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MATTERS OF SEPARATION

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1. Extending in some respects, sharpening in others, results in the literature, we establish here that:

(1) *Every classically valid wff A of* QC=, *the first-order quantificational calculus with identity, is provable by means of axiom schemata* A1-A3 *and rule* R1 *in Table I, plus the axiom schemata and rules of that table for only such of the logical symbols* '$\sim$', '&', '$\vee$', '$\equiv$', '$\forall$', '$\exists$', *and* '=' *as occur in A,*

(2) *Every intuitionistically valid wff A of* QC= *is provable by means of axiom schemata* A1-A2 *and rule* R1 *in Table I, plus the axiom schemata and the seven rules of that table for only such of the logical symbols in question as occur in A.*

In the first of our two theorems R2 is to serve as rule for '$\forall$'; in the second, R2 or R2′ according as '&' occurs or not in A.

TABLE I

Axiom schemata

For '$\supset$':	A1.	$A \supset (B \supset A)$
	A2.	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
	A3.	$((A \supset B) \supset A) \supset A$
For '$\sim$':	A4.	$(A \supset B) \supset (\sim B \supset \sim A)$
	A5.	$A \supset \sim\sim A$
	A6.	$\sim\sim A \supset (\sim A \supset B)$
For '&':	A7.	$(A \,\&\, B) \supset A$
	A8.	$(A \,\&\, B) \supset B$
	A9.	$A \supset (B \supset (A \,\&\, B))$
For '$\vee$':	A10.	$A \supset (A \vee B)$
	A11.	$B \supset (A \vee B)$
	A12.	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
For '$\equiv$':	A13.	$A \supset ((A \equiv B) \supset B)$
	A14.	$A \supset ((B \equiv A) \supset B)$
	A15.	$(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$
For '$\forall$':	A16.	$(\forall X)\, A \supset A(Y/X)$
	For '$\exists$': A17.	$A(Y/X) \supset (\exists X)\, A$

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For '=': A18. $X = X$
A19. $X = Y \supset (A \supset A(Y//X))$, *where* A *is an atomic wff of* QC=.

Attendant substitution conventions: (i) In A16-17 $A(Y/X)$ is to be like A except for containing free Y wherever A contains free X. (ii) In A19 $A(Y//X)$ is to be like A except for containing (free) Y at zero or more places where A contains (free) X.

Rules

For '$\supset$': R1. *From* A *and* $A \supset B$ *to infer* B.

For '$\forall$': R2. *From* $A \supset B$ *to infer* $A \supset (\forall X)B$, *so long as* X *does not occur free in* A.

R2′. *From* $A \supset (B \supset C)$ *to infer* $A \supset (B \supset (\forall X)C)$, *so long as* X *does not occur free in either one of* A *and* B.

For '$\exists$': R3. *From* $A \supset B$ *to infer* $(\exists X)A \supset B$, *so long as* X *does not occur free in* B.

The earliest forerunner of (2) is probably a result of Curry's in [1], which differs from (2) in only three minor respects: (i) A is restricted throughout to be a wff of QC, the first-order quantificational calculus without identity, (ii) '$\equiv$' is ignored, being treated as a defined sign, and (iii) R2 serves in all cases as rule for '$\forall$', the extra axiom schema

B1. $(\forall X)(A \supset B) \supset (A \supset (\forall X)B)$, *where* X *does not occur free in* A,

being thrown in when '&' does not occur in A.[1] The earliest anticipation of (1) that we know of is a theorem of Kleene's in [4], p. 459, to the effect that if a wff A of QC is classically valid, then A is provable by means of axiom schemata A1-A2, the following two axiom schemata (for '$\sim$'):

B2. $(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$
B3. $\sim\sim A \supset A$,

rule R1, plus the axiom schemata and rules of *Table I* for only such of the four logical symbols '&', '$\vee$', '$\forall$', and '$\exists$' as occur in A. Like Curry, Kleene ignores '$\equiv$', uses R2 as his one rule for '$\forall$', and calls on axiom schema B1 (redundant, it so happens, in the presence of A1-A2, B1-B2, and R1) when '&' does not occur in A. A partial forerunner of (1) and (2) is of course Kanger's [3], which gives proof of both theorems for the case where A is a wff of SC, the sentential calculus.[2]

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1. The last footnote on p. 288 of [1] suggests that $B1'$. $(\forall X)(A \supset B) \supset ((\exists X)A \supset B)$, where X does not occur free in B, is also needed in the absence of '&', but this is probably unintended since $B1'$ is provable by means of A1-A2, A16, R1, and R3.

