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TABLEAU METHODS OF PROOF FOR MODAL LOGICS

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1 *Introduction*: In [1] Fitch proposed a new proof proceedure for several standard modal logics. The chief characteristic of this was the inclusion in the object language, of symbols representing worlds in Kripke models. In this paper we incorporate the device into a tableau proof system and it is seen that the resulting (propositional) proof system is highly analogous to a classical first order tableau system, with the modal operators behaving like quantifiers. Exploiting this similarity, a tableau completeness proof for first order logic directly becomes a Kripke completeness proof for modal logic, and Smullyan's fundamental theorem of quantification theory (a Herbrand-like theorem) [7] has its analog. Indeed, more than analogy is at work here; from an appropriate abstract point of view certain modal logics, first order classical and intuitionistic logic, and various infinitary logics may be treated simultaneously, an approach due to R. Smullyan and developed in a forthcoming monograph (see [8] for a preliminary version).

We will treat only tableau proof systems and some familiarity with [7] is presumed. In addition to being metatheoretically interesting, specific tableau systems we give for S5, S4, T, B, DS4, DT, and K are quite simple to use. The extension of these systems to first order systems is straightforward, and is discussed briefly in the last section.

2 *Kripke model theory*: In this section we present Kripke's model theory for propositional modal logics to establish notation and terminology. [4] and [5] are the basic references.

By a general model structure we mean a pair, $\langle G, R\rangle$, where G is a non-empty set and R is a binary relation on G. If R is an equivalence relation, $\langle G, R\rangle$ is an S5 *model structure*. If R is reflexive and transitive, $\langle G, R\rangle$ is an S4 *model structure*. If R is reflexive and symmetric, $\langle G, R\rangle$ is a B *model structure*. If R is reflexive, $\langle G, R\rangle$ is a T *model structure*. If R satisfies the condition: for any $\Gamma \epsilon G$ there is some $\Delta \epsilon G$ such that $\Gamma R \Delta$, $\langle G, R\rangle$ is a DT *model structure*. If R satisfies the previous condition, and is transitive, $\langle G, R\rangle$ is a DS4 *model structure*. If R is transitive, we call $\langle G, R\rangle$ a KS4 *model structure*. Finally, if R has no special conditions placed on it, we call $\langle G, R\rangle$ a KT *model structure* (sometimes just called K).

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Let L be one of the logics S5, S4, B, T, DS4, DT, KS4, or KT, and let $\langle G, R\rangle$ be an L-model structure. The triple $\langle G, R, \vdash\rangle$ is called an *L-model* if $\vdash$ is a relation between elements of G and formulas such that for any $\Gamma, \Gamma^* \in G$,

(1) $\Gamma \vdash (X \wedge Y) \Longleftrightarrow \Gamma \vdash X$ and $\Gamma \vdash Y$
(2) $\Gamma \vdash (X \vee Y) \Longleftrightarrow \Gamma \vdash X$ or $\Gamma \vdash Y$
(3) $\Gamma \vdash (\sim X) \Longleftrightarrow \Gamma \nvdash X$ (i.e. not-$\Gamma \vdash X$)
(4) $\Gamma \vdash (X \supset Y) \Longleftrightarrow \Gamma \nvdash X$ or $\Gamma \vdash Y$
(5) $\Gamma \vdash \Box X \Longleftrightarrow$ for every $\Gamma^* \in G$ such that $\Gamma R \Gamma^*$, $\Gamma^* \vdash X$
(6) $\Gamma \vdash \Diamond X \Longleftrightarrow$ for some $\Gamma^* \in G$ such that $\Gamma R \Gamma^*$, $\Gamma^* \vdash X$.

A formula X is called *valid* in the L-model $\langle G, R, \vdash\rangle$ if $\Gamma \vdash X$ for all $\Gamma \in G$.

