It is always possible to say, "If these formulas are theorems, and those rules are applied, then this formula is also a theorem." The preceding statement will either be true or false, and determining which it is is a logical matter. For logic studies implication and inference, and the statement in question asserts a certain implication. But the formal system by itself need not be considered a logical system. Logic is concerned with the general principles of inference and implication, and the development of the formal system in question may require that certain inferences be made —of course it is a logical matter to determine if these inferences are correct. But there is no more reason to consider the formal system a logical system than there is to say that politics is a branch of logic because inferences and arguments are sometimes made in that field.

The usefulness of formal systems for logic is that they can be used to analyze certain fundamental concepts, and to study the inter-relationships between them. (Perhaps the most important relation here is that of implication). Formal systems can also be used to express a philosophical view— as Leśniewski employed Mereology. While Leśniewski espouses a kind of nominalism, this is not expressed by the logical systems of Protothetic and Ontology. There is nothing about these systems that is inconsistent with Leśniewski's nominalism, but nothing about them is inconsistent with the denial of his view either. The concepts treated in Protothetic and Ontology are very basic ones. In the formal systems these concepts are separated from the ambiguity and confusion which characterize concepts expressed in ordinary language. The formal system allows us to study the structure of certain statements, and to formulate logical laws. This task would be much more difficult if we restricted ourselves to ordinary language.

If there is agreement about the fundamental character of the concepts treated in Protothetic and Ontology, as well as the fact that these concepts are necessary for any senseful talk about the world, then there should be no dispute about the advantages of dealing with these concepts by means of the formal systems of Protothetic and Ontology. It is important to realize that a formal logical system must be related to ordinary language, although the purpose for devising this system is to overcome certain shortcomings of ordinary language (shortcomings from the standpoint of logic). The formal system makes it possible to demonstrate that if we restrict ourselves to certain concepts, then this statement and that statement imply this other statement; from these statements we can infer that one. There is no reason to consider the formal system as furnishing the outlines of some ideal language—the formal system (if it is meaningful) must begin with ordinary language and should serve to elucidate and clarify ordinary language. Leśniewski does seem to have viewed his formal systems as the bare bones of an ideal language. However, disregarding this, it is possible to agree that his logical systems accomplish the legitimate purposes of formal logical systems.

Leśniewski did not consider Mereology to be a logical system. This is because the basic concept of Mereology concerns a relation between objects in the world (i.e., the relation of part to whole). The basic term of Protothetic is either that for material implication or for material equivalence— this is used to relate linguistic entities but not objects in the world. And the basic term of Ontology is used for saying that a certain individual is a something-or-other. This does not concern a relation between objects in the world, for giving an object a name is not the same as stating that one non-linguistic object is characterized with respect to another non-linguistic object. Protothetic and Ontology seem to be properly logical because the concepts they deal with are in some way linguistic ones, while the relation of part to whole is in no way linguistic. It would seem that if Leśniewski is to be a consistent Nominalist, he must insist that there is something unreal about a relation in the world. However, Leśniewski does seem to think that one object can genuinely be part of another. Even though he believes that the world consists of individuals, some individuals also consist of other individuals. Leśniewski's development and presentation of Mereology illustrates how a formal system may be used to express a philosophical view. Leśniewski felt that the world consists of objects, and that any segment of the world can be taken as an object. The axioms of Mereology are statements of this view. And by developing the formal system (i.e., by adding theses), it is possible to show the consequences of such a view. One consequence, which Leśniewski did not expect, was that the terms '**Ing**' and '**El e**' are equivalent to one another. If someone accepts Leśniewski's view, then he cannot regard one object as being composed in a unique fashion from certain others. Any object which contains others as parts can be taken apart in many different ways.

Leśniewski's view of the world is defective in that it does not recognize structure. And this defect in his view is the very reason why Mereology has so little utility as a foundation for mathematics. No one can understand the world who ignores the element of structure, or pattern, or organization, that is found there. Those objects which are parts of a larger object are arranged in a certain way to form this larger object. To ignore this arrangement is to ignore the feature which makes the larger object the kind of object that it is.

Leśniewski ignored the element of structure. But the world contains structures. Even formal systems, which, though they are man-made, are genuinely objects in the world, have structures. In using his logical systems as foundations for mathematics, Leśniewski was forced to substitute the structure of these systems for the structure he ignored in the world. It was his failure to recognize structure when he set up Mereology which is responsible for the failure of Mereology to replace set theory. Leśniewski was forced to rely on Ontology in setting up mathematics.