2. Except A9, borrowed from Robinson's [6], A1-A12 are the very axiom schemata that Kanger uses in [3]. Robinson notes in [6] that $(A \supset \sim B) \supset (B \supset \sim A)$ and $\sim A \supset (A \supset B)$ can do duty for all three of A4-A6.

2. For proof of (1) consider first the case where A contains no occurrence of '=' and hence is a wff of QC.

It is shown in [5] that every classically valid sequent of the sort

$$A_1, A_2, \ldots, A_n \rightarrow B,$$

where $A_1, A_2, \ldots, A_n (n \geq 0)$, and B are wffs of QC, is provable by means of the axiom schema

$$K, A, L \rightarrow A \qquad \text{(Ax)}$$

and the intelim rules of *Table II* for only such of the seven logical symbols '$\supset$', '$\sim$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' as occur in the sequent.[3]

TABLE II

	Introduction rules	*Elimination rules*
For '$\supset$':	$\dfrac{K, A \rightarrow B}{K \rightarrow A \supset B}$	$\dfrac{K \rightarrow A \supset B \quad K \rightarrow (A \supset C) \supset A}{K \rightarrow B}$
For '$\sim$':	$\dfrac{K, A \rightarrow B \quad K, A \rightarrow \sim B}{K \rightarrow \sim A}$	$\dfrac{K \rightarrow \sim\sim A}{K \rightarrow A}$
For '&':	$\dfrac{K \rightarrow A \quad K \rightarrow B}{K \rightarrow A \,\&\, B}$	$\dfrac{K \rightarrow A \,\&\, B}{K \rightarrow A} \quad \dfrac{K \rightarrow A \,\&\, B}{K \rightarrow B}$
For '$\vee$':	$\dfrac{K \rightarrow A}{K \rightarrow A \vee B} \quad \dfrac{K \rightarrow B}{K \rightarrow A \vee B}$	$\dfrac{K, A \rightarrow C \quad K, B \rightarrow C \quad K \rightarrow A \vee B}{K \rightarrow C}$
For '$\equiv$':	$\dfrac{K, A \rightarrow B \quad K, B \rightarrow A}{K \rightarrow A \equiv B}$	$\dfrac{K \rightarrow A \quad K \rightarrow (C \equiv A) \equiv (C \equiv B)}{K \rightarrow B}$
For '$\forall$':	$\dfrac{K \rightarrow A}{K, L \rightarrow (\forall X)A}$	$\dfrac{K \rightarrow (\forall X)A}{K \rightarrow A(Y/X)}$
For '$\exists$':	$\dfrac{K \rightarrow A(Y/X)}{K \rightarrow (\exists X)A}$	$\dfrac{K, L \rightarrow (\exists X)A \quad K, A \rightarrow B}{K, L \rightarrow B}$
For '$\forall$' and '$\vee$':	$\dfrac{K \rightarrow A \vee B}{K, L \rightarrow (\forall X)A \vee B}$	

Attendant restrictions: (i) In the introduction rule for '$\forall$' the variable X is not to occur free in any wff in K. (ii) In the elimination rule for '$\exists$' and the introduction rule for '$\forall$' and '$\vee$', X is not to occur free in any wff in K nor in B.

3. In four out of five cases the quantificational rules of *Table I* are simplifications (patterned after rules in Fitch's [2]) of their counterparts in [5]. As the reader may wish to verify, they permit proof of exactly the same sequents as their counterparts in [5] do.

Now let the wff-associate of a sequent of the sort $\rightarrow B$ be B, that of a sequent of the sort $A_1 \rightarrow B$ be $A_1 \supset B$, that of a sequent of the sort A_1, $A_2 \rightarrow B$ be $A_1 \supset (A_2 \supset B)$, and so on. It is easily verified that the wff-associate of any sequent of the above sort $K, A, L \rightarrow A$ is provable by means of A1-A2 and R1. It can also be verified (see section 4 for three sample cases) that (i) if a sequent S follows from another sequent S_1, or two other sequents S_1 and S_2, or three other sequents S_1, S_2, and S_3 by application of an intelim rule of *Table II* for one or (as in the case of the introduction rule for '∀' and 'v') two of the logical symbols '⊃', '~', '&', 'v', '≡', '∀', and '∃', and (ii) the wff-associate of S_1, or the wff-associates of S_1 and S_2, or the wff-associates of S_1, S_2, and S_3 are provable by means of a set α of axiom-schemata and rules from *Table I*, then the wff-associate of S is provable by means of α, A1-A3, R1, and the axiom schemata and rules of *Table I* for the one symbol or the two symbols in question.

