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A THEOREM ON n-TUPLES WHICH IS EQUIVALENT TO THE WELL-ORDERING THEOREM

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Using a form of the well-ordering theorem which is due to A. Levy [3] it is possible to generalize a result of B. Sobociński [6] and prove the following theorem: *For all natural numbers n and k such that $n > 2$ and $1 < k < n$ the following proposition is equivalent to the well-ordering theorem.*

$\mathbf{P}(n, k)$: *For each set x which is not finite there exists a family N of unordered n-tuples of elements of x such that each unordered k-tuple of elements of x is a subset of exactly one of the elements of N.*

W. Sierpiński [5] proved that the axiom of choice implies $\mathbf{P}(3, 2)$ and B. Sobociński [6] proved that $\mathbf{P}(3, 2)$ implies the axiom of choice. Moreover, unknown to us, W. Frascella has also been working on this problem. In [1] Frascella proved that for each natural number $n > 2$, $\mathbf{P}(n, n-1)$ is equivalent to the axiom of choice and in [2] he proved the main results of this paper. However, Frascella's proofs are considerably different from ours.

Theorem 1. The well-ordering theorem implies that for all natural numbers n and k such that $n > 2$ and $1 < k < n$, $P(n, k)$ holds.

Proof: Let x be any set which is not finite and let n and k be natural numbers satisfying the hypotheses. By the well-ordering theorem there is an initial ordinal number ω_α such that $x \approx \omega_\alpha$. Let K be the set of all unordered k-tuples of elements of x. Then, it is also true that $K \approx \omega_\alpha$. (For example, we may well-order K by a relation R defined as follows: if $u, v \in K$, $u R v \Longleftrightarrow [(\max u < \max v)$ or $(\max u = \max v = w$ and $\max (u \sim \{w\}) < \max (v \sim \{w\}))$ or . . . or $(\max u = \max v$ and $\max (u \sim \{w\} = \max (v \sim \{w\})$ and . . . and $\min u \leq \min v)\}]$.) Let $K = \{k_\beta : \beta < \omega_\alpha\}$. In a similar manner we can well-order the set T of all unordered n-tuples of elements of x, so we also have $T \approx \omega_\alpha$. Let $T = \{t_\beta : \beta < \omega_\alpha\}$.

Now, we shall construct a subset N of T which satisfies $\mathbf{P}(n,k)$. Let $T_0 = \phi$. Suppose $T_\gamma \subseteq T$ has the property that for all $\beta < \gamma < \omega_\alpha$, k_β is a subset of exactly one element of T_γ and for all β such that $\gamma \leq \beta < \omega_\alpha$, k_β is

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a subset of at most one element of T_γ. If k_γ is a subset of exactly one element of T_γ define $T_{\gamma+1} = T_\gamma$. If k_γ is not a subset of any element of T_γ, let $t_\gamma^!$ be the smallest element s of T such that $k_\gamma \subseteq s$, but for all $\beta < \gamma$, $k_\beta \nsubseteq s$. (We can always find such an element in T because the set

$$S = \{t \epsilon T : k_\gamma \subseteq t \text{ and } (\forall u)(u \epsilon t \rightarrow (\exists \beta)(\beta < \gamma \text{ and } u \epsilon k_\beta \sim k_\gamma))\} \prec \omega_\alpha.$$

Since $P = \{t \epsilon T : k_\gamma \subseteq t\} \approx \omega_\alpha$, there is an $s \epsilon P \sim S$. Any such s cannot contain as a subset any k_β with $\beta < \gamma$.) Now, define $T_{\gamma+1} = T_\gamma \cup \{t_\gamma^!\}$ and if γ is a limit ordinal $T_\gamma = \bigcup_{\beta<\gamma} T_\beta$. Clearly $N = \bigcup_{\gamma<\omega_\alpha} T_\gamma$ is the required set.