3 *Modal tableau systems*: In this section let L be one of the logics S5, S4, T, B, DS4, or DT (we treat KS4 and KT later). Let $\langle G_0, R_0\rangle$ be a general model structure, fixed for the rest of this section. If X is a modal formula and $P \in G_0$, we call PX (P followed by X) a *prefixed formula*. Let S be a set of prefixed formulas, $\langle G, R, \vdash\rangle$ an L-model, and I a mapping from a subset of G_0 to G; we call I an *interpretation* for the set F of prefixes of formulas in S if the domain of I is F and for any P and Q in F, if PR_0Q, then $I(P)RI(Q)$. If, moreover, $PX \in S$ implies $I(P) \vdash X$, we call $\langle\langle G, R, \vdash\rangle, I\rangle$ an *L-realization* of S. We say S is *L-realizable* if it has an *L-realization*.

Let $\varphi(x, y)$ be a function from $G_0 \times 2^{G_0}$ to 2^{G_0}; we call φ a *selection function* for $\langle G_0, R_0\rangle$ if, for each finite subset F of G_0 and each $P \in F$, $\varphi(P, F)$ is a non-empty subset of G_0 such that if $Q \in \varphi(P, F)$, PR_0Q. Let φ be a selection function for $\langle G_0, R_0\rangle$; we call $\langle\langle G_0, R_0\rangle, \varphi\rangle$ a *tableau structure* for L if

(1) $\langle G_0, R_0\rangle$ is a countable model structure for L,

(2) if S is any finite set of prefixed formulas which is L-realizable and F is the set of prefixes of formulas in S, then:

(a) if $P\Box X \in S$ or $P\Diamond X \in S$ and $Q \in \varphi(P, F)$, $S \cup \{QX\}$ is L-realizable,

(b) If $P{\sim}\Box X \in S$ or $P{\sim}\Diamond X \in S$ and $Q \in \varphi(P, F)$, $S \cup \{Q \sim X\}$ is L-realizable.

We now show how a tableau proof system for L may be constructed from any tableau structure for L. In **4** we give a specific tableau structure for L which is easy to use.

Let $\langle\langle G_0, R_0\rangle, \varphi\rangle$ be a fixed tableau structure for the logic L. We assume the reader is familiar with the first order tableaus of [7] and, in particular, with the α, β, γ, δ notation. We change the definition of α and β slightly, and introduce ν and π rules, analogous to the γ and δ rules.

We will use X and Y to represent (propositional modal) formulas. We will call the elements of G_0 *prefixes* (they are analogous to parameters in first order logic), and we will use P and Q to represent them. Since in this tableau proof system we work only with prefixed formulas, we redefine α, β and their components as follows:

α	α_1	α_2
$P(X \wedge Y)$	PX	PY
$P\sim(X \vee Y)$	$P\sim X$	$P\sim Y$
$P\sim(X \supset Y)$	PX	$P\sim Y$
$P\sim\sim X$	PX	PX

β	β_1	β_2
$P(X \vee Y)$	PX	PY
$P\sim(X \wedge Y)$	$P\sim X$	$P\sim Y$
$P(X \supset Y)$	$P\sim X$	PY

We also define ν (necessary) and π (possible) prefixed formulas and their instances as follows:

ν	$\nu(Q)$
$P\Box X$	QX
$P\sim\Diamond X$	$Q\sim X$

π	$\pi(Q)$
$P\Diamond X$	QX
$P\sim\Box X$	$Q\sim X$

Let PX be any prefixed formula. By an L*-tableau* we mean any tree constructed as follows. Begin by placing PX at the origin. If an α formula occurs on a branch, α_1 and α_2 may be added to the end of the branch. If a β formula occurs on a branch we may extend it by adding to the end of the branch two branches, one with β_1 and one with β_2. Suppose a π formula occurs on a branch; let P be the prefix of π and let F be the set of prefixes occurring in formulas on the branch. If $Q \in \varphi(P, F)$ we may add $\pi(Q)$ to the end of the branch. We think of φ as selecting prefixes in the relation R_0 to P, but which are otherwise unrestricted by the other prefixes on the branch. Thus, we will say briefly, we may add $\pi(Q)$ to a branch containing π for any related, unrestricted Q. Finally, suppose a ν formula occurs on a branch; let P be the prefix of ν and let F be the set of prefixes occurring in formulas on the branch. If $Q \in \varphi(P, F)$, we may add $\nu(Q)$ to the end of the branch; also if $Q \in F$ and PR_0Q, we may add $\nu(Q)$ to the end of the branch. Briefly, we may add $\nu(Q)$ to a branch containing ν for any related Q which is either unrestricted or has been used.

