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Notre Dame Journal of Formal Logic
Volume VIII, Number 3, July 1967

A HENKIN COMPLETENESS THEOREM FOR T

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In [1] A. Bayart uses a method similar to that of Henkin [2] to prove a completeness theorem for the S5 modal predicate calculus.[1] We show how this method can be adapted to give completeness results for first order quantificational T and S4 with the *Barcan* formula.[2] T is a modal predicate calculus with propositional variables p, q, r . . . etc., individual variables x, y, z . . . etc., individual constants u_1, u_2, u_3 . . . etc., and predicate variables ϕ, ψ, χ etc., $\sim$, $\vee$, the universal quantifier and L (the necessity symbol). We assume usual formation rules and definitions of $\supset$, $.$, $\equiv$, $\exists$, and M. T has the following axioms and axiom schemata,

PC some set sufficient for the propositional calculus

LA1 $Lp \supset p$
LA2 $L(p \supset q) \supset (Lp \supset Lq)$

$\forall_1$ $(a)\alpha \supset \beta$ where a is an individual variable and β differs from α only in having some individual symbol b (variable or constant) everywhere where a occurs free in α provided a in α does not occur within the scope of (b).
B (the *Barcan* formula) $(x)L\alpha \supset L(x)\alpha$ where α is any wff. and the following rules of transformation; Uniform substitution for propositional variables provided no variable is bound as a result of substitution. (If PC and LA1, LA2 are formulated as schemata this rule, and the propositional variables, are unnecessary)

MP $\vdash \alpha, \vdash \alpha \supset \beta \rightarrow \vdash \beta$
LR1 (Necessitation) $\vdash \alpha \rightarrow \vdash L\alpha$
$\forall_2$ $\vdash \alpha \supset \beta \rightarrow \vdash \alpha \supset (a)\beta$ where a is some variable not free in α.

We obtain S4 by adding LA3 $Lp \supset LLp$ and S5 by adding LA4 $\sim Lp \supset L \sim Lp$ (If we have LA4 we may drop the *Barcan* formula; cf. [6]).

We say that a formula is *closed* (a cwff) if it contains no free variable. Where Λ is a set of formulae and β a wff we say that $\Lambda \vdash \beta$ iff there is some finite subset of Λ, $\{\alpha_1, \ldots, \alpha_n\}$ such that $(\alpha_1 \ldots \alpha_n) \supset \beta$. The following are derivable;

T1 (The Deduction Theorem) *If* Λ, $\alpha \vdash \beta$ *then* $\Lambda \vdash (\alpha \supset \beta)$.

Received November 14, 1965

T2 *Where* $\wedge$ *is a set of wffs and* β *is a wff and* β' *is obtained from* β *by replacing some variable x wherever it occurs free in* β *by some individual symbol not in* β *or in any member of* $\wedge$ *then if* $\wedge \vdash \beta'$ *then* $\wedge \vdash (x)\beta$

T3 (subs eq.) *If* $\vdash \alpha \equiv \beta$ *and* γ *differs from* δ *only in having* α *in some of the places where* δ *has* β *then* $\vdash \gamma \equiv \delta$ (*and hence* $\vdash \gamma \longleftrightarrow \vdash \delta$)

T4 (L-distribution) $L(p\,.\,q) \equiv (Lp\,.\,Lq)$

T5 $(Lp\,.Mq) \supset M(p\,.q)$

T6 $\vdash \alpha \rightarrow \vdash M\beta \supset M(\beta\,.\,\alpha)$

T7 (The *Barcan* formula) $M(\exists x)\alpha \equiv (\exists x)M\alpha$

We define validity for T as follows[3]. Assume two truth values *1* and *0*. Assume a domain D of individuals $u_1, u_2, \ldots u_i, \ldots$ etc. We take u_1, u_2 etc. as the individual constants also, letting them designate themselves. Assume also a set W of 'worlds' $x_1, x_2, \ldots, x_i, \ldots$ etc. and a reflexive relation R over W. **V** is a T-*assignment*, giving a formula α the value *1* or *0* in a world x_i iff it satisfies the following;