An example of what I mean by Leśniewski's use of the structure of his logical systems is the following. Consider the multiplication of two numbers. The most natural way to regard multiplication (although it will not work for infinite numbers) is the following: when we say that six apples is two times three apples, we think of dividing the apples into groups of three, and then counting the number of groups, which is two. But in order to make such a definition, we must be able to regard both the apples and the groups of apples as objects whose names belong to the same semantical category (i.e., we must be able to regard them as objects in the same sense of 'object'). This kind of definition of multiplication cannot be given in Mereology, because objects can be taken apart in too many different ways (most classes do not have a definite number of members); and the definition cannot be given in Ontology, because there is no way in Ontology to regard the groups as objects. Numbers are defined in Ontology rather than Mereology—to give a definition of multiplication similar to that outlined above, it is necessary to define two different kinds of numbers. The basic use of (ontological) numerical expressions is illustrated in

There are three apples.

Primitive numerical statements have the form

$n\{a\}$

(which is to say, "There are *n* *a*'s.") The basic kind of numerals (or numbers) count objects. For our definition of multiplication we need a kind of numeral that counts predicates. With this kind of numeral, we can make statements of the form

$n\{f\}$,

where '*f*' is a functor which takes predicates (noun expressions) as arguments. ('$n\{f\}$' means that there are *n* noun expressions which are *f*.) It is now possible to define multiplication in the natural way, as follows,

$$m \times n\{a\} \equiv (\exists f).m\{f\} \,\&\, (b)(f\{b\} \supset (b \subset a)) \,\&\, n\{a\}.$$ [63]

A definition would have been much simpler if it were possible to regard groups of objects (apples) as objects themselves, having the same status as the original objects. This could be done if it were possible to regard a group as an object constituted in a unique way by component objects—but this unique way would have to be a structure of some kind. By introducing two kinds of numbers, we are utilizing the structure of Ontology instead of explicity recognizing structure in the world.

I feel, then, that Leśniewski's philosophical position is mistaken in that it does not take account of structure. And this mistake deprives Mereology of significance. But other aspects of Leśniewski's position (especially with regard to abstract entities) I regard as correct. And with Mereology, he has at least given us an illuminating example of the way that a formal system can be used to express and to clarify a philosophical position.

NOTES

1. E. C. Luschei, *The Logical Systems of Leśniewski*, p. 26.
2. Tarski has called it "intuistic formalism." *Cf.* Z. Jordan, "The Development of Mathematical Logic and of Logical Positivism in Poland between the two Wars." *Polish Science and Learning*, No. 6, p. 25.
3. S. Leśniewski, quoted by Z. Jordan, *op. cit.*, p. 26.
4. Z. Jordan *op. cit.*, p. 26.
5. Quoted by E. C. Luschei, *op. cit.*, p. 149.
6. I. M. Bocheński, "The Problem of Universals," *The Problem of Universals*, p. 47.

7. E. C. Luschei, *op. cit.*, p. 53. Luschei, in a note (47, pp. 313-315), compares different formal systems and different languages to different kinds of maps of the world. Just as one type of map (e.g., one which shows the location of different ethnic groups) is appropriate for one purpose and not another (the map just suggested would be of little help in understanding the weather in different areas), so one language may be appropriate for one purpose but not another.

   Using this metaphor, Leśniewski's position might be explained by saying that his intuition is like knowledge of the general principles of map-making. His formal systems are like instructions concerning the most general principles of effective map-making.

8. S. Leśniewski, "O podstawach matematyki," *Przegląd Filozoficzny*, XXX(1927), 2-3, p. 167. Hereafter, this article, or that much of it which occurs in volume 30, will be referred to as PF. My knowledge of the article "O podstawach matematyki," except that much of it which presents Mereology (which is not found in volume 30) is based on a translation made for me by Mrs. E. J. Scott, although I made some corrections to this with the aid of a dictionary and a Polish grammar.

9. Leśniewski, in Mereology, does have a use for the word 'class'—but this use is unlike that of the ordinary logician.

10. S. Leśniewski, PF, p. 185.

11. S. Leśniewski, PF, p. 166.

12. *Ibid.*, p. 167.