Our tableau rules may be summed up as follows:

$$\frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad\qquad \frac{\beta}{\beta_1 \mid \beta_2}$$

$$\frac{\nu}{\nu(Q)} \quad \textit{for any related used or unrestricted } Q$$

$$\frac{\pi}{\pi(Q)} \quad \textit{for any related unrestricted } Q$$

A branch of a tableau is called *closed* if it contains PY and $P\sim Y$ for some formula Y and some prefix P. A tableau is called closed if each branch is closed. X is an L*-theorem* if there is a closed L-tableau for $P\sim X$, for some prefix P.

Intuitively, PX means X is true in the world P. As in all tableau systems, proof is by refutation: X is a theorem if the assumption that $\sim X$ is true in some world (beginning the tableau with $P\sim X$) produces a contradiction (the tableau closes).

4 *Specific tableau systems*: We give specific tableau structures for the logics of the last section which are easy to work with, and we illustrate their use.

To produce a system for S5, let G_0 be the collection of all positive integers and let R_0 hold between any two integers. If F is a finite subset of G_0, and $P \in F$, let $\varphi(P, F)$ be simply $G_0 - F$. We leave it to the reader to show that $\langle\langle G_0, R_0\rangle, \varphi\rangle$ is a tableau structure for S5.

For all the other logics of **3**, take G_0 to be the collection of all finite sequences of positive integers. If F is a finite subset of G_0 and $P \in F$, take $\varphi(P, F)$ to be the set of all $Q \in G_0$ such that (1) P is an initial segment of Q, (2) Q is exactly one integer longer than P, and (3) Q is not an initial segment of any element of F. By changing the definition of R_0 we produce tableau structures for the various logics. For S4, let PR_0Q mean P is an initial segment of Q (P may be Q itself). For T, let PR_0Q mean P is an initial segment of Q, and Q is at most one integer longer than P. For B, let PR_0Q mean that either P is an initial segment of Q, and Q is at most one integer longer than P, or Q is an initial segment of P, and P is at most one integer longer than Q. To get DS4 and DT, change the corresponding definitions of R_0 above so that PR_0P never holds.

Using the above tableau structure for B, the following is a proof that $\Diamond\Box X \supset X$ is a B-theorem (the numbers on the right are only for explanation are not part of the proof.)

$1 \sim(\Diamond\Box X \supset X)$	(1)
$1\Diamond\Box X$	(2)
$1 \sim X$	(3)
$1, 1\Box X$	(4)
$1X$	(5)
closure	

In this proof lines 2 and 3 are from line 1 by the α rule, 4 is from 2 by the π rule, and 5 is from 4 by the ν rule. For a more complicated example, this time with branching, the following is a T-tableau proof of $(\Box X \wedge \Box Y) \supset \Box(X \wedge Y)$.

	$1 \sim[(\Box X \wedge \Box Y) \supset \Box(X \wedge Y)]$		(1)
	$1(\Box X \wedge \Box Y)$		(2)
	$1 \sim\Box(X \wedge Y)$		(3)
	$1\Box X$		(4)
	$1\Box Y$		(5)
	$1, 1 \sim(X \wedge Y)$		(6)
$1, 1 \sim X$	(7)	$1, 1 \sim Y$	(8)
$1, 1X$	(9)	$1, 1Y$	(10)
closure		closure	

In the above proof, 2 and 3 are from 1 by α, 4 and 5 are from 2 by α, 6 is from 3 by π, 7 and 8 are from 6 by β, 9 is from 4 by ν, and 10 is from 5 by ν.

