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IMPLICATIONLESS WFFS IN IC

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Let Σ be the set of all wffs. of the Intuitionist Propositional Calculus (hereafter **IC**.) Let Σ_1, $\Sigma_1 \subset \Sigma$, be the set of wffs. which contain only the conjunction and negation signs. Similarly let Σ_2, $\Sigma_1 \subset \Sigma_2 \subset \Sigma$, be the set of wffs. which do not contain the implication sign. For wffs. $P_i \varepsilon \Sigma_1$ we have the well-known "representation theorem" of Gödel, [1], based on a result of Glivenko, that

$$\vdash_{\mathbf{IC}} P_i \ \textit{iff} \ \vdash_{\mathbf{HA}} P_i$$

where **HA**. is the classical propositional calculus. An analogous representation theorem for Σ_2 can be shown to follow from a result of Jankov, [2]. We note firstly.

THEOREM 1 There is no finite characteristic model for Σ_2

Proof Consider the wff $A \equiv \bigvee_{\substack{i<j \\ 2\leq j\leq k}} \neg(a_i \wedge \neg a_j)$ and proceed exactly as in Gödel's proof, *cf.* [1], that there exists no finite characteristic model for Σ.

LEMMA 1 Every wff $P_i \varepsilon \Sigma_2$ *is equivalent to a wff* $A_k \varepsilon \Sigma_2$ *where* A_k *is of the form* $\bigvee_{1\leq i\leq k} a_i$ *and each* $a_i \varepsilon \Sigma_1$.

Proof By induction on the number of connectives in P_i using the equivalence $\neg(a \vee b) \equiv \neg a \wedge \neg b$ and the distributive laws.

LEMMA 2 For every wff $P_i \varepsilon \Sigma_2$, $\vdash_{\mathbf{IC}} P_i$ *iff* $A_k^{\leq}$ *vanishes identically in* $\Gamma(\mathbf{B}^k)$ *where* A_k *is the normal form of* P_i *as defined in the preceding lemma,* $A_k^{\leq}$ *is the lattice polynomial (for lattice background, see* [3]) *corresponding to* A_k *and* $\Gamma(B^k)$ *is the lattice obtained by applying the Jaśkowski operation* Γ *to the direct product of the 2-element Boolean lattice with itself k times.*

Proof If $\vdash_{\mathbf{IC}} P_i$ then $A_k^{\leq}$ will vanish identically in $\Gamma(\mathbf{B}^k)$ since $\Gamma(\mathbf{B}^k)$ is a finite distributive lattice. For the converse, suppose P_i is not a theorem

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of **IC**. Then for no i, $1 \leqslant i \leqslant k$, $\nvdash_{\mathbf{IC}} a_i$ and hence by Gödel's result, for no i, $1 \leqslant i \leqslant k$, $\nvdash_{\mathbf{HA}} a_i$. Let $\mathbf{V}_i$ be the refuting evaluation of each a_i with respect to the two-element Boolean lattice. Then $A_k^{<}$ will fail in the lattice $\Gamma(\mathbf{B}^k)$ under the evaluation $<\mathbf{V}_1\mathbf{V}_2\ldots\mathbf{V}_k>$ to each a_i.

THEOREM 2 For all $P_i \varepsilon \Sigma_2$, $\vdash_{\mathbf{IC}} P_i$ *iff* $\vdash_{\mathbf{MC}} P_i$ *where* **MC** *is the calculus obtained from* **IC** *by adding the wff*

$$[\neg\neg a \wedge (a \supset b) \wedge ((b \supset a) \supset a)] \supset b$$

as a new axiom

Proof Follows directly from lemma 2 and the result of Jankov that $\Gamma(\mathbf{B}^k)$, $K = 1, 2, \ldots\ldots\ldots$ is a characteristic model for **MC**.

It may be noted that all the connectives in **MC** are independent. It is easy to show that in general for any superconstructive system (in the sense of Jankov) the connectives will be independent if its characteristic model has a submodel isomorphic to $\Gamma(\mathbf{B}^2)$

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