13. *Ibid.*

14. Leśniewski published his Terminological Explanations in various journals, but they have been collected and translated into English by E. C. Luschei, in *The Logical Systems of Leśniewski.*

15. In Leśniewski's systems, one speaks of theses rather than theorems. Any formula which is an axiom, is derived from an axiom or other thesis according to the rules, is introduced by the rule for introducing definitions, or is introduced by the rule for introducing theses of extensionality, is a thesis.

16. Leśniewski gives such a definition for Protothetic in a footnote to "Über Definitionen in der sogennanten Theorie der Deduktion," *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, Classe 3, XXIV(1931), pp. 300-302, n. 1. Luschei presents this definition on pp. 207-210.

17. Since the T.E.s contain variables, they also require quantifiers if they are to be senseful statements. I have omitted the quantifiers for the sake of convenience.

18. If this T.E. were presented with symbols instead of words (but using symbols different from those used by Leśniewski), it would look like

$$(AB).(A\varepsilon\text{int}(B)) \equiv .(B\varepsilon\text{expression})\&(A\varepsilon\text{word})\&(A\varepsilon\text{in}(B))\&$$
$$(A\varepsilon\mathcal{N}(\text{first word in } B))\&(A\varepsilon\mathcal{N}(\text{last word in } B))$$

   This is found in Luschei, *op. cit.*, p. 178. (In writing formulas, I abbreviate parentheses according to the convention explained by A. Church, *Introduction to Mathematical Logic*, pp. 79-80).

19. E. C. Luschei, *op. cit.*, p. 180.

20. To say that there are potentially an infinite number is just to say that there is no last system—another can always be constructed. In a similar fashion, a system which employs axiom schemata does not contain an infinite number of axioms, the number is potentially infinite.

21. B. Sobociński, "On the Single Axioms of Protothetic I," *Notre Dame Journal of Formal Logic*, I(1960), p. 57.

22. I am not using the symbolism devised by Leśniewski. His is a totally unfamiliar symbolism, and there would be no advantage to using it here. Even Leśniewski did not employ this symbolism for informal presentations. According to Sobociński, it was "introduced by him mostly in order to formulate the rules of procedure in the most precise way." (B. Sobociński, "On the Single Axioms of Protothetic I," *Notre Dame Journal of Formal Logic*, I(1960), 1, p. 52.) In this paper, I will use the customary (Peano-Russell) symbols—except that I use '&' for conjunction—in the customary order. I will use ordinary parentheses for as many purposes as I can, even to enclose quantifiers. These will be abbreviated according to the convention of Church, as cited in note 18.

23. A. N. Whitehead, *The Axioms of Projective Geometry*, p. 2. Quoted by J. H. Woodger, *The Axiomatic Method in Biology*, p. 5.

24. Sobociński calls them creative, as does Słupecki, in "St. Leśniewski's protothetics," *Studia Logica*, I(1953), p. 51. Luschei prefers 'fruitful' to 'creative'—*op. cit.*, pp. 132-133.

25. A thesis of extensionality cannot be added for terms of a given category until these terms have appeared as arguments of a proposition-forming functor for one argument. Suppose that '**Eqv**(pqr)' is a propositional function all three of whose arguments belong to the category of propositions. And let '**Prp**(f)' be a propositional function whose arguments belong to the same category as '**Eqv**.' Then the following is a legitimate thesis if added to the system:

$$(fg):((pqr)f(pqr)\equiv g(pqr)):\equiv.(\phi).\phi(f)\equiv\phi(g).$$

26. Leśniewski's theory of semantical categories is closest to the simple theory of types (if provision is made for the different categories of Protothetic).

27. S. Leśniewski, "Grundzüge eines neuen Systems der Grundlagen der Mathematik," *Fundamenta Mathematica*, XIV(1929), p. 14.

28. This fact is overlooked by Jerzy Słupecki in "St. Leśniewski's protothetics," *Studia Logica*, I(1953), pp. 44-111. He repeats this error in "S. Leśniewski's calculus of names," *Studia Logica* III(1955). pp. 7-71.

29. '0' is the symbol for falsity. It can be defined

$$0\equiv(p).p.$$

30. Consider, for example, the law of quantity of functions, which is,

$$(f).((p).f(p)\equiv \mathbf{as}(p))\vee((p).f(p)\equiv\mathbf{vr}(p))\vee((p).f(p)\equiv\mathbf{fl}(p))\vee ((p).f(p)\equiv\sim(p)),$$

where the three unfamiliar functors are assertium, verum, and falsum. This example is taken from J. Słupecki, "St. Leśniewski's protothetics," *Studia Logica*, I(1953), p. 56.