5 *Correctness of the tableau systems*: Again let L be one of the logics S5, S4, T, B, DS4, or DT, and let $\langle\langle G_0, R_0\rangle, \varphi\rangle$ be a fixed tableau structure for L. The following two lemmas are straightforward.

Lemma 1: *Let S be a set of prefixed formulas.*
(a) *If $S \cup \{\alpha\}$ is L-realizable, so is $S \cup \{\alpha, \alpha_1, \alpha_2\}$*
(b) *If $S \cup \{\beta\}$ is L-realizable, so is one of $S \cup \{\beta, \beta_1\}$ or $S \cup \{\beta, \beta_2\}$.*

Lemma 2: *Let $S \cup \{\nu\}$ be a finite set of prefixed formulas, let F be the set of prefixes occurring in it, let P be the prefix of ν, let $Q \in F$, and let PR_0Q. Then if $S \cup \{\nu\}$ is L-realizable, so is $S \cup \{\nu, \nu(Q)\}$.*

Call a *branch* of a tableau L-realizable if the set of formulas on it is L-realizable; call an *L-tableau* L-realizable if some branch of it is. By the above two lemmas together with the definition of tableau structure for L, a simple modification of the proof in [7] to show the corresponding result for first order logic shows

Theorem: *If T is an L-realizable L-tableau and T' results from T by the application of a single tableau rule, T' is L-realizable.*

Corollary: *If X is provable by an L-tableau, X is valid in all L-models.*

Proof: Suppose X is provable, but in the L-model $\langle G, R, \vdash\rangle$, for some $\Gamma \in G$, $\Gamma \nvdash X$ (so $\Gamma \vdash {\sim} X$). A proof of X begins with $P \sim X$ (for some prefix P). But if I is any function from G_0 to G such that $I(P) = \Gamma$, $\langle\langle G, R, \vdash\rangle, I\rangle$ L-realizes $\{P \sim X\}$, so by the above theorem we must have a closed L-realizable L-tableau, which is not possible.

6 *Modal Hintikka sets*: Let L and $\langle\langle G_0, R_0\rangle, \varphi\rangle$ be as in the last section. Let S be a set of prefixed formulas, and let F be the set of prefixes occurring in S. We call S an *L-Hintikka set* if

(0) *for no atomic formula A and for no prefix P do both PA and $P \sim A$ belong to S*
(1) $\alpha \in S \Rightarrow \alpha_1 \in S$ *and* $\alpha_2 \in S$
(2) $\beta \in S \Rightarrow \beta_1 \in S$ *or* $\beta_2 \in S$
(3) $\nu \in S \Rightarrow \nu(Q) \in S$ *for all $Q \in F$ in the relation R_0 to the prefix of ν, and there are $Q \in F$ in the relation R_0 to the prefix of ν*
(4) $\pi \in S \Rightarrow \pi(Q) \in S$ *for some $Q \in F$ in the relation R_0 to the prefix of π.*

We will show

Theorem: *Any L-Hintikka set is L-realizable.*

First, let L be one of the logics S5, S4, T, or B, and let S be an L-Hintikka set. Define an L-model $\langle G, R, \vdash\rangle$ as follows. Let $G = F$, the set of prefixes in S. Let $R = R_0 \restriction F$. Then $\langle G, R\rangle$ is an L-model structure. If A is atomic, let $P \vdash A$ if $PA \in S$, and extend $\vdash$ to all formulas so that $\langle G, R, \vdash\rangle$ is an L-model (this can be done in one and only one way.) It is easy to show, by induction on the degree of X that if $PX \in S$ then $P \vdash X$, and hence S, is L-realizable by the L-model $\langle G, R, \vdash\rangle$.

