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DECISION PROBLEM IN THE CLASSICAL LOGIC

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The important problem of decision in mathematical logic has been studied by many authors; it was resolved for propositional calculus. For functional calculus, Church demonstrated that there was no model by which we can determine whether a well-formed formula of the predicate calculus is true or false.

In classical logic a formula "α" is a tautology if for the propositional variables:

$$p_1, p_2, \ldots, p_n$$

in "α" we can make correspond the truth values:

$$a_1, a_2, \ldots, a_n$$

(where each of "a_i" are constant, $\mathbf{v}$ = *true*, and $\mathbf{f}$ = *false*) and the substitution of the variable p_i by a_i conduct to "α" true). In our paper we shall say that "α" is a tautology if its logical value is $\mathbf{v}$, where *logical value* of a formula means the result which we get making the substitution of the propositional variables by $\mathbf{v}$ or $\mathbf{f}$ in all possible ways and making all the operations connected.

The purpose of this article is to present a new method for the resolution of the decision problem, a method which is an immediate result of our studies on normal forms in propositional calculus. The work is treated in this way: I. For forms made with equivalence. II. For forms made with equivalence, negation, reciprocity. III. For forms made with equivalence, reciprocity and alternation. IV. For a general form of classical logic.

For all these we use the notation of *J. Łukasiewicz*. The idea of form is defined in this way:

1. Each propositional variable is a form;
2. If "α" is a form and "F" is a unary functor, then "$F\alpha$" is a form;
3. If "α" and "β" are forms and "F" is a binary functor, then "$F\alpha\beta$" is a form. The set of the forms made by the means of the functors $F_1, F_2, \ldots, F_n$ is to be written: $\mathbf{S}(F_1, F_2, \ldots, F_n)$. For simplicity, we denote by S the set of all forms from classical logic. Two forms "α" and "β" are equipollent

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$$\alpha \sim \beta$$

if $E\alpha\beta$ is a tautology.

§1. In the work [1] we considered the axioms:

$$A1. \; EEpqEqp$$
$$A2. \; EEEpqrEpEqr,$$

and we proved that the set of consequences is non-contradictory and complete. The rules of deduction are: 1. The rule of substitution; and 2. The "modus ponens" rule.

$$\begin{array}{l} \vdash \alpha \\ \vdash \underline{E\alpha\beta} \\ \vdash \beta \end{array}$$

Remark: If "α" is a tautology then the form "β", which we obtain applying the rule of substitution of the "modus ponens" rule from "α" is also a tautology. If "β" is not a tautology then "α" also is not a tautology.

The axioms *A1-A2* being tautologies, it means that all the consequences (theses) which we obtain by application of rules of deduction are also tautologies.

The normal forms of the system: Each form $\alpha\varepsilon\mathbf{S}(E)$ which contains $2n$ propositional variables p_i, q_i $(i = 1, 2, \ldots, n)$ admits the normal form

$$\mathbf{N}_1(E) = E^{n-1}Ep_1q_1Ep_2q_2 \ldots\ldots Ep_nq_n$$

Each $\alpha\varepsilon\mathbf{S}(E)$ which contains $2n+1$ propositional variables p_i; q_j $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n+1)$ admits the normal form

$$\mathbf{N}_2(E) = E^{n-1}Ep_1q_1Ep_2q_2 \ldots\ldots Ep_n\, Eq_n\, q_{n+1}$$

in which the letters p_i, q_i may be permuted and $\alpha \sim \mathbf{N}_1(E)$; $\alpha \sim \mathbf{N}_2(E)$.

Remark 2. If: $\alpha \sim \beta$ and $\alpha = \mathbf{v}$ then also $\beta = \mathbf{v}$, and $\alpha = \mathbf{f}$ then also $\beta = \mathbf{f}$.

Indeed: If $\alpha \sim \beta$ then $E\alpha\beta = \mathbf{v}$ and therefore if $E\mathbf{v}\beta = \mathbf{v}$, then $\beta = \mathbf{v}$, and if $E\mathbf{f}\beta = \mathbf{v}$, then $\beta = \mathbf{f}$.

Theorem I. The form "α" is a tautology only if each propositional variable of "α" occurs an even number of times.

We show that each form "α" which contains each propositional variable an even number of times is a tautology. The form "α" admits indeed the normal form:

$$\mathbf{N}_1(E) = E^{\sum\limits_i^n h_i - 1}(Ep_1p_1)^{h_1}(Ep_2p_2)^{h_2} \ldots\ldots (Ep_np_n)^{h_n}$$

and

$$\alpha \sim \mathbf{N}_1(E)$$

But $\mathbf{N}_2(E)$ is a tautology because:

$$Ep_i p_i = \mathbf{v} \qquad (i = 1, 2, \ldots, n)$$

and therefore:

$$E^{\sum_{i}^{n} h_i - 1} (\mathbf{v})^{h_1} (\mathbf{v})^{h_2} \ldots\ldots (\mathbf{v})^{h_n} = \mathbf{v}$$

and therefore $\alpha = \mathbf{v}$, that is a tautology.

Now let us show that other forms $\alpha \varepsilon \mathbf{S}(E)$ having at least one propositional variable which occurs an odd number of times, it is not a tautology. Indeed take the variable p_{n+1} of "α", which occurs $2h+1$ times.