Take then the wff A of (1). Since A is presumed to be classically valid, then the corresponding sequent $\rightarrow A$ is sure to be classically valid as well. Hence there is sure to be a proof of $\rightarrow A$ by means of the one axiom schema and the intelim rules of *Table II* for only such of the logical symbols '⊃', '~', '&', 'v', '≡', '∀', and '∃' as occur in $\rightarrow A$; Hence there is sure to be for each entry $K_i \rightarrow B_i$ in the proof in question of $\rightarrow A$ a proof of the wff-associate of $K_i \rightarrow B_i$ by means of A1-A3, R1, and the axiom schemata and rules of *Table I* for only such of the logical symbols '~', '&', 'v', '≡', '∀', and '∃' as occur in $\rightarrow A$. Hence, in particular, there is sure to be a proof of A (the wff-associate of $\rightarrow A$) by means of A1-A3, R1, and the axiom schemata and rules of *Table I* for only such of the logical symbols '~', '&', 'v', '≡', '∀', and '∃' as occur in $\rightarrow A$ and hence in A.[4]

3. Consider then the case where A contains at least one occurrence of '='. Since A is presumed to be classically valid and since the axiom schemata and rules of *Table I* permit proof of every classically valid wff of QC=, there is sure to be a column of wffs of QC= that closes with A and counts as a proof of A by means of the axiom schemata and rules of *Table I.* Now let $B_1, B_2, \ldots$, and B_n ($n \geq 0$) be in any order all the entries in the column in question that are of the sort A18 or the sort A19 in *Table I.* By virtue of the Deduction Theorem

$$B_1 \supset (B_2 \supset (\ldots \supset (B_n \supset A) \ldots))$$

is sure to be provable by means of the axiom schemata and rules of *Table I minus* A18-A19. Hence so is the result

$$B_1' \supset (B_2' \supset (\ldots \supset (B_n' \supset A') \ldots))$$

of turning in every component of $B_1 \supset (B_2 \supset (\ldots \supset (B_n \supset A) \ldots))$ of the sort $X = Y$ for one of the sort $F(X, Y)$, where F is any two-place predicate variable of QC that is foreign to $B_1 \supset (B_2 \supset (\ldots \supset (B_n \supset A) \ldots))$. But

4. The argument is reminiscent of arguments in [1], [3], [4], and [5].

$B_1' \supset (B_2' \supset (\ldots \supset (B_n' \supset A') \ldots))$ is a wff of QC, and—being provable by means of the axiom schemata and rules of *Table I* minus A18-A19—is sure to be classically valid. Hence by the case covered in section 2 $B_1' \supset (B_2' \supset (\ldots \supset (B_n' \supset A') \ldots))$ is sure to be provable by means of A1-A3, R1, and the axiom schemata and rules of *Table I* for only such of the logical symbols '$\sim$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' as occur in $B_1' \supset (B_2' \supset (\ldots \supset (B_n' \supset A')) \ldots))$. Hence clearly $B_1 \supset (B_2 \supset (\ldots \supset (B_n \supset A) \ldots))$ is sure to be provable by means of A1-A3, R1, and the axiom schemata and rules of *Table I* for only such of the symbols in question as occur in $B_1 \supset (B_2 \supset (\ldots \supset (B_n \supset A) \ldots))$. Hence A is sure to be provable by means of A1-A3, A18-A19, R1, and the axiom schemata and rules of *Table I* for only such of the symbols in question as occur in one or more of $B_1, B_2, \ldots, B_n$, and A. But none of '$\sim$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' occurs in anyone of $B_1, B_2, \ldots,$ and B_n; and '=', which does occur in each one of $B_1, B_2, \ldots,$ and B_n, is presumed to occur in A. Hence A is sure to be provable by means of A1-A3, R1, and the axiom schemata and rules of *Table II* for only such of the logical symbols '$\sim$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' as occur in A.[5]

4. The three sample cases that we promised to work out in detail are the introduction rule for '$\forall$' (= $\forall$I), the introduction rule for '$\forall$' and '$\vee$' (= $\forall I_{\vee}$), and the elimination rule for '$\exists$' (= $\exists$E). Throughout α is to be an arbitrary set of axiom schemata and rules from *Table I.*

Lemma 1. $((A \supset B) \supset B) \supset ((A \supset (\forall X)B) \supset B)$ *is provable by means of* A1-A2, A16, *and* R1.