Next, let L be either DS4 or DT, and let S be an L-Hintikka set. Now define an L-model $\langle G, R, \vdash\rangle$ as follows. Again let $G = F$. If $P \in F$, call P *terminal* if for no $Q \in F$ does PR_0Q. Let the relation R' be such that $PR'P$ for all terminal $P \in F$. Let $R = R_0 \restriction F \cup R'$. Then $\langle G, R\rangle$ is an L-model structure. Now we may proceed as before.

7 *Completeness of the tableau systems*: Again, let L be one of S5, S4, T, B, DS4, or DT, and let $\langle\langle G_0, R_0\rangle, \varphi\rangle$ be a tableau structure for L. We begin with a description of a systematic tableau proceedure, taken from the first order analog in [7]. It involves designating certain occurrences of prefixed formulas as 'used'.

Let PX be a prefixed formula. Begin an L-tableau by placing PX at the origin. Then apply the four branch extension rules systematically as follows.

Suppose at the *nth* stage the tableau we have constructed is closed, then stop. Also if every occurrence of each non-atomic formula on the tableau has been used, stop. Otherwise, choose an occurrence of a prefixed formula as close to the origin of the tree as possible, say QY, which has not yet been used, and extend the tableau as follows: for each branch through that occurrence of QY,

(1) if QY is an α, add α_1 and α_2 to the end of the branch;

(2) if QY is a β, split the branch and add β_1 to the end of one resulting branch and β_2 to the other;

(3) if QY is a π, take the first (recall G_0 is countable) unrestricted element Q' of G_0 such that QR_0Q', and add $\pi(Q')$ to the end of the branch;

(4) If QY is a ν, let $Q_1', \ldots, Q_n'$ be the prefixes occurring on the branch such that QR_0Q_i' (if any) and add $\nu(Q_1'), \ldots, \nu(Q_n')$ and ν to the end of the branch; if there are no such prefixes, let Q' be the first unrestricted element of G_0 such that QR_0Q', and add $\nu(Q')$ and ν to the end of the branch.

Having done the above for each branch through QY, declare that occurrence of QY used. This concludes the $n + 1st$ stage of the systematic proceedure.

Now let X be a formula. Choose some P from G_0 and, using the above procedure, construct a systematic L-tableau beginning with $P \sim X$. If the tableau closes at some stage, X is an L-theorem. If the proceedure does not produce a closed L-tableau, we will generate a finite or infinite tree which must have an open branch. It is straighforward that an open branch of a systematically completely constructed L-tableau is an L-Hintikka set (and contains $P \sim X$). Then by **6**, in some L-model $\langle G, R, \vdash\rangle$, for some $\Gamma \in G$, $\Gamma \vdash \sim X$, $\Gamma \nvdash X$, so X is not valid in all L-models. Thus we have

Theorem: *If X is valid in all L-models, X is provable by an L-tableau (indeed, by a systematically constructed one).*

8 *The K logics*: In [3] we gave a first order tableau system which proved just those formulas which were classically valid in all domains, including the empty domain. This is the first order analog of the two K logics whose model theory we gave in **2**. For this section let L be one of the logics KS4 or KT, let $\langle G_0, R_0\rangle$ be an L-model structure, and let φ be a selection function for $\langle G_0, R_0\rangle$. Changing a definition from **3** somewhat, we now call $\langle\langle G_0, R_0\rangle, \varphi\rangle$ a *tableau structure for L* if

(1) $\langle G_0, R_0\rangle$ is a countable L model structure

(2) if S is any finite set of prefixed formulas which is L-realizable (as in **3**) and F is the set of prefixes of formulas in S, then if $\pi \epsilon S$ and $Q \epsilon \varphi(P, F)$ where P is the prefix of π, $S \cup \{QX\}$ is L-realizable.

Using a tableau structure for L we construct an L-tableau system just as in **3**, but we make a change in the ν rule: if ν occurs on a branch we may add $\nu(Q)$ for any related, used Q (i.e. we eliminate the possiblity of adding an unrestricted Q).

Specific L tableau structures are easy to come by; for KS4 the tableau structure for DS4 of **4** will work, similarly for KT the DT tableau structure works.