i) If p is a propositional variable then for every $x_i \in W$ $\mathbf{V}(p, x_i) = 1$ or $\mathbf{V}(p, x_i) = 0$
ii) Every individual variable is assigned an individual.
iii) For n-adic predicate variable ϕ and n-tuple $\langle a_1, \ldots, a_n \rangle$ of D $\mathbf{V}[\phi(a_1, \ldots, a_n), x_i] = 1$ or 0. (i.e. ϕ is assigned a set of n-tuples in each world.)
iv) For any wff α and any $x_i \in W$ $\mathbf{V}(\sim\alpha, x_i) = 1$ iff $\mathbf{V}(\alpha, x_i) = 0$, otherwise 0.
v) For any wffs α and β and any $x_i \in W$ $\mathbf{V}((\alpha \vee \beta), x_i) = 1$ iff either $\mathbf{V}(\alpha, x_i) = 1$ or $\mathbf{V}(\beta, x_i) = 1$ otherwise 0
vi) For any wff α and any $x_i \in W$ $\mathbf{V}((x)\alpha, x_i) = 1$ iff for every α' differing from α in having some constant replacing free x everywhere in α, $\mathbf{V}(\alpha', x_i) = 1$, otherwise 0.
vii) For every wff α and every $x_i \in W$ $\mathbf{V}(L\alpha, x_i) = 1$ iff $\mathbf{V}(\alpha, x_j) = 1$ for every $x_j R x_i$, otherwise 0.

A formula α is valid iff for every $x_i \in W$, every reflexive R and every T-assignment **V**; $\mathbf{V}(\alpha, x_i) = 1$. That every theorem of T is valid follows from seeing that all the axioms are valid and that the rules are validity-preserving. We show that every valid formula is a theorem.

A formula α is consistent iff $\sim\alpha$ is not a theorem. A formula α is satisfiable iff $\sim\alpha$ is not valid. A set of formulae is consistent if it contains no finite subset $\{\alpha_1, \ldots \alpha_n\}$ such that $\vdash \sim (\alpha_1, \ldots, \alpha_n)$.

We show that given a consistent formula $\mathcal{K}$, $\mathcal{K}$ is satisfiable. We show how to construct from $\mathcal{K}$, a series of maximal consistent sets[4] representing the 'real' world and 'possible' worlds related to the real world.

We first define the notion of a **C**-*form*

1) Where α is a wff containing x as its only free variable then $(\exists x)\alpha \supset \alpha$ is a **C**-form.

2) If α is a C-form and β is a cwff then $M\beta \supset M(\beta \,.\, \alpha)$ is a C-form.

Clearly any C-form will have only one free variable (say x.) Further all C-forms are enumerable. Where α is a C-form then α' is a *C-formula* of that form if some individual contant u replaces free x everywhere in $\alpha . u$ is called the *replacing constant*. Clearly every C-formula is closed.

Lemma I. *Where α is a C-form containing free x then* $\vdash (\exists x)\alpha$.

Proof by induction on the construction of C-forms. If α is $(\exists x)\beta \supset \beta$ then $(\exists x)\alpha$ is $(\exists x)[(\exists x)\beta \supset \beta]$ a theorem of quantification theory. If $\vdash (\exists x)\alpha$ then, by T6 $\vdash M\beta \supset M(\beta \,.\, (\exists x)\alpha)$. Now β is closed and so contains no free x. Hence $\vdash M\beta \supset M(\exists x)(\beta \,.\, \alpha)$, hence by the Barcan formula (T7) and T3 $\vdash M\beta \supset (\exists x)M(\beta \,.\, \alpha)$, hence ($\beta$ closed) $\vdash (\exists x)[M\beta \supset M(\beta \,.\, \alpha)]$. Hence by induction the lemma holds for all C-forms. QED.

Lemma II. *Where Λ is a consistent set of formulae and α' is a C-formula whose replacing constant does not occur in any member of Λ or in the C-form α of α' then α' can be consistently added to Λ.*

Since the replacing constant does not occur in α or in any member of Λ then by T2 if $\Lambda \vdash \sim\alpha'$ (i.e. if α' cannot be consistently added to Λ) then $\Lambda \vdash (x)\sim\alpha$, i.e. $\Lambda \vdash \sim(\exists x)\alpha$. Hence by Lemma I Λ is inconsistent, contrary to hypothesis. QED

Given some consistent cwff $\mathscr{N}$ let Γ_1 be a maximal consistent set of cwffs, containing $\mathscr{N}$, constructed as follows. For each C-form α add some C-formula α' whose replacing constant does not occur in α or earlier in the construction of Γ_1. By Lemma II the set will remain consistent at each stage. Then increase the set to a maximal consistent set.

A set of cwffs Λ is said to have the *C-property* iff for every C-form α there is in Λ a C-formula of that form. Clearly Γ_1 has the C-property. We show that where Γ_i is a maximal consistent set of cwffs with the C-property we may construct, for each cwff α such that $M\alpha \;\varepsilon\; \Gamma_i$, a maximal consistent set Γ_j, containing α, with the C-property and such that for every cwff $L\beta \;\varepsilon\; \Gamma_i$, $\beta \;\varepsilon\; \Gamma_j$. Γ_j is called a *subordinate* of Γ_i.