Proof: $(\forall X)B \supset B$ is provable by means of A16. Hence Lemma 1.

Lemma 2. *If* $A \supset (B \supset C)$ *is provable by means of* α, *then* $A \supset (B \supset (\forall X)C)$ *is provable by means of* α, A1-A3, A16, R1, *and* R2, *so long as* X *does not occur free in either one of* A *and* B.

Proof: Suppose $A \supset (B \supset C)$ is provable by means of α. Since $(A \supset (B \supset C)) \supset (((A \supset (B \supset (\forall X)C)) \supset C) \supset C)$ is provable by means of A1-A3 and R1, then $((A \supset (B \supset (\forall X)C)) \supset C) \supset C$ is provable by means of α, A1-A3, and R1. Hence in view of Lemma 1 $((A \supset (B \supset (\forall X)C)) \supset (\forall X)C) \supset C$ is provable by means of α, A1-A3, A16, and R1. Suppose next that X does not occur free in either one of A and B. Then $((A \supset (B \supset (\forall X)C)) \supset (\forall X)C) \supset (\forall X)C$, which follows from $((A \supset (B \supset (\forall X)C)) \supset (\forall X)C) \supset C$ by application of R2, is provable by means of α, A1-A3, A16, R1, and R2. But $(((A \supset (B \supset (\forall X)C)) \supset (\forall X)C) \supset (\forall X)C) \supset (A \supset (B \supset (\forall X)C))$ is provable by means of A1-A3 and R1. Hence Lemma 2.[6]

5. A like argument obviously goes through for any predicate constant other than '=' whose axiom schemata are all of the sort $A_1 \supset (A_2 \supset (\ldots \supset (A_n \supset B) \ldots))$, where $A_1, A_2, \ldots, A_n$ $(n \geq 0)$, and B are atomic.

6. The two conditionals $(A \supset (B \supset C)) \supset (((A \supset (B \supset (\forall X)C)) \supset C) \supset C)$ and $(((A \supset (B \supset (\forall X)C)) \supset (\forall X)C) \supset (\forall X)C) \supset (A \supset (B \supset (\forall X)C))$, though provable by means of A1-A3 and R2, are not provable by means of A1-A2 and R1 alone. Hence B1 (see Lemma 3) will call for a fresh proof in section 5.

Lemma 3. $(\forall X)(A \supset B) \supset (A \supset (\forall X)B)$, *where* X *does not occur free in* A, *is provable by means of* A1-A3, A16, R1, *and* R2.

Proof: $(\forall X)(A \supset B) \supset (A \supset B)$ is provable by means of A16. Hence Lemma 3 by Lemma 2.

Theorem 1. *If the wff-associate* $B_1 \supset (B_2 \supset (\ldots \supset (B_m \supset A)\ldots))$ *of* B_1, $B_2, \ldots, B_m \rightarrow A$ *is provable by means of* α, *then the wff-associate* $B_1 \supset (B_2 \supset (\ldots \supset (B_m \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_n \supset (\forall X)A)\ldots)))) \ldots))$ *of* $B_1, B_2, \ldots, B_m, C_1, C_2, \ldots, C_n \rightarrow (\forall X)A$ *is provable by means of* α, A1-A3, A16, R1, *and* R2, *so long as* X *does not occur free in anyone of* $B_1, B_2, \ldots$, *and* B_m. $(\forall \text{I})$

Proof by mathematical induction on m. *Base Case*: Suppose A is provable by means of α. Then $(p \supset p) \supset A$ is provable by means of α, A1-A2, and R1. Hence $(p \supset p) \supset (\forall X)A$, which follows from $(p \supset p) \supset A$ by application of R2, is provable by means of α, A1-A2, R1, and R2. Hence so is $C_1 \supset (C_2 \supset (\ldots \supset (C_n \supset (\forall X)A)\ldots))$.
Inductive Case: Suppose $B_1 \supset (B_2 \supset (\ldots \supset (B_m \supset A)\ldots))$ is provable by means of α, and X does not occur free in anyone of $B_1, B_2, \ldots$, and B_m. Then by the hypothesis of the induction (with n equal to 0) $B_1 \supset (B_2 \supset (\ldots \supset (\forall X)(B_m \supset A)\ldots))$ is provable by means of α, A1-A3, A16, R1, and R2. But in view of Lemma 3 $(B_1 \supset (B_2 \supset (\ldots \supset (\forall X)(B_m \supset A)\ldots))) \supset (B_1 \supset (B_2 \supset (\ldots \supset (B_m \supset (\forall X)A)\ldots)))$ is provable by means of A1-A3, A16, R1, and R2. Hence $B_1 \supset (B_2 \supset (\ldots \supset (B_m \supset (\forall X)A)\ldots))$ is provable by means of α, A1-A3, A16, R1, and R2. Hence so is $B_1 \supset (B_2 \supset (\ldots \supset (B_m \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_n \supset (\forall X)A)\ldots)))) \ldots))$.