Using a result of A. Levy [3] we can give a relatively short proof of the converse. Levy has shown that for each natural number $m > 0$ the following statement is equivalent to the well-ordering theorem.

$\mathsf{Q}(m)$: *Every set is the union of a well-ordered family of finite sets each of which has at most m elements.*

Theorem 2. If $\mathsf{P}(n,k)$ holds for some natural numbers n and k such that $n > 2$ and $1 < k < n$ then $\mathsf{Q}(n - k)$ holds.

Proof: Suppose x is a set which is not finite and n and k are natural numbers such that $n > 2$ and $1 < k < n$. (Clearly, it is sufficient to prove $\mathsf{Q}(n - k)$ for non-finite sets.) Let y be a well-ordered set such that $y \cap x = \phi$ and $y \not\lesssim w$ where

$$w = \{u : u \subseteq x \text{ and } \overline{\overline{u}} = n - k\}.$$

(For example, let y be a set such that $\overline{\overline{y}} = \aleph(2^{\overline{\overline{x}}})$, where for each cardinal number m, $\aleph(m)$ is Hartog's aleph, the smallest aleph which is $\nleq m$.) By hypothesis $\mathsf{P}(n,k)$ holds for $x \cup y$. Let N be a set of n-tuples of elements of $x \cup y$ such that each k-tuple of elements of $x \cup y$ is a subset of exactly one element of N. For each $u \epsilon x$ let

$$N_u = \{v \epsilon N : u \epsilon v \text{ and } \overline{\overline{v \cap y}} \geqslant k - 1\}.$$

Then there is a $v \epsilon N_u$ such that $\overline{\overline{v \cap y}} \geqslant k$. For suppose not. Let t be any $(k - 1)$-element subset of y. Then $t \cup \{u\}$ is a subset of exactly one element v of N_u. Moreover, each element of N_u contains exactly one subset $t \cup \{u\}$ where t is a $(k - 1)$-element subset of y. Consequently,

$$N_u \approx H = \{t : t \subseteq y \text{ and } \overline{\overline{t}} = k - 1\}.$$

Since $\overline{\overline{y}}$ is an aleph, $H \approx y$. But the mapping which assigns to each $t \epsilon H$ the set $v \sim (t \cup \{u\})$, where v is the unique element of N_u such that $t \cup \{u\} \subseteq v$, is a 1 - 1 mapping from H into w. Thus, we would have $y \lesssim w$, which is impossible.

Now, let

$$M = \{v : v \epsilon N \text{ and } \overline{\overline{v \cap y}} \geqslant k\}$$

and let

$$L = \{v \cap y : v \epsilon M\}.$$

We have shown that for each $u \epsilon x$, $M \cap N_u \neq \phi$. Furthermore, since each element of L has at least k elements, each element of L is contained in exactly one element of M. Clearly, L can be well-ordered. For each $t \epsilon L$, let $F(t) = v \sim t$, where v is the unique element of M such that $t \subseteq v$. Then F yields a well-ordering of a collection of subsets of x each of which has at most $n - k$ elements and whose union is x. Thus $Q(n - k)$ holds.

We can strengthen our result slightly and prove the following statement is equivalent to the well-ordering theorem:

For each set x which is not finite there exist natural numbers n and k, $n > 2$ and $1 < k < n$ and there exists a family N of unordered n-tuples of elements of x such that each unordered k-tuple of elements of x is a subset of exactly one of the elements of N.

Using the proof of Theorem 1 we can show that the well-ordering theorem implies this proposition. The well-ordering theorem follows in essentially the same manner as in Theorem 2. For we can choose y so that $x \cap y = \phi$, y can be well-ordered and

$$y \not\leq^* \{s : s \subseteq x \text{ and } s \text{ is finite}\}.$$

Then the proof of Theorem 2 yields a well-ordered family of subsets of x, each having at most m elements for some finite m, and whose union is x. But by a result given in [4] (WE 6, p 1), this implies the well-ordering theorem.

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