Proofs of completeness and correctness are analogous to the first order work in [3] and we leave the adaptation of **5**, **6**, **7** to the reader.

9 *A slight generalization*: For use in the next section we describe a mild generalization of the above tableau systems, in which we are able to treat *truth functional* combinations of prefixed formulas. Let L now be one of S5, S4, T, B, DS4, or DT (a similar generalization of KS4 and KT is possible, but can not be used in the next section, so we do not discuss it), and let $\langle\langle G_0, R_0\rangle, \varphi\rangle$ be a tableau structure for L. We call truth-functional combinations of prefixed formulas *generalized formulas*, and we use U and V to represent them. (We continue to use X and Y to represent formulas.) We re-define α and β as follows:

α	α_1	α_2
$(U \wedge V)$	U	V
$\sim(U \vee V)$	$\sim U$	$\sim V$
$\sim(U \supset V)$	U	$\sim V$
$\sim\sim U$	U	U

β	β_1	β_2
$(U \vee V)$	U	V
$\sim(U \wedge V)$	$\sim U$	$\sim V$
$(U \supset V)$	$\sim U$	V

The definitions of ν and π are unchanged. The α, β, ν, and π rules are still as in **3**, but we add *prefix reduction rules*:

$$\frac{P(X \wedge Y)}{(PX \wedge PY)} \qquad \frac{P(X \vee Y)}{(PX \vee PY)} \qquad \frac{P(X \supset Y)}{(PX \supset PY)} \qquad \frac{P{\sim}X}{\sim PX}$$

$$\frac{\sim P(X \wedge Y)}{\sim(PX \wedge PY)} \qquad \frac{\sim P(X \vee Y)}{\sim(PX \vee PY)} \qquad \frac{\sim P(X \supset Y)}{\sim(PX \supset PY)} \qquad \frac{\sim P{\sim}X}{\sim\sim PX}$$

The definition of closure of a branch is changed to: a branch is called *closed* if it contains U and $\sim U$ for some generalized formula U. U is called an *L-theorem* if there is a closed L-tableau for $\sim U$. (Now, X is an L-theorem if PX is, where P is any prefix.)

We may also generalize the notion of realization. Thus, let S be a set of generalized formulas, and let I be an interpretation for the prefixes in S

in the L-model $\langle G, R, \vdash\rangle$. We call PX *true* under this interpretation if $I(P) \vdash X$. We call α true if α_1 and α_2 are both true, and β true if one of β_1 or β_2 is true. We call S L-realizable if all the formulas of S are true under some L-interpretation.

Using the above methods we may show: a generalized formula U is L-provable if and only if U is true in all L-interpretations for its prefixes.

We note for use in the next section that from the definition of tableau structure for L we may show

Lemma: *Let S be any finite, L-realizable set of generalized formulas; then if $\nu \in S$ (respectively $\pi \in S$) and Q is unrestricted with respect to S, and related to the prefix of ν (respectively π), then $S \cup \{\nu(Q)\}$ (respectively $S \cup \{\pi(Q)\}$) is L-realizable.*

10 *The fundamental theorem*: Again let L be one of S5, S4, T, B, DS4, or DT, and let $\langle\langle G_0, R_0\rangle, \varphi\rangle$ be a tableau structure for L. In this section we show *Smullyan's fundamental theorem of quantification theory* [7] has an analog for L. Roughly, it says that in a proof the modal part of the argument can be separated from the truth functional part in a natural way.