Lemma 4. $(\forall X)(A \vee B) \supset ((\forall X)A \vee B)$, *where* X *does not occur free in* B, *is provable by means of* A1-A3, A10-A12, A16, R1, *and* R2.

Proof: $(\forall X)(A \vee B) \supset (A \vee B)$ is provable by means of A16, and $(A \vee B) \supset ((B \supset A) \supset A)$ provable by means of A1-A3, A10-A12, and R1. Hence $(\forall X)(A \vee B) \supset ((B \supset A) \supset A)$ is provable by means of A1-A3, A10-A12, A16, R1, and R2. Hence in view of Lemma 1 so is $(\forall X)(A \vee B) \supset ((B \supset (\forall X)A) \supset A)$. Hence so is $(\forall X)(A \vee B) \supset ((B \supset (\forall X)A) \supset (\forall X)A)$, which follows from $(\forall X)(A \vee B) \supset ((B \supset (\forall X)A) \supset A)$ by application of R2. But $((B \supset (\forall X)A) \supset (\forall X)A) \supset ((\forall X)A \vee B)$ is provable by means of A1-A3, A10-A12, and R1. Hence Lemma 4.

Theorem 2. *If the wff-associate* $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (A \vee B))\ldots))$ *of* $C_1, C_2, \ldots, C_m \rightarrow A \vee B$ *is provable by means of* α, *then the wff-associate* $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (D_1 \supset (D_2 \supset (\ldots \supset (D_n \supset ((\forall X)A \vee B))\ldots)))) \ldots))$ *of* $C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_n \rightarrow (\forall X)A \vee B$ *is provable by means of* α, A1-A3, A10-A12, A16, R1, *and* R2, *so long as* X *does not occur free in anyone of* $C_1, C_2, \ldots, C_m$, *and* B. $(\forall \text{I}_{\vee})$

Proof: Suppose $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (A \vee B))\ldots))$ is provable by means of α, and X does not occur free in anyone of $C_1, C_2, \ldots$, and C_m. Then in view of Theorem 1, $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (\forall X)(A \vee B))\ldots))$ is provable by means of α, A1-A3, A16, R1, and R2. Suppose next that X does not occur free in B. Then in view of Lemma 4, $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset ((\forall X)A \vee B)) \ldots))$ is provable by means of α, A1-A3, A10-A12, A16, R1, and R2. Hence so is $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (D_1 \supset (D_2 \supset (\ldots \supset (D_n \supset ((\forall X)A \vee B)) \ldots)))) \ldots))$.

Theorem 3. *If the wff-associates* $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (D_1 \supset (D_2 \supset (\ldots \supset (D_n \supset (\exists X)A) \ldots)))) \ldots))$ *and* $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (A \supset B)) \ldots))$ *of* $C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_m \rightarrow (\exists X)A$ *and* $C_1, C_2, \ldots, C_m, A \rightarrow B$ *are provable by means of* α, *then the wff-associate* $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (D_1 \supset (D_2 \supset (\ldots \supset (D_n \supset B) \ldots)))) \ldots))$ *of* $C_1, C_2, \ldots, C_m, D_1, D_2, \ldots, D_n \rightarrow B$ *is provable by means of* α, A1-A2, R1, *and* R3, *so long as* X *does not occur free in anyone of* $C_1, C_2, \ldots, C_m$, *and* B. ($\exists$E)

Proof: Suppose $C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (A \supset B)) \ldots))$ is provable by means of α. Then $A \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset B) \ldots)))$ is provable by means of α, A1-A2, and R1. Suppose next that X does not occur free in anyone of $C_1, C_2, \ldots, C_m$, and B. Then $(\exists X)A \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset B) \ldots)))$, which follows from $A \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset B) \ldots))$ by application of R3, is provable by means of α, A1-A2, R1, and R3. Hence so is $(C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (D_1 \supset (D_2 \supset (\ldots \supset (D_n \supset (\exists X)A) \ldots)))) \ldots))) \supset (C_1 \supset (C_2 \supset (\ldots \supset (C_m \supset (D_1 \supset (D_2 \supset (\ldots \supset (D_n \supset B) \ldots)))) \ldots)))$. Hence Theorem 3.

