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PROBABILITY AS DEGREE OF POSSIBILITY

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Let L be a set of sentences closed under disjunction and negation. Let S be a system of possible worlds. For p in L and W in S, let "$W(p) = \mathbf{T}$" mean that p is true in W. We assume that

(1) If $W(p \vee q) = \mathbf{T}$, then $W(p) = \mathbf{T}$ or $W(q) = \mathbf{T}$.
(2) If $W(p) = \mathbf{T}$, then $W(-p) \neq \mathbf{T}$.
(3) If p is truth functionally (**tf**) valid (and hence necessary), then $W(p) = \mathbf{T}$ for every W in S.

(S could be identified with a system of Hintikka model sets of sentences of L.)

Let N be the number of worlds in S, and assume for now that N is *finite* and positive. For any p in L, let $\mathsf{N}(p)$ be the number of worlds W in S such that $W(p) = \mathbf{T}$. Define the probability that p by

$$\pi(p) = \mathsf{N}(p)/\mathsf{N}.$$

Then for any p in L

(4) $0 \leq \pi(p) \leq 1$
(5) $\Diamond p$ *iff* $\pi(p) > 0$
(6) $\Box p$ *iff* $\pi(p) = 1$.

Now

(7) *If* $W(p) = \mathbf{T}$, *then* $W(p \vee q) = \mathbf{T}$,

for $-p \vee (p \vee q)$ is **tf** valid, so by (3), $W(-p \vee (p \vee q)) = \mathbf{T}$, and then by (1), $W(-p) = \mathbf{T}$ or $W(p \vee q) = \mathbf{T}$. But $W(p) = \mathbf{T}$, so by (2), $W(-p) \neq \mathbf{T}$, and thus $W(p \vee q) = \mathbf{T}$. Next:

(8) *If* $-\Diamond(p \wedge q)$, *then* $\pi(p \vee q) = \pi(p) + \pi(q)$.

If $W(p \vee q) = \mathbf{T}$, then by (1) and the impossibility of $(p \wedge q)$, exactly one of p, q is true in W. By (7), if $W(p) = \mathbf{T}$ or $W(q) = \mathbf{T}$, then $W(p \vee q) = \mathbf{T}$. Hence $\{W \epsilon S \mid W(p \vee q) = \mathbf{T}\}$ is the disjoint union of $\{W \epsilon S \mid W(p) = \mathbf{T}\}$ and $\{W \epsilon S \mid W(q) =$

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$\mathbf{T}\}$. Thus $\mathrm{N}(p \vee q) = \mathrm{N}(p) + \mathrm{N}(q)$, from which (8) follows. But (4), (6), (8), are basically Kolmogorov's first three axioms for probability.

In the denumerable case, let $S = \langle W_1, W_2, \ldots \rangle$ be a sequence of type ω of possible worlds. For each n, let S_n be the initial segment $\langle W_1, W_2, \ldots, W_n \rangle$ of S. For each p in L and each n, let $\mathrm{N}(p, n)$ be the number of worlds W in S_n such that $W(p) = \mathbf{T}$. Define the probability that p by

$$\pi(p) = \lim_{n \to \infty} \mathrm{N}(p, n)/n$$

if this limit exists; otherwise $\pi(p)$ is undefined. If $\pi(p)$ exists, then

(9) $0 \leq \pi(p) \leq 1$
(10) *If* $\pi(p) > 0$, *then* $\Diamond p$
(11) *If* $\Box p$, *then* $\pi(p) = 1$

As before, if it is impossible that $(p \wedge q)$, then for each n,

$$\mathrm{N}(p \vee q, n) = \mathrm{N}(p, n) + \mathrm{N}(q, n)$$

so, assuming convergence,

(12) $\pi(p \vee q) = \lim ((\mathrm{N}(p, n)/n) + (\mathrm{N}(q, n)/n)) = \pi(p) + \pi(q)$.

But (9), (11), (12) are again basically Kolmogorov's axioms. In this way, we may construe probability as measuring uncertainty as to which is the actual world among the vast range of possible worlds.

At least two objections might be made against the above construction. First, as defined, probability is not truth functional. That is, we may have both

$$p \equiv q \text{ and } \pi(p) \neq \pi(q).$$

For example

(13) 1 = 1

and

(14) Neutrinos have no mass

are both true and hence materially equivalent. But (13) is necessary, so its probability is one, while (14) is contingent so its probability may be less than one. However, this fact might be an advantage of the construction since in some intuitive sense of "likely," it does not seem counter-intuitive to say that it is more likely that (13) is (or would be) true than (14). (Of course, *if* $\Box(p \equiv q)$, *then* $\pi(p) = \pi(q)$.)

Second, extending the above point, consider the denumerable case and let W_a be the actual world. Then there could be a p such that $W_a(p) = \mathbf{T}$ but for all $W \neq W_a$, $W(p) \neq \mathbf{T}$. Such a sentence p in L would be true but have $\pi(p) = 0$. However, precisely because this is an extension of the above point, it too may be an advantage of the construction. For if in some sense of "likely," some truths are more likely than others, then some truths might have a likelihood of zero; consider Ripley's *Believe It or Not*.

Finally, this construction makes sense of ascriptions of non-zero, non-unit probabilities to sentences describing events as particular as one wishes; for such sentences can be true in many possible worlds and false in many others.

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