We have a normal form:

$$\mathbf{N}_2(E) = E^{\sum_{1}^{n} h_i} (Ep_1p_1)^{h_1} (Ep_2p_2)^{h_2} \ldots\ldots (Ep_np_n)^{h_n} (Ep_{n+1}p_{n+1})^{h_{n+1}} p_{n+1}$$

$$= E\omega p_{n+1}$$

where

$$\omega = E^{\sum_{1}^{n+1} h_i} (Ep_1p_1)^{h_1} (Ep_2p_2)^{h_2} \ldots\ldots (Ep_np_n)^{h_n} (Ep_{n+1}p_{n+1})^{h_{n+1}}$$

is a tautology. Then we have $\omega = \mathbf{v}$ and therefore

$$E\omega p_{n+1} = E\mathbf{v}p_{n+1} = \begin{cases} \mathbf{v} & \text{if} \quad p_{n+1} = \mathbf{v} \\ \mathbf{f} & \text{if} \quad p_{n+1} = \mathbf{f} \end{cases}$$

and therefore:

$$\mathbf{N}_2(E) = \begin{cases} \mathbf{v} & \text{if} \quad p_{n+1} = \mathbf{v} \\ \mathbf{f} & \text{if} \quad p_{n+1} = \mathbf{f} \end{cases}$$

and:

$$\alpha = \begin{cases} \mathbf{v} & \text{if} \quad p_{n+1} = \mathbf{v} \\ \mathbf{f} & \text{if} \quad p_{n+1} = \mathbf{f} \end{cases}$$

That is, "α" is not a tautology.

Conclusion. Each form $\alpha \varepsilon \mathbf{S}(E)$ is a decision only if each propositional variable of "α" is included by an even number (we shall call a tautological formula a decision).

§2. In the work [2] we take the axiomatic system $\mathbf{L}(E,R,N)$ given by the axioms:

A1. $EEpqEqp$
A2. $EEEpqrEpEqr$,
A3. $EERpqRrsEEpqErs$,
A4. $ENpERppp$,

in which the functors E, R and N are defined by the matrix

p, q	Epq	Rpq	Np
f, f	v	f	v
v, f	f	v	
f, v	f	v	
v, v	v	f	f .

We have demonstrated that this system is non-contradictory and incomplete.

The normal forms of the system. Each form: $\alpha \varepsilon \mathbf{S}(E,R,N)$ which includes $2n$ propositional variables p_i, q_i ($i = 1, 2, \ldots, n$) admits a normal form of the type

$$\mathbf{N}_1(E,R) = E^{n-1} E p_1 q_1 E p_2 q_2 \ldots\ldots E p_n q_n$$

if the number of the functors "R" and "N" which are included in "α" is an even number, and a normal form of the type

$$\mathbf{N}_2(E,R) = R^{n-1} (R p_1 q_1)(R p_2 q_2) \ldots\ldots (R p_n q_n)$$

if the functors "R" and "N" appear an odd number of times. The form "α" is equipollent with $\mathbf{N}_1(E, R)$, respectively, $\mathbf{N}_2(E, R)$, and the letters p_i, q_i may be permuted. Each form $\alpha \varepsilon \mathbf{S}(E,R,N)$ which contains $2n+1$ variables p_i, q_j ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n, n+1$) admits a normal form of the type

$$\mathbf{N}_3(E,R) = E^{n-1} (E p_1 q_1)(E p_2 q_2) \ldots\ldots (E p_n q_{n+1})$$

if "α" includes an even number of the functors R and N and a normal form of the type

$$\mathbf{N}_4(E,R) = R^{n-1} (R p_1 q_1)(R p_2 q_2) \ldots\ldots (R p_n R q_n q_{n+1})$$

if "α" contains an odd number of the functors "R" and "N"; $\mathbf{N}_3(E,R)$ being equipollent with the form "α" and the letters p_i, q_j being able to be transferred.

Theorem II. A form "α" is a tautology only if each propositional variable occurs an even number of times and the number of the functors "R" and "N" is also an even number.

Indeed, if "α" contains each propositional variable an even number of times and the number of the functors "R" and "N" is also an even number, then "α" admits the normal form

$$\mathbf{N}_1(E,R) = E^{\sum_1^n h_i - 1} (E p_1 p_1)^{h_1} (E p_2 p_2)^{h_2} \ldots\ldots (E p_n p_n)^{h_n}$$

which is a tautology according to *Theorem* 1. If "α" contains each propositional variable an even number of times, and the number of functors "R" and "N" is an odd number, then "α" admits the normal form

$$\mathbf{N}_2(E,R) = R^{\sum_1^n h_i - 1} (R p_1 p_1)^{h_1} (R p_2 p_2)^{h_2} \ldots\ldots (R p_n p_n)^{h_n}$$

which is not a tautology, i.e. $\mathsf{N}_2(E,R) = \mathbf{f}$. Indeed, the form

$$E\ \mathsf{N}_2(E,R)\ Rpp$$

is a tautology according to the preceding case. But:

$$Rpp = \mathbf{f} \text{ and } \mathsf{N}_2(E,R) = Rpp = \mathbf{f}$$

and therefore $\mathsf{N}_2(E,R) = \mathbf{f}$ and $\alpha = \mathbf{f}$, that is "α" is not a tautology.

If "α" contains at least one propositional variable which occurs an odd number of times then "α" is not a tautology. We have two subcases:

1. The number of the functors "R" and "N" is an even number. In this case "α" admits the normal form

$$\mathsf{N}_3(E,R) = E^{\sum_1^n h_i} (Ep_1p_1)^{h_1}(Ep_2p_2)^{h_2} \ldots\ldots (Ep_{n+1}p_{n+1})^{h_{n+1}} p_{n+1}$$

then according to *Theorem I* "α" is not a tautology.