5. Proof of (2) is essentially like that of (1), except for using another result from [5], this one to the effect that every intuitionistically valid sequent of the sort $A_1, A_2, \ldots, A_n \rightarrow B$ is provable by means of the axiom schema $K, A, L \rightarrow A$ and the intelim rules of Table III for only such of the seven logical symbols '$\supset$', '$\sim$', '&', '$\vee$', '$\equiv$', '$\forall$', and '$\exists$' as occur in the sequent.

TABLE III

Introduction rules: Same as in *Table I* minus $\forall I_{\vee}$.

Elimination rules:

For '&', '$\vee$', '$\forall$', and '$\exists$': Same as in *Table II*.

For '$\supset$': $$\frac{K \rightarrow A \quad K \rightarrow A \supset B}{K \rightarrow B}$$

For '$\sim$': $$\frac{K \rightarrow \sim A \quad K \rightarrow \sim\sim A}{K \rightarrow A}$$

For '$\equiv$': $$\frac{K \rightarrow A \quad K \rightarrow A \equiv B}{K \rightarrow B} \quad \frac{K \rightarrow A \quad K \rightarrow B \equiv A}{K \rightarrow B}$$

To restrict ourselves again to quantificational matters, $\exists$E can be handled as in section 4. $\forall$I, on the other hand, calls for fresh treatment, since our proof of B1 in section 4 (see Lemma 3) makes use of A3. Proof of B1 by means of A1-A2, A7-A9, A16, R1, and R2 is readily had. We do not know, however, of any proof of B1 by means of A1-A2, A16, R1, and R2 alone, nor for that matter of any proof of B1 by means of A1-A2, A16, R1, R2, and the axiom schemata and rules of Table I for anyone of '$\sim$', '$\vee$', '$\equiv$', and '$\exists$'; and hence, in every case in which the wff A of (2) contains no '&', resort to R2′, which of course delivers B1 at a stroke.

Lemma 5. (a) B1 *is provable by means of* A1-A2, A7-A9, A16, R1, *and* R2. (b) B1 *is provable by means of* A16 *and* R2′.

Proof: (a) $(\forall X)(A \supset B) \supset (A \supset B)$ is provable by means of A16. But $((\forall X)(A \supset B) \supset (A \supset B)) \supset (((\forall X)(A \supset B) \,\&\, A) \supset B)$ is provable by means of A1-A2, A7-A9, and R1. Hence $((\forall X)(A \supset B) \,\&\, A) \supset B$ is provable by means of A1-A2, A7-A9, A16, and R1. Hence $((\forall X)(A \supset B) \,\&\, A) \supset (\forall X)B$, which follows from $((\forall X)(A \supset B) \,\&\, A) \supset B$ by application of R2, is provable by means of A1-A2, A7-A9, A16, R1, and R2. But $(((\forall X)(A \supset B) \,\&\, A) \supset (\forall X)B) \supset ((\forall X)(A \supset B) \supset (A \supset (\forall X)B))$ is provable by means of A1-A2, A7-A9, and R1. Hence $(\forall X)(A \supset B) \supset (A \supset (\forall X)B)$ is provable by means A1-A2, A7-A9, A16, R1, and R2.
(b) $(\forall X)(A \supset B) \supset (A \supset B)$, from which $(\forall X)(A \supset B) \supset (A \supset (\forall X)B)$ follows by application of R2′, is provable by means of A16. Hence (b).

Theorem 4. *If the wff-associate of* $B_1, B_2, \ldots, B_m \rightarrow A$ *is provable by means of* α, *then the wff-associate of* $B_1, B_2, \ldots, B_m, C_1, C_2, \ldots, C_n \rightarrow (\forall X)A$, *where* X *does not occur free in anyone of* $B_1, B_2, \ldots,$ *and* B_m, *is provable by means of* α, A1-A2, A16, R1, *and* R2 *when* A7-A9 *belong to* α, *otherwise by means of* α, A16, *and* R2′.

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