If S is a (finite) set of generalized formulas, we call U *L-compatible* with S if any L-interpretation for the prefixes in S can be extended to an L-interpretation for the prefixes of U as well. By a *regular* (*generalized*) *formula* we mean any formula of the form $\nu \supset \nu(Q)$ (respectively $\pi \supset \pi(Q)$), where Q is related to the prefix of ν, (respectively of π). By a *regular sequence for U* we mean any (finite) sequence of regular formulas such that (1) if $\nu \supset \nu(Q)$ (respectively $\pi \supset \pi(Q)$) occurs in the sequence, ν (respectively π) is compatible with the set of preceeding generalized formulas and U, (2) if $\pi \supset \pi(Q)$ occurs in the sequence, Q is unrestricted with respect to the set of prefixes in π, preceeding formulas, and U, and (3) if $\nu \supset \nu(Q)$ occurs in the sequence, either Q already occurs in ν, or in a preceeding generalized formula, or in U, or Q is unrestricted with respect to the set of prefixes in ν, preceeding formulas, and U. By a *regular set for U* we mean a set R whose terms can be arranged in a regular sequence for U. If S is a set of generalized formulas, by $\hat{S}$ we mean the conjunction of the formulas in S. By an *augmented regular set for U* we mean a finite set $S \cup R$ where R is a regular set for U, $\hat{S}$ is compatible with $R \cup \{U\}$, and the elements of S are of the form $V \supset V'$ where V' results from the application of a prefix reduction rule to V. We now proceed to show the following

Theorem: *For any generalized formula U, U is an L-theorem if and only if there is some R, an augmented regular set for U, such that $\hat{R} \supset U$ is a classical tautology.*

Lemma: *Let S be an L-realizable set of generalized formulas. Then*

(a) *if ν is compatible with S, $\nu \supset \nu(Q)$ is regular, and Q occurs in $S \cup \{\nu\}$, then $S \cup \{\nu \supset \nu(Q)\}$ is L-realizable,*

(b) *if ν (respectively π) is compatible with S, $\nu \supset \nu(Q)$ (respectively $\pi \supset \pi(Q)$) is a regular formula, and Q is unrestricted with respect to $S \cup \{\nu\}$ (respectively $S \cup \{\pi\}$) then $S \cup \{\nu \supset \nu(Q)\}$ (respectively $S \cup \{\pi \supset \pi(Q)\}$) is L-realizable,*

(c) *if V is compatible with S and V' results from V by the application of a prefix reduction rule, $S \cup \{V \supset V'\}$ is L-realizable.*

Proof: We show only (b). Suppose $\pi \supset \pi(Q)$ is regular and Q is unrestricted with respect to $S \cup \{\pi\}$. S is L-realizable, so there is an L-interpretation I in some L-model $\langle G, R, \vdash \rangle$ making all the formulas of S true. Since π is compatible with S, I can be extended to an interpretation I' for $S \cup \{\pi\}$. Certainly all the formulas of S are still true under I'. If π is false, $\pi \supset \pi(Q)$ is true and we are done. If π is true, $S \cup \{\pi\}$ is L-realizable, and hence so is $S \cup \{\pi, \pi(Q)\}$. Thus $S \cup \{\pi \supset \pi(Q)\}$ is L-realizable.

Let us call a set R augmented regular for a set S of generalized formulas if R is augmented regular for $\hat{S}$.

Lemma: *If S is L-realizable and R is augmented regular for S, $R \cup S$ is L-realizable.*

Proof: By induction on the number of elements in R, using the above lemma.

Theorem: *If R is an augmented regular set for the generalized formula U and $\hat{R} \supset U$ is an L-theorem, so is U.*

Proof: Since R is augmented regular for U, R is also augmented regular for $\sim U$. If U is not an L-theorem, $\{\sim U\}$ must be L-realizable, so by the above lemma, $R \cup \{\sim U\}$ is also. But then $\sim(\hat{R} \supset U)$ is L-realizable, contradicting the fact that $\hat{R} \supset U$ is an L-theorem.

Corollary: *If R is an augmented regular set for U and $\hat{R} \supset U$ is a tautology, U is an L-theorem.*

Proof: If $\hat{R} \supset U$ is a tautology it is provable by the classical α and β tableau rules, and hence by the α and β rules of the generalized system above. Thus $\hat{R} \supset U$ is an L-theorem, and by the above theorem, we are done.