Let the initial member of Γ_j be α. α is consistent for if not $\vdash \sim\alpha$ hence $\vdash L \sim\alpha$ hence $\vdash \sim M\alpha$. But $M\alpha \;\varepsilon\; \Gamma_i$ and Γ_i is consistent.

Given the first n members of Γ_j as $\alpha, \alpha_1, \ldots, \alpha_{n-1}$ form the $n{+}1$'th by taking the n'th C-form β_n. By the C-property of Γ_i there will be some C-formula β_n' of that form such that $[M(\alpha \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_{n-1}) \supset M(\alpha \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_{n-1} \,.\, \beta_n)] \;\varepsilon\; \Gamma_i$. Let β_n be the $n{+}1$'th member of Γ_j. Hence Γ_j has the C-property. Further, since $M\alpha \;\varepsilon\; \Gamma_i$ then for any finite subset of C-forms there will be a C-formula of each form in some set $\{\alpha_1, \ldots, \alpha_k\}$ of C-formula such that $M(\alpha_1 \,.\, \ldots \,.\, \alpha_k) \;\varepsilon\; \Gamma_i$. Now add to Γ_j every formula β such that $L\beta \;\varepsilon\; \Gamma_i$. The set remains consistent for suppose not, then for some finite subset of Γ_j $\vdash \sim(\beta_1 \,.\, \ldots \,.\, \beta_n \,.\, \alpha \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_k)$ hence $\vdash \sim M(\beta_1 \,.\, \ldots \,.\, \beta_n \,.\, \alpha \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_k)$ *where* $L\beta_1, \ldots, L\beta_n \;\varepsilon\; \Gamma_i$ and $M(\alpha \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_k) \;\varepsilon\; \Gamma_i$. But by T4 and T5 we have; $[L\beta_1 \,.\, \ldots \,.\, L\beta_n \,.\, M(\alpha \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_k) \supset M(\beta_1 \,.\, \ldots \,.\, \beta_n \,.\, \alpha \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_k)$. Hence if $\{\beta_1, \ldots, \beta_n, \alpha, \alpha_1, \ldots, \alpha_k\}$ were inconsistent then $\{L\beta_1, \ldots, L\beta_n, M(\alpha \,.\, \alpha_1 \,.\, \ldots \,.\, \alpha_k)\}$

would be inconsistent, i.e., Γ_i would be inconsistent contrary to hypothesis. Finally increase Γ_j to a maximal consistent set of cwffs. Hence Γ_j is a maximal consistent set of cwffs with the **C**-property such that for some $M\alpha \ \varepsilon \ \Gamma_i \ \alpha \ \varepsilon \ \Gamma_j$ and for every $L\beta \ \varepsilon \ \Gamma_i$, $\beta \ \varepsilon \ \Gamma_j$. For every Γ_i construct such a Γ_j for each $M\alpha \ \varepsilon \ \Gamma_i$.

We now give a T-assignment which gives $\mathcal{N}$ the value *1* for some x_i. **V** is the following assignment. For each propositional variable p, $\mathbf{V}(p,x_i) = 1$ iff $p \ \varepsilon \ \Gamma_i$, otherwise *0*. For each n-adic predicate variable ϕ, $\mathbf{V}[\phi(a_1, \ldots, a_n, x_1] = 1$ iff $\phi(a_1, \ldots, a_n) \ \varepsilon \ \Gamma_i$, otherwise *0*. Let R be a relation such that x_jRx_i if Γ_j is a subordinate of Γ_i, (i.e. if Γ_j is constructed from an initial member α such that $M\alpha \ \varepsilon \ \Gamma_i$) or is Γ_i (so that R is reflexive).