2. The number of the functors "R" and "N" is an odd number. In this case the form "α" admits the normal form

$$\mathsf{N}_1(E,R) = R^{\sum_1^{n+1} h_i} (Rp_1p_1)^{h_1}(Rp_2p_2)^{h_2} \ldots\ldots (Rp_{n+1}p_{n+1})^{h_{n+1}} p_{n+1} = R\omega p_{n+1}$$

where

$$\omega = R^{\sum_1^{n+1} h_i - 1} (Rp_1p_1)^{h_1}(Rp_2p_2)^{h_2} \ldots\ldots (Rp_{n+1}p_{n+1})^{h_{n+1}} = \mathbf{f}$$

It follows

$$\mathsf{N}_1(E,R) = R\omega p_{n+1} = R\mathbf{f}p_{n+1} = \begin{cases} \mathbf{v} & \text{if } p_{n+1} = \mathbf{f} \\ \mathbf{f} & \text{if } p_{n+1} = \mathbf{v} \end{cases}$$

and therefore $\mathsf{N}_1(E,R)$ is not a tautology and therefore "α" is not a tautology.

Conclusion. Each form $\alpha \varepsilon \mathsf{S}(E,R,N)$ is a decision only if each propositional variable of "α" occurs an even number of times and the number of the functors "N" and "R" is also an even number.

§3. In [3] we considered the axioms:

A1. EEpqEqp
A2. EEEpqrEpEqr
A3. EKKpqrKKqpr
A4. EKKpqrKpKqr
A5. EKKppqKpq
A6. EApqEEpqKpq
A7. EARpqrEEAprAqrr.

This system is non-contradictory and incomplete. The following forms

1. *EKRpqrRKprKqr*
2. *EKEpqrEEKprKqrr*
3. *EAEpqrEAprAqr*

are theses (tautologies). Indeed:

1. $KRpqr \sim EERpqrARpqr \sim EERpqrEEAprAqrr \sim EERpqEAprAqr$
$\sim EERpqEEprKprEEqrKqr \sim RE^{6}prqrpqKprKqr \sim RKprKqr$

that is the form *1.* (We used the thesis $EKpqEEpqApq$ of sub-system $\mathbf{L}(E,K,A)$).

2. We have:

$$AEpqr \sim EEEpqrKEpqr \sim E^{3}pqrE^{2}KprKqrr \sim E^{2}pqEKprKqr$$
$$\sim EEpqEEEprAprEqrAqr \sim E^{7}ppqqrrAprAqr \sim EAprAqr$$

and therefore the form *3.*

3. We have:

$$KRRpqrr \sim RKRpqrKrr \sim RRKprKqrKrr \sim EEKprKqrKrr \sim EEKprKqrr$$

and therefore form *2.*

The functors "E", "R", "A", "K" are defined by the matrix:

p, q	Kpq	Apq	Epq	Rpq
f, f	f	f	v	f
v, f	f	v	f	v
f, v	f	v	f	v
v, v	v	v	v	f

Theorem III. Each form $\alpha \varepsilon \mathbf{S}(K)$ *which contains* $\sum_{i=1}^{n} h_i$ *propositional variables* $p_i (i = 1, 2, \ldots, n)$ *each letter* p_i *being included respectively* h_i *times admits a normal form of the type*:

$$\mathbf{N}(K) = K^{n-1} p_1 p_2 \ldots . p_n$$

and in this form all the letters p_i *can be permuted* (*cf.* [4]).

Theorem IV. Each form $\alpha \varepsilon \mathbf{S}(A)$ *which contains* $\sum_{i=1}^{n} h_i$ *propositional variables* $p_i (i = 1, 2, \ldots, n)$ *each letter* p_i *being included respectively* h_i *times admits the normal form of the type*

$$\mathbf{N}(A) = A^{n-1} p_1 p_2 \ldots . p_n,$$

and in this form all letters p_i *can be transferred.*

Theorem V. Each form $\alpha \varepsilon \mathbf{S}(E,K,A,R)$ *admits a normal form of the type*:

$$\mathbf{N}(E,R,A) = E^{k} R^{h} \alpha_1 \alpha_2 \ldots . \alpha_{h+k+1},$$

in which $\alpha_i \varepsilon \mathbf{S}(A)$

$$\alpha \sim \mathbf{N}(E,R,A).$$

Theorem VI. Two forms $\alpha, \beta \varepsilon \mathbf{S}(A)$ *are equipolent if each variable p of of "*α*" is a variable of "*β*" and vice versa.*

Theorem VII. Each form $\alpha \varepsilon \mathbf{S}(E,R,K,A)$ *is a tautology only if it admits the normal form of the type*

$$*\mathbf{N}(E,A) = E^{\sum_1^h 2n_i - 1}(\alpha_1^1\alpha_2^1 \ldots . \alpha_{2n_1}^1)(\alpha_1^2\alpha_2^2 \ldots . \alpha_{2n_2}^2) \ldots .. (\alpha_1^h \alpha_2^h \ldots . \alpha_{2n_h}^h)$$

in which $\alpha_1^i \sim \alpha_2^i \sim \ldots .. \sim \alpha_{2n_i}^i \qquad (i = 1, 2, \ldots, n)$

We shall prove that if "α" permits the normal form $*\mathbf{N}(E,A)$ then it is a tautology and if "α" does not permit a normal form of the type $*\mathbf{N}(E,A)$ then it is not a tautology.

