224
Notre Dame Journal of Formal Logic
Volume XIII, Number 2, April 1972
NDJFAM

NECESSITY AND TICKET ENTAILMENT

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In [1], Anderson introduces the system $\mathbf{P}_I$, i.e. the implicational fragment of the system $\mathbf{P}$ of "ticket entailment," for which the following axiom schemas are given:

$\mathbf{P}_I$1. $A \to A$
$\mathbf{P}_I$2. $A \to B \to . B \to C \to . A \to C$
$\mathbf{P}_I$3. $A \to B \to . C \to A \to . C \to B$
$\mathbf{P}_I$4. $(A \to . A \to B) \to . A \to B.$

$\to\mathbf{E}$ (*modus ponens*) is the sole primitive inference rule of $\mathbf{P}_I$. A theory of necessity cannot be developed in $\mathbf{P}_I$ (as in $\mathbf{E}_I$, i.e. the implicational fragment of $\mathbf{E}$) via the definition

$$\mathsf{N}A =_{df} A \to A \to A$$

since $A \to A \to A \to A$ (i.e. $\mathsf{N}A \to A$) is not provable in $\mathbf{P}_I$. In [2], the question is raised whether there is any function f of a single variable A definable in $\mathbf{P}_I$ which makes $f(A)$ look like "necessarily A," i.e. such that

(1) $\vdash f(A) \to A$
(2) $\dashv A \to f(A)$
(3) *if* $\vdash A$ *then* $\vdash f(A)$
(4) *if* $\vdash A \to B$ *then* $\vdash f(A) \to f(B)$.

In [3, §6], the question is raised again with slightly different conditions on f: (1)-(3) above, and

(5) $\vdash A \to B \to . f(A) \to f(B)$.

This last formulation of the question is answered by the following

Theorem. *There is no function f definable in $\mathbf{P}_I$ satisfying conditions* (1)-(3) *and* (5).

Proof. Assume on the contrary that there is such a function. Consider the matrix (with designated values 2 and 3)

Received January 29, 1971

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
*2	0	3	2	3
*3	0	0	0	3

It is easy to verify that this matrix satisfies $\mathbf{P}_I$1-$\mathbf{P}_I$4 and $\rightarrow$**E**. If f is to be definable in terms of $\rightarrow$, there must also be a matrix

A	$f(A)$
0	l
1	m
2	n
3	p

such that each of l, m, n, and p is a member of $\{0, 1, 2, 3\}$ and such that the two matrices together satisfy (1), (3), and (5). $f(A)$ must be distinct from A to satisfy (2), so $f(A)$ must be an entailment. Since entailments never take the value 1, we have it that

(6) $l, m, n, p \in \{0, 2, 3\}$.

It is immediate that

(7) $n \neq 3$
(8) $m \neq 3$

if we are to have (1). Now, consider the following row of a truth-table for (5)

A	B	$A \rightarrow B \rightarrow . f(A) \rightarrow f(B)$		
2	1	3	n	m
			*	

A '3' must be entered in the starred column to insure that (5) takes a designated value for this assignment of values to A and B. Given (6), (7), and (8), we can have a '3' here only if $n = 0$. But this falsifies (3), since $A \rightarrow A$ is a theorem of $\mathbf{P}_I$ and $f(A \rightarrow A)$ is not (for $A = 2$, $A \rightarrow A = 2$, so $f(A \rightarrow A) = 0$). Thus there is no such f.

REFERENCES

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[3] Anderson, Alan Ross, and Nuel D. Belnap, Jr., *Entailment*, Forthcoming.

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