We thus have half of the fundamental theorem. The converse follows very simply from the tableau construction. We first make a change in the ν and π rules. In these, when we required the prefix Q to be unrestricted, it was with respect to the set of formulas on the *branch*; we now require that Q be unrestricted with respect to the set of formulas on the *tree*. Clearly this makes no change in the set of L-theorems.

Now suppose U is provable. Construct a closed tableau for $\sim U$ (using the above restriction) and simultaneously construct a set S, and a sequence R_s of regular formulas as follows. Suppose we have completed the *nth* stage in the tableau construction. If the $n + 1st$ step is to add $\nu(Q)$ to a branch using the ν rule, also add $\nu \supset \nu(Q)$ to the end of the sequence R_s of regular formulas. Similarly if the $n + 1st$ step is to add $\pi(Q)$ to a branch using the π rule, also add $\pi \supset \pi(Q)$ to the end of the sequence R_s. If the $n + 1st$ step is to add V' to a branch containing V, using a prefix reduction rule, add $V \supset V'$ to the set S. Clearly the sequence R_s resulting from the completed tableau is a regular sequence for U, so if R is the set of formulas in the sequence R_s, R is a regular set for U; hence $S \cup R$ is an augmented regular set for U. Moreover, it is easy to see $(\hat{S} \wedge \hat{R}) \supset U$ is provable using only the α and β rules, and so is a tautology.

11 *First order tableau systems*: The model theory for the logics treated above can be generalized to the first order case (in several ways). Corresponding to this, appropriate γ and δ rules can be introduced into the tableau proof systems above. Rather than treat the general case, we work only with S4, and we give, without proof, tableau systems for S4 with and without the Barcan formula (the Barcan formula, $\Diamond(\exists x)A(x) \supset (\exists x)\Diamond A(x)$ makes first order Kripke S4 models "constant domain" models; see [9]); for first order S4 models without the Barcan formula, see [6].

Let $\langle\langle G_0, R_0\rangle, \varphi\rangle$ be a tableau structure for S4. We revert now to the simpler tableau system of **3**. We assume we have a countable set of parameters, $a, b, c, \ldots$, (distinct from bound variables, $x, y, z, \ldots$). Adapting a definition from [7] we define γ and δ formulas and their instances as follows.

γ	$\gamma(a)$
$P(\forall x)A(x)$	$PA(a)$
$P\sim(\exists x)A(x)$	$P\sim A(a)$

δ	$\delta(a)$
$P(\exists x)A(x)$	$PA(a)$
$P\sim(\forall x)A(x)$	$P\sim A(a)$

We add to the α, β, ν, π rules of **3** the following γ and δ rules:

$\dfrac{\gamma}{\gamma(a)}$ *for any parameter a*

$\dfrac{\delta}{\delta(a)}$ *for any new parameter a (i.e. not yet used on the branch)*

The resulting tableau system is first order S4 with the Barcan formula. For first order S4 without the Barcan formula we must complicate matters somewhat. Let us suppose we have associated with each prefix $P \in G_0$ a distinct countable set of parameters, $a_P, b_P, c_P, \ldots$ Then instead of the above rules, let us add:

$\dfrac{\gamma}{\gamma(a_p)}$ *where* a_p *is any parameter associated with* P, *such that if* Q *is the prefix of* γ, PR_0Q.

$\dfrac{\delta}{\delta(a_p)}$ *where* P *is the prefix of* δ, *and* a_p *is new to the branch.*

The resulting tableau system is first order S4, without the Barcan formula.

The work of [7] may be combined with that of **10** to produce a 'double' fundamental theorem in which first the first order part of a proof, then the modal part is separated out.

We also remark that the tableau system for intuitionistic logic of [2] can be modified to use prefixed signed formulas. If we use the devices introduced above for first order S4 we may produce tableau systems for both regular intuitionistic logic and for 'constant domain' intuitionistic logic. Also versions of the fundamental theorem may be shown. Finally, devices similar to those of [3] may be used, producing variants of modal and intuitionistic logic in which the domains of the Kripke models may be empty.

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