31. B. Russell, *Principles of Mathematics*, p. 91.

32. B. Russell, "Mathematical Logic as Based on a Theory of Types," *Logic and Knowledge*, p. 65.

33. *Ibid.*

34. B. Sobociński, in a conversation.

35. W. V. Quine, *Word and Object*, p. 163.

36. S. Leśniewski, PF, p. 187.

37. W. V. Quine, "Logic and the Reificiation of Universals," *From a Logical Point of View*, p. 102.

38. *Ibid.*

39. *Ibid.*, p. 103.

40. *Ibid.*, p. 109.

41. *Ibid.*

42. B. Sobociński, "L'analyse de l'antinomie russellienne par Leśniewski," *Methodos*, II(1950), 6-7, p. 240.

44. Actually the formula in the text is not well-formed. What should be said is that there are no theses of the form

$$(\phi)(\exists A)(A\varepsilon\phi)$$

or

$$(\exists A\ \phi)(A\varepsilon\ \phi).$$

45. W. V. Quine, *Mathematical Logic*, p. 162.

46. The formula

$$A = B$$

is defined

$$(AB).(A=B)\equiv.(A\varepsilon B)\&(B\varepsilon A).$$

47. N. Goodman, *The Structure of Appearance*, p. 42.

48. '$\mathcal{N}$' is defined by

$$(Aa).(A\varepsilon\mathcal{N}[a])\equiv.(A\varepsilon A)\&\sim(A\varepsilon a).$$

49. B. Sobociński, "Studies in Leśniewski's Mereology," *Yearbook for 1954-55 of the Polish Society of Arts and Sciences Abroad*, p. 5ff.

50. This is the paradox which Leśniewski states and eliminates in PF, p. 182ff.

51. B. Sobociński, "L'analyse de l'antinomie russellienne par Leśniewski," *Methodos*, II(1950), 6-7, p. 239.

52. S. Leśniewski, PF, p. 200.

53. S. Leśniewski, PF, p. 191.

54. G. Frege, *Kritische Beleuchtung einiger Punkte in E. Schröders Vorlesungen über die Algebra der Logik*, pp. 444-445, quoted by S. Leśniewski in PF, p. 198. Russell quotes this argument in PM, vol. I, *51, p. 340.

55. S. Leśniewski, PF, p. 190.

56. R. Clay, Doctoral Thesis in Mathematics at the University of Notre Dame (1961), ch. 2.

57. S. Leśniewski, PF, p. 166.

58. S. Leśniewski, "Grundzüge eines neuen Systems der Grundlagen der Mathematik," *Fundamenta Mathematicae*, XIV(1929), p. 14.

59. It is interesting to compare Leśniewski's view with that offered by Wittgenstein in *Tractatus Logico-Philosophicus*. Wittgenstein writes, "A function cannot be its own argument because the functional sign already contains the prototype of its own argument and it cannot contain itself." (3.333, p. 57) In Wittgenstein's view, the distinction between functor (function) and argument is based on a distinction in the world. But he recognizes that the paradox has the nature of a grammatical mistake—there are not different types of individuals in the world.

60. B. Sobociński, "L'analyse de l'antinomie russellienne par Leśniewski," *Methodos*, II(1950), p. 240.

61. The functor '$\rightleftarrows$' is defined

$$(f).\rightleftarrows\{f\}\equiv.\rightrightarrows\{f\}\,\&\,\leftleftarrows\{f\}$$

where '$\rightrightarrows$' and '$\leftleftarrows$' are defined

$$(f).\rightrightarrows\{f\}\equiv(abc).f\{ab\}\,\&\,f\{ac\}\supset(b\circ c)$$
$$(f).\leftleftarrows\{f\}\equiv(abc).f\{ac\}\,\&\,f\{bc\}\supset(a\circ b)$$

62. W. V. Quine, "On What There Is," *From a Logical Point of View*, p. 13.

63. The universal quantifier which belongs at the beginning of this definition is omitted for the sake of convenience. This definition is actually inadequate because it gives no sense to the commutative law for multiplication. A corrected definition is so complicated that it would confuse the point I am trying to make.

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