Case 1. "α" admits the normal form $*\mathbf{N}(E,A)$. But according to *Theorem IV*, the forms

$$\alpha_1^i \sim \alpha_2^i \sim \ldots . \sim \alpha_{2n_i}^i \sim \mathbf{N}_i(A) \qquad (i = 1, 2, \ldots, h)$$

admit the normal forms $\mathbf{N}_i(A)$ and:

$$\alpha_1^i \sim \alpha_2^i \sim \ldots . \sim \alpha_{2n_i} \sim \mathbf{N}_i^2(A) \qquad (i = 1, 2, \ldots, h).$$

It follows that

$$*\mathbf{N}(E,A) \sim E^{\sum_1^h 2n_i - 1} [\mathbf{N}_1(A)]^{2n_1} [\mathbf{N}_2(A)]^{2n_2} \ldots .. [\mathbf{N}_h(A)]^{2n_h}$$

and

$$\alpha \sim *\mathbf{N}(E,A).$$

But the form

(1) $\quad E^{\sum_1^h 2n_i - 1} (p_1)^{2n_2}$

is a tautology. We make in the form (1) the following substitution:

$$p_i/\mathbf{N}_i(A) \qquad (i = 1, 2, \ldots, h)$$

and we obtain the form "ω". It follows that "ω" is a tautology and it is "α".

Case 2. The form "α" does not admit the normal form $*\mathbf{N}(E,A)$.

It means that "α" admits one of the following normal forms

I. $\mathbf{N}(E,A) = E^M \prod_{i=1}^{h} \prod_{j=1}^{2n_i} \alpha_j^i \prod_{l=1}^{m} \beta_l . \qquad [\alpha_j^i, \beta_l \varepsilon \mathbf{S}(A)]$

and in which

$$M = \sum_{i=1}^{h} 2n_i + m - 1$$

The forms $\prod_{j=1}^{2n_i} \alpha_j \, (i = 1, 2, \ldots, n)$ are equipolent and the forms $\beta_1, \beta_2, \ldots, \beta_i$ are not equipolent.

II. $\mathbf{N}(R,A) = R^{\sum_1^h 2n_i - 1} (\alpha_1^1 \alpha_2^1 \ldots \alpha_{2n_1}^1) \ldots .. (\alpha_1^h \alpha_2^h \ldots .. \alpha_{2n_h}^h)$

$$\alpha_1^i \sim \alpha_2^i \sim \ldots\ldots \sim \alpha_{2n_i}^i \qquad (i = 1, 2, \ldots, h)$$

III. $\mathsf{N}(E,R,A) = E^M R^N(\alpha_1^1\alpha_2^1\ldots\alpha_{2n_1}^1)\ldots(\alpha_1^h\alpha_2^{h}\ldots\alpha_{2n_h}^h)(\beta_1\beta_2\ldots\beta_m)$

in which

$$M + N = \sum_{i=1}^{h} 2n_i + m - 1$$

$$\alpha_1^i \sim \alpha_2^i \sim \ldots\ldots \sim \alpha_{2n_i}^i \qquad (i = 1, 2, \ldots, h)$$

and the forms $\beta_1, \beta_2, \ldots, \beta_m$ are not mutually equipolent.

I. Let

$$\omega = E^{\sum_1^h 2n_i} (\alpha_1^1\alpha_2^1\ldots\alpha_{2n_1}^1)(\alpha_2^2\alpha_2^2\ldots\alpha_{2n_2}^2)\ldots(\alpha_1^h\alpha_2^h\ldots\alpha_{2n_h}^h)$$

We have

$$\mathsf{N}(E,A) = E\omega E^{m-1}\beta_1, \beta_2, \ldots \beta_m$$

But "ω" is a tautology and according to the "modus ponens" rule, it follows that:

$$\gamma = E^{m-1}\beta_1\beta_2\ldots\beta_m$$

By the first remark $\mathsf{N}(E,A)$ is not a tautology if "α" is not. We have two cases:

1. $m = 2k + 1$. In this case replacing all the propositional variables in the form "β_i" by "p" and according to the formula $App \sim p$ we have

$$\gamma' = E^{2k}(p)^{2k}p = E\mathbf{v}p = \begin{cases} \mathbf{v} & \text{if } p = \mathbf{v} \\ \mathbf{f} & \text{if } p = \mathbf{f} \end{cases}$$

and therefore "γ'" is not a tautology, also "γ" and $\mathsf{N}(E,A)$ and "α".

2. $m = 2k$. Let be $p_1\varepsilon\beta_1$ and $p_1 \not\varepsilon \beta_2$. We make the substitution p_1 by q and the other propositional variables by p. According to the formulas

$$App \sim p; \quad AAppq \sim Apq; \quad A^{2k}(p)^{2k}q \sim Apq$$

it follows that:

$$E^{2k}(Apq)^{2k}q \sim q = \begin{cases} \mathbf{v} & \text{if } q = \mathbf{v} \\ \mathbf{f} & \text{if } q = \mathbf{f} \end{cases}$$

whence we have

$$\gamma' = E^{2k+1}(Apq)^{2k+1}q \sim EApqq = \begin{cases} \mathbf{v} & \text{if } p = \mathbf{f}, q = \mathbf{f} \\ \mathbf{v} & \text{if } p = \mathbf{v}, q = \mathbf{v} \\ \mathbf{v} & \text{if } p = q, q = \mathbf{v} \\ \mathbf{f} & \text{if } p = \mathbf{v}, q = \mathbf{f} \end{cases}$$

and therefore "γ'" is not a tautology, also "γ" and $\mathsf{N}(E,A)$ and "α".

II. "α" permits the form $\mathbf{N}(R,A)$.

According to the theorem IV, we have

$$\alpha_i^j \sim \mathbf{N}_j(A) \qquad (j = 1, 2, \ldots, h)$$

and therefore

$$\mathbf{N}(R,A) = R^{\sum_1^h 2n_i - 1} \; [\mathbf{N}_1(A)]^{2n_1} [\mathbf{N}_2(A)]^{2n_2} \ldots . [\mathbf{N}_h(A)]^{2n_h}$$

But

$$E\,\mathbf{N}(R,A)Rpp$$

is a tautology, therefore

$$\mathbf{N}(R,A) \sim Rpp$$

and $\mathbf{N}(E,A)$ is not a tautology and also "α".