Lemma III. *For any cwff* α $\mathbf{V}(\alpha,x_i) = 1$ *iff* $\alpha \ \varepsilon \ \Gamma_i$, *otherwise 0.*

Proof by induction on the construction of α. Since each Γ_i is maximal consistent and has the **C**-property and since where $(\exists x)\beta$ is a cwff then $(\exists x)\beta \supset \beta$ is a **C**-form there is some β' having a constant wherever β has free x such that $(\exists x)\beta \supset \beta' \ \varepsilon \ \Gamma_i$. Hence if $(\exists x)\beta \ \varepsilon \ \Gamma_i$ then $\beta' \ \varepsilon \ \Gamma_i$. Thus by induction as in [2] p. 163 we may show that the lemma holds for truth functions and quantification. Suppose that α has the form $L\beta$. By the induction hypothesis $\mathbf{V}(\beta,x_i) = 1$ iff $\beta \ \varepsilon \ \Gamma_i$(for every Γ_i). We have to show that $\mathbf{V}(L\beta, x_i) = 1$ iff $L\beta \ \varepsilon \ \Gamma_i$ (otherwise *0*). Suppose $L\beta \ \varepsilon \ \Gamma_i$, then for every Γ_j subordinate to Γ_i (and for Γ_i)$\beta \ \varepsilon \ \Gamma_j$. Hence (induction hypothesis) for every x_jRx_i $\mathbf{V}(\beta, x_j) = 1$. Hence $\mathbf{V}(L\beta, x_i) = 1$.

Suppose $L\beta \not\varepsilon \ \Gamma_i$. Then ($\Gamma_i$ maximal) $\sim L\beta \ \varepsilon \ \Gamma_i$. Hence $M{\sim}\beta \ \varepsilon \ \Gamma_i$. Hence for some Γ_j subordinate to $\Gamma_i, \sim\beta \ \varepsilon \ \Gamma_j$. Hence (induction hypothesis) $\mathbf{V}(\sim\beta,x_j) = 1$, hence $\mathbf{V}(\beta,x_j) = 0$. But x_jRx_i. Hence $\mathbf{V}(L\beta,x_i) = 0$. Hence the lemma holds. QED

Thus for any cwff α $\mathbf{V}(\alpha,x_i) = 1$ iff $\alpha \ \varepsilon \ \Gamma_i$. But $\mathcal{N} \ \varepsilon \ \Gamma_1$. Hence $\mathbf{V}(\mathcal{N},x_1) = 1$. Hence $\mathcal{N}$ is satisfiable. Hence any consistent cwff is satisfiable. Now if any cwff α is valid then $\sim\alpha$ is not satisfiable, hence inconsistent, hence $\vdash\alpha$. Further since any formula is valid iff its universal closure is valid and a theorem iff its universal closure is a theorem then for any formula α if α is valid then $\vdash\alpha$. I.e. T is complete. QED

We can extend this result to S4 and S5. The only change in the definition of validity is that R is transitive and reflexive for S4 and an equivalence relation for S5 (v. [4]). By LA3 and the maximal consistency of Γ_i if $L\beta \ \varepsilon \ \Gamma_i$ then $LL\beta \ \varepsilon \ \Gamma_i$ and hence $L\beta \ \varepsilon \ \Gamma_j$. (Of course 'consistent' now means consistent in S4 and S5 respectively.) Since if $L\beta$ appears in any set then it appears also in every subordinate of that set, an assignment can be constructed as before, but in which R is also transitive. For S5 we need, in addition, that R is symmetrical, i.e., we need to show that if Γ_j is subordinate to Γ_i then if $L\beta \ \varepsilon \ \Gamma_j$ then $L\beta \ \varepsilon \ \Gamma_i$. Suppose not; then ($\Gamma_i$ maximal) $\sim L\beta \ \varepsilon \ \Gamma_i$, then by LA4 ($\Gamma_i$ max) $L \sim L\beta \ \varepsilon \ \Gamma_i$. Hence $\sim L\beta \ \varepsilon \ \Gamma_j$, hence $L\beta \not\varepsilon \ \Gamma_j$.

A simpler construction (essentially the one used in [1]) can be given for S5 in which all sets are subordinates of Γ_1 and the only **C**-forms which need to be considered are $(\exists x)\beta \supset \beta$ and $M\alpha \supset M(\alpha.((\exists x)\beta \supset \beta))$.

NOTES

1. Kripke [3] has also proved the completeness of the S5 predicate calculus, using the method of semantic tableaux. In [4] he considers a semantics for quantificational T (M), S4 and the *Brouwersche* system which would lead to similar completeness results.

2. For the *Barcan* formula v. [5] p. 2 axiom number 11. For the propositional system T v. [7].

3. This is based on the semantics given in [4] though Kripke assumes a different domain of individuals for each world and thus gives a semantics which does not, as ours does, verify the Barcan formula. For a detailed account of these methods applied to propositional logics v. [8].

4. v. [2] A set Γ of cwffs is maximal consistent iff Γ is consistent and for every cwff α either $\alpha \epsilon \Gamma$ or $\sim\alpha \epsilon \Gamma$. Any consistent set of cwffs can be increased to a maximal consistent set by the process described in [2] p. 162.

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