III. "α" admits the normal form $\mathbf{N}(E,R,A)$. In this case N is an odd number. We have two possibilities:

1. $M = 2k$. Then we have:

$$\begin{aligned}\mathbf{N}(E,R,A) = {} & E^{2k}R^N(\alpha_1^1\alpha_2^2\ldots.\alpha_{2n_1}^1)(\alpha_1^2\alpha_2^2\ldots.\alpha_{2n_2}^2)\ldots.(\alpha_1^h\,\alpha_2^h\ldots.\alpha_{2n_h}^h)\\ & (\beta_1\beta_2\ldots.\beta_l) \sim R^{2k+N}(\alpha_1^1\alpha_2^1\ldots.\alpha_{2n_1}^1)(\alpha_1^2\alpha_2^2\ldots.\alpha_{2n_2}^2)\ldots..\\ & (\alpha_1^h\,\alpha_2^h\ldots.\alpha_{2n_h}^h)(\beta_1\beta_2\ldots.\beta_l)\end{aligned}$$

in which l is an even number because $2k+N$ is an odd number. It follows that $N + 2k + 1$ is an even number and therefore we have

$$\begin{aligned}\mathbf{N}(E,R,A) \sim {} & E^{N+2k+1}(\alpha_1^1\alpha_2^2\ldots.\alpha_{2n_1}^1)(\alpha_1^2\alpha_2^2\ldots.\alpha_{2n_2}^2)\ldots.(\alpha_1^h\,\alpha_2^h\ldots.\alpha_{2n_h}^h)\\ & (\beta_1\beta_2\ldots.\beta_{l-2}R\beta_{l-1}\beta_l) \sim E\gamma E^{l-2}(\beta_1\beta_2\ldots.\beta_{l-2})R\,\beta_{l-1}\beta_l\end{aligned}$$

in which

$$\gamma = E^{\sum_1^h 2n_i - 1}(\alpha_1^1\alpha_2^1\ldots.\alpha_{2n_1}^1)(\alpha_1^2\alpha_2^2\ldots.\alpha_{2n_2}^2)\ldots.(\alpha_1^h\,\alpha_2^h\ldots.\alpha_{2n_h}^h)$$

is a tautology, and therefore, according to the "modus ponens" rule, we shall deduce

$$\gamma' = E^{l-2}(\beta_1\beta_2\ldots.\beta_{l-2})R\beta_{l-1}\beta_l$$

Let be $p_1\varepsilon\beta_1$ and $p_1\not\varepsilon\beta_2$. We make the substitution for p by q and for the other variables by p. Then we have

$$E^{l-2}(Apq)^{2n}(p)^{l-2n-2}Rpp \sim EpRpp = Ep\mathbf{f} = \begin{cases}\mathbf{f} & \text{if } p = \mathbf{v}\\ \mathbf{v} & \text{if } p = \mathbf{f}\end{cases}$$

whence we have

$$\gamma'' = E^{l-2}(Apq)^{2n+1}(p)^{l-2n-2}Rpp \sim EApqRpp = EApq\mathbf{f} = \begin{cases}\mathbf{v} & \text{if } p = \mathbf{f};\ q = \mathbf{f}\\ \mathbf{f} & \text{if } p = \mathbf{v};\ q = \mathbf{v}\\ \mathbf{f} & \text{if } p = \mathbf{f};\ q = \mathbf{v}\\ \mathbf{f} & \text{if } p = \mathbf{v};\ q = \mathbf{f}\end{cases}$$

and therefore "γ'''" is not a tautology and also "γ''", $\mathbf{N}(E,R,A)$ and "α".

2. $M = 2k + 1$. In this case $M + N$ is an even number and therefore h is an odd number. We make the substitution of all propositional variables by p and there results a form of the type

$$\gamma = ER^{M+N-1}(p)^{2n_1}(p)^{2n_2}\dots(p)^{2nh}(p)^{l-1}p \sim ERppp = \begin{cases} \mathbf{v} & \text{if} \quad p = \mathbf{f} \\ \mathbf{f} & \text{if} \quad p = \mathbf{v} \end{cases}$$

and therefore is not a tautology and also $\mathbf{N}(E,R,A)$ and "α".

Conclusion. Each form $\alpha \varepsilon \mathbf{S}(E,R,A,K)$ is a decision only if "α" admits the normal form

$$*\mathbf{N}(E,A) = E^{\sum_1^h 2n_i - 1}(\alpha_1^1\alpha_2^1\dots\alpha_{2n_1}^1)(\alpha_1^2\alpha_2^2\dots\alpha_{2n_2}^2)\dots(\alpha_1^h\alpha_2^h\dots\alpha_{2n_h}^h)$$

where $\alpha_1^i \varepsilon \mathbf{S}(A)$ and $\alpha_1^i \sim \alpha_2^i \sim \dots \sim \alpha_{2n_i}^i \qquad (i = 1, 2, \dots, h)$

§4. We consider the functors of the classical logic defined as follows:

p, q	Apq	Kpq	Cpq	Epq	Rpq	Dpq	Hpq	Bpq
f, f	**f**	**f**	**v**	**v**	**f**	**f**	**f**	**v**
v, f	**v**	**f**	**f**	**f**	**v**	**v**	**f**	**v**
f, v	**v**	**f**	**v**	**f**	**v**	**f**	**v**	**f**
v, v	**v**	**v**	**v**	**v**	**f**	**f**	**f**	**v**

Mpq	Gpq	$\overline{M}pq$	$\overline{G}pq$	$\overline{A}pq$	$\overline{K}pq$	Vpq	Fpq	Np
v	**f**	**f**	**v**	**v**	**v**	**v**	**f**	**v**
f	**f**	**v**	**v**	**f**	**v**	**v**	**f**	
v	**v**	**f**	**f**	**f**	**v**	**v**	**f**	
f	**v**	**v**	**f**	**f**	**f**	**v**	**f**	**f**

The system $\mathbf{L}$ is the set of consequences of the axioms of the system $\mathbf{L}(E,R,A,K)$ and of the axioms

A8. $EACpqrAEApqqr$
A9. $EADpqrARApqqr$
A10. $EAHpqrARApqpr$
A11. $EAGpqrAEEppqr$
A12. $EAMpqrAERpqqr$
A13. $EANprAERpppr$
A14. $EA\overline{A}pqrANApqr$
A15. $EA\overline{K}pqrANKpqr$
A16. $EA\overline{G}pqrANGpqr$
A17. $EA\overline{M}pqrANMpqr$
A18. $EABpqrAEApqpr$

For demonstration of the theorems we use the following theses:
The forms

1. $EEAprAqrAEpqr$

2. *EAppp*

are theses of the system $\mathsf{L}(E,R,A,K)$

*1p/Cpq, q/EApqq, * EA8-3r/ECpqEApqq*

3. *AECpqEApqqECpqEApqq*

*2p/ECpqEApqq * E3-4*

4. *ECpqEApqq*

*1p/Dpq, q/RApqq * EA9-5r/EDpqRApqq*

5. *AEDpqRApqqEDpqRApqq*

*2p/EDpqRApqq * E5-6*

6. *EDpqRApqq*

*1p/Hpq, q/RApqp * EA10-7r/EHpqRApqp*

7. *AEHpqRApqpEHpqRApqp*.

*2p/EHpqRApqp * E7-8*

8. *EHpqRApqp*.

*1p/Bpq, q/EApqp * EA18-9r/EBpqEApqp*

9. *AEBpqEApqpEBpqEApqp*

*2p/EBpqEApqp * E9-10*

10. *EBpqEApqp*

*1p/Gpq, q/EEppq * EA11-11r/EGpqEEppq*

11. *AEGpqEEppqEGpqEEppq*

*2p/EGpqEEppq * E11-12*

12. *EGpqEEppq*

*1p/Mpq, q/ERpqq * EA12-13r/EMpqERpqq*

13. *AEMpqERpqqEMpqERpqq*

*2p/EMpqERpqq * 13-14*

14. *EMpqERpqq*

*1p/Np, q/ERppp * EA13-15r/ENpERppp*

15. *AENpERpppENpERppp*

*2p/ENpERppp * E15-16*

16. *ENpERppp*

$1p/\bar{A}pq,\ q/NApq\ *\ EA14\text{-}17r/E\bar{A}pqNApq$

17. $AE\bar{A}pqNApqE\bar{A}pqNApq$

$2p/E\bar{A}pqNApq\ *\ E17\text{-}18$

18. $E\bar{A}pqNApq$

$1p/\bar{K}pq,\ q/NKpq\ *\ EA15\text{-}19r/E\bar{K}pqNKpq$

19. $AE\bar{K}pqNKpqE\bar{K}pqNKpq$

$2p/E\bar{K}pqNKpq\ *\ E19\text{-}20$

20. $E\bar{K}pqNKpq$

$1p/\bar{G}pq,\ q/NGpq\ *\ EA16\text{-}21r/E\bar{G}pqNGpq$

21. $AE\bar{G}pqNGpqE\bar{G}pqNGpq$

$2p/E\bar{G}pqNGpq\ *\ E21\text{-}22$

22. $E\bar{G}pqNGpq$.

$1p/\bar{M}pq,\ q/NMpq\ *\ EA17\text{-}23r/E\bar{M}pqNMpq$

23. $AE\bar{M}pqNMpqE\bar{M}pqNMpq$

$2p/E\bar{M}pqNMpq\ *\ E23\text{-}24$

24. $E\bar{M}pqNMpq$

According to these theses we have the following forms

$Cpq \sim EApqq$ (I)
$Dpq \sim RApqq$ (II)
$Hpq \sim RApqp$ (III)
$Bpq \sim EApqp$ (IV)
$Gpq \sim EEppq$ (V)
$Mpq \sim ERpqq$ (VI)
$Np \sim ERppp$ (VII)
$\bar{A}pq \sim NApq \sim ERApqApqApq$ (VIII)
$\bar{K}pq \sim NKpq \sim ERKpqKpqKpq$ (IX)
$\bar{G}pq \sim NGpq \sim ERGpqGpqGpq$ (X)
$\bar{M}pq \sim NMpq \sim ERMpqMpqMpq$ (XI)

and therefore each form $\gamma \varepsilon \mathbf{S}$ is an equipolent with a form of the type $E\alpha\beta$ or $R\alpha\beta$.

We denote by **S** the set of forms made with all the functors of classical logic.

Theorem VIII. Each form α is equipolent with a form $\beta \varepsilon \mathbf{S}(E,R,A)$.

To demonstrate this theorem we use the theses:

a. $EEpqEEprEqr$
b. $EEpqERprRqr$

which are the theses of the system $\mathbf{N}(E,R)$

$ap/C\alpha\beta,\ q/EA\alpha\beta\beta,\ r/\gamma * E4\ p/\alpha,\ q/\beta - a_1$
a_1. $EEC\alpha\beta\gamma EEA\alpha\beta\beta\gamma$
$bp/C\alpha\beta,\ q/EA\alpha\beta\beta,\ r/\gamma * E4 p/\alpha,\ q/\beta - b_1$
b_1. $ERC\alpha\beta\gamma REA\alpha\beta\beta\gamma$
$ap/D\alpha\beta,\ q/RA\alpha\beta\beta,\ r/\gamma * E6p/\alpha,\ q/\beta - a_2$
a_2. $EED\alpha\beta\gamma ERA\alpha\beta\beta\gamma$
$bp/D\alpha\beta,\ q/RA\alpha\beta\beta,\ r/\gamma * E6p/\alpha,\ q/\beta - b_2$
b_2. $ERD\alpha\beta\gamma RRA\alpha\beta\beta\gamma$
$ap/H\alpha\beta,\ q/RA\alpha\beta\alpha,\ r/\gamma * E8p/\alpha,\ q/\beta - a_3$
a_3. $EEH\alpha\beta\gamma ERA\alpha\beta\alpha\gamma$
$bp/H\alpha\beta,\ q/RA\alpha\beta\alpha,\ r/\gamma * E8p/\alpha,\ q/\beta - b_3$
b_3. $ERH\alpha\beta\gamma RRA\alpha\beta\alpha\gamma$
$ap/B\alpha\beta,\ q/EA\alpha\beta\alpha,\ r/\gamma * E10p/\alpha,\ q/\beta - a_4$
a_4. $EEB\alpha\beta\gamma EEA\alpha\beta\alpha\gamma$
$bp/B\alpha\beta,\ q/EA\alpha\beta\alpha,\ r/\gamma * E10p/\alpha,\ q/\beta - b_4$
b_4. $ERB\alpha\beta\gamma REA\alpha\beta\alpha\gamma$
$ap/G\alpha\beta,\ q/EE\alpha\alpha\beta,\ r/\gamma * E12p/\alpha,\ q/\beta - a_5$
a_5. $EEG\alpha\beta\gamma EEE\alpha\alpha\beta\gamma$
$bp/G\alpha\beta,\ q/EE\alpha\alpha\beta,\ r/\gamma * E12p/\alpha,\ q/\beta - b_5$
b_5. $ERG\alpha\beta\gamma REE\alpha\alpha\beta\gamma$
$ap/M\alpha\beta,\ q/ER\alpha\beta\beta,\ r/\gamma * E14\ p/\alpha,\ q/\beta - a_6$
a_6. $EEM\alpha\beta\gamma EER\alpha\beta\beta\gamma$
$bp/M\alpha\beta,\ q/ER\alpha\beta\beta,\ r/\gamma * E14\ p/\alpha,\ q/\beta - b_6$

b_6. $ERM\alpha\beta\gamma RER\alpha\beta\beta\gamma$

$\quad ap/N\alpha,\ q/ER\alpha\alpha\alpha,\ r/\beta\ *\ E16\ p/\alpha,\ q/\beta - a_7$

a_7. $EEN\alpha\beta EER\alpha\alpha\alpha\beta$

$\quad bp/N\alpha,\ q/ER\alpha\alpha\alpha,\ r/\beta\ *\ E16\ p/\alpha,\ q/\beta - b_7$

b_7. $ERN\alpha\beta RER\alpha\alpha\alpha\beta$

$\quad ap/\bar{A}\alpha\beta,\ q/NA\alpha\beta,\ r/\gamma\ *\ E18\ p/\alpha,\ q/\beta - a_8$

a_8. $EE\bar{A}\alpha\beta\gamma ENA\alpha\beta\gamma$

$\quad bp/\bar{A}\alpha\beta,\ q/NA\alpha\beta,\ r/\gamma\ *\ E18\ p/\alpha,\ q/\beta - b_8$

b_8. $ER\bar{A}\alpha\beta\gamma RNA\alpha\beta\gamma$

$\quad ap/\bar{K}\alpha\beta,\ q/NK\alpha\beta,\ r/\gamma\ *\ E20\ p/\alpha,\ q/\beta - a_9$

a_9. $EE\bar{K}\alpha\beta\gamma ENK\alpha\beta\gamma$

$\quad bp/\bar{K}\alpha\beta,\ q/NK\alpha\beta,\ r/\gamma\ *\ E20\ p/\alpha,\ q/\beta - b_9$

b_9. $ER\bar{K}\alpha\beta\gamma RNK\alpha\beta\gamma$

$\quad ap/G\alpha\beta,\ q/NG\alpha\beta,\ r/\gamma\ *\ E22\ p/\alpha,\ q/\beta - a_{10}$

a_{10}. $EE\bar{G}\alpha\beta\gamma ENG\alpha\beta\gamma$

$\quad bp/\bar{G}\alpha\beta,\ q/NG\alpha\beta,\ r/\gamma\ *\ E22\ p/\alpha,\ q/\beta - b_{10}$

b_{10}. $ERG\alpha\beta\gamma ENG\alpha\beta\gamma$

$\quad ap/\bar{M}\alpha\beta,\ q/NM\alpha\beta,\ r/\gamma\ *\ E24\ p/\alpha,\ q/\beta - a_{11}$

a_{11}. $EE\bar{M}\alpha\beta\gamma ENM\alpha\beta\gamma$

$\quad bp/\bar{M}\alpha\beta,\ q/NM\alpha\beta\ ,\ r/\gamma\ *\ E24\ p/\alpha,\ q/\beta - b_{11}$

b_{11}. $ER\bar{M}\alpha\beta\gamma RNM\alpha\beta\gamma$

According to these theorems we have the following formulas

$EC\alpha\beta\gamma \sim EEA\alpha\beta\beta\gamma$ (XII)

$RC\alpha\beta\gamma \sim REA\alpha\beta\beta\gamma$ (XIII)

$ED\alpha\beta\gamma \sim ERA\alpha\beta\beta\gamma$ (XIV)

$RD\alpha\beta\gamma \sim RRA\alpha\beta\beta\gamma$ (XV)

$EH\alpha\beta\gamma \sim ERA\alpha\beta\alpha\gamma$ (XVI)

$RH\alpha\beta\gamma \sim RRA\alpha\beta\alpha\gamma$ (XVII)

$EB\alpha\beta\gamma \sim EEA\alpha\beta\alpha\gamma$ (XVIII)

$RB\alpha\beta\gamma \sim REA\alpha\beta\alpha\gamma$ (XIX)

$EG\alpha\beta\gamma \sim EEE\alpha\alpha\beta\gamma$ (XX)

$RG\alpha\beta\gamma \sim REE\alpha\alpha\beta\gamma$ (XXI)

$EM\alpha\beta\gamma \sim EER\alpha\beta\beta\gamma$ (XXII)

$RM\alpha\beta\gamma \sim RER\alpha\beta\beta\gamma$ (XXIII)

$EN\alpha\beta \sim EER\alpha\alpha\alpha\beta$ (XXIV)

$RN\alpha\beta \sim RER\alpha\alpha\alpha\beta$ (XXV)

$E\bar{A}\alpha\beta\gamma \sim ENA\alpha\beta\gamma$ (XXVI)

$R\bar{A}\alpha\beta\gamma \sim RNA\alpha\beta\gamma$ (XXVII)

$E\bar{K}\alpha\beta\gamma \sim ENK\alpha\beta\gamma$ (XXVIII)

$R\bar{K}\alpha\beta\gamma \sim RNK\alpha\beta\gamma$ (XXIX)

$E\bar{G}\alpha\beta\gamma \sim ENG\alpha\beta\gamma$ (XXX)

$R\bar{G}\alpha\beta\gamma \sim RNG\alpha\beta\gamma$ (XXXI)

$E\bar{M}\alpha\beta\gamma \sim ENM\alpha\beta\gamma$ (XXXII)

$R\bar{M}\alpha\beta\gamma \sim RNM\alpha\beta\gamma$ (XXXIII)

On the other hand, according to the axioms *A8–A18* we have

$ACαβγ \sim AEAαββγ$ (XXXIV)
$ADαβγ \sim ARAαββγ$ (XXXV)
$AHαβγ \sim ARAαβαγ$ (XXXVI)
$ABαβγ \sim AEAαβαγ$ (XXXVII)
$AGαβγ \sim AEEααβγ$ (XXXVIII)
$AMαβγ \sim AERαββγ$ (XXXIX)
$A\bar{A}αβγ \sim ANAαβγ$ (XXXX)
$A\bar{K}αβγ \sim ANKαβγ$ (XXXXI)
$A\bar{G}αβγ \sim ANGαβγ$ (XXXXII)
$A\bar{M}αβγ \sim ANMαβγ$ (XXXXIII)

We continue with the demonstration of the theorem. Any form can belong to one of the following types

1. $Eαβ$	2. $Rαβ$	3. $Aαβ$
4. $Kαβ$	5. $Cαβ$	6. $Dαβ$
7. $Hαβ$	8. $Bαβ$	9. $Gαβ$
10. $Mαβ$	11. $Nα$	12. $\bar{A}αβ$
13. $\bar{K}αβ$	14. $\bar{G}αβ$	15. $\bar{M}αβ$

each form containing m_i ($i = 1, 2, \ldots, n$) functors $C, D, H, B, G, M, N, \bar{A}, \bar{K}, \bar{G}$ and $\bar{M}$.

As to the formulas (I)-(XI), the forms of the type 5-15 are equipolent with the forms of the type 1 and 2.

According to commutation and association of the functors E, R, the forms of type 1 and 2 can be of one of the following particular types

1.1. $ECαβγ$	1.2. $EDαβγ$	1.3. $EHαβγ$
1.4. $EBαβγ$	1.5. $EGαβγ$	1.6. $EMαβγ$
1.7. $ENαβ$	1.8. $E\bar{A}αβγ$	1.9. $E\bar{M}αβγ$
1.10. $E\bar{K}αβγ$	1.11. $E\bar{G}αβγ$	2.1. $RGαβγ$
2.2. $RDαβγ$	2.3. $RHαβγ$	2.4. $RBαβγ$
2.5. $RGαβγ$	2.6. $RMαβγ$	2.7. $RNαβ$
2.8. $R\bar{A}αβγ$	2.9. $R\bar{K}αβγ$	2.10. $R\bar{G}αβγ$
2.11. $R\bar{M}αβγ$		

According to the formulas (XII)-(XXXII), it results that these forms are equipolent with the forms of the type 1 and 2, in which $C, D, H, B, G, M, \bar{N}, \bar{A}, \bar{K}, \bar{G}$ and $\bar{M}$ are containing of m_i-j ($i = 1, 2, \ldots, n$).

Repeating this argument a finite number of times it results that the forms of the types 1.1-1.11, 2.1-2.11 are equipolent with the forms of the set $\mathbf{S}(E,R,A)$.

The forms of the type 3 are further specified as of the following types:

3.1. $ACαβγ$	3.2. $ADαβγ$	3.3. $AHαβγ$
3.4. $ABαβγ$	3.5. $AGαβγ$	3.6. $ANαβ$
3.7. $AMαβγ$	3.8. $A\bar{A}αβγ$	3.9. $A\bar{K}αβγ$
3.10. $A\bar{G}αβγ$	3.11. $A\bar{M}αβγ$	

According to the formulas (XXXIV)-(XXXXIII) these forms will be equipolent with forms which contain one less occurrence of the functors C-$\bar{M}$.

Repeating the reasoning we come to the conclusion that each form of the set **S** is equipolent with a form of the set $\alpha\varepsilon$**S**(E,R,A) and hence it admits a normal form of the type **N**(E,R,K) or **N**(E,R,A).

Conclusion. A form "α" is a tautology if and only if its normal form is of the type ***N**(E,A) of §3.

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