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TRANSITIVITY, SUPERTRANSITIVITY AND INDUCTION

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In [3] and [6] Montague and Tarski, respectively, published without proofs the following general principles of induction as theorems of Zermelo-Fraenkel set theory (ZF).

A. (Montague) *If the formula* $\varphi(x)$ *does not contain the variable* y *and the relation* R *is well-founded then*

$$(y)(y \in \mathsf{Fld}R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in \mathsf{Fld}R \rightarrow \varphi(y)).$$

B. (Tarski) *If the formula* $\varphi(x)$ *does not contain the variable* y *then*

$$(y)[(x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(\varphi(y)).$$

In this paper we shall present, in Gödel-Bernays set theory (GB), some results concerning general principles of induction, relating them to **A** and **B** above. In section **1** we list our notation and definitions. In section **2**, since Montague and Tarski published their results without proofs, we shall for the sake of completeness give our proofs of their results. We shall also prove the following general induction principle:

C. *If the formula* $\varphi(x)$ *does not contain the variable* y *and* A *is a transitive class then*

$$(y)[y \in A \wedge (x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(y \in A \rightarrow \varphi(y)).$$

We also show that it is necessary that A be transitive for this principle to hold. We also present a transitive decomposition formula for classes. In section **3** we introduce the notion of supertransitivity for classes, give examples of supertransitive classes and discuss the relationship between supertransitivity and transitivity. Finally in section **4** we present a supertransitive decomposition formula for classes and prove the following induction principle for supertransitive classes.

D. *Let* A *be a supertransitive class and* $\varphi(x)$ *be a formula not containing the variable* y. *If* φ *has the property that for every sequence of sets* $\{a_n\}_{n<\omega}$

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in A *such that* $a_{n+1} \subset a_n$ *for every* $n \geqq 0$, *there is at least one integer* m *such that* $\varphi(a_m)$, *then*

$$(y)\,[y \in A \wedge (x)\,(x \subset y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)\,(y \in A \rightarrow \varphi(y))\,.$$

We also show that if $\mathrm{card}(A_0^s)$ is finite then it is necessary that A be supertransitive for this principle to hold. The notation A_0^s is defined in section **4**.

1. *Notation and Terminology*. The small Latin letters $a, b, c, \ldots$ are used for elements of classes; i, m, n are reserved for non-negative integers and x, y, u, v for variables. The capital Latin letters $A, B, C, \ldots$ are used for classes but R will always represent a relation. As usual $\emptyset$ denotes the null set and the symbols $\subseteq$ and $\subset$ denote inclusion and proper inclusion respectively.

$$\bigcup_{n=0}^{\infty} A_n = \{x \mid (\exists m)\,(m \geqq 0 \wedge x \in A_m)\}\,.$$

$$\bigcap_{n=0}^{\infty} A_n = \{x \mid (m)\,(m \geqq 0 \rightarrow x \in A_m)\}\,.$$

$\varphi \rightarrow \psi$ will denote material implication, while $\varphi \Rightarrow \psi$ will denote inferential implication. We use the following definitions:

(a) A *relation* R is a class of ordered pairs $\langle x, y\rangle$ such that $x \in$ *domain* R and $y \in$ *range* R.
(b) R is *well-founded* if and only if there is no sequence of sets $\{\langle a_{n+1}, a_n\rangle\}_{n<\omega}$ in R.
(c) R is *internal* if and only if $(u)\,(v)\,(x)\,((xRu \equiv xRv) \equiv u = v)$.
(d) $\mathrm{Fld}\,R$ (the *field* of R) = domain $R \cup$ range R.
(e) If i is an integer, define the *power of* A, $\mathcal{P}^i(A)$ as follows:
(i) If $i = 0$, $\mathcal{P}^i(A) = A$ for A a set or a class. (ii) If A is a set, $\mathcal{P}(A) = \{x \mid x \subseteq A\}$. (iii) If A is a class, $\mathcal{P}(A) = \{x \mid x \subseteq A$ and x is a set$\}$. (iv) If $i > 0$, $\mathcal{P}^i(A) = \mathcal{P}(\mathcal{P}^{i-1}(A))$.
(f) A is *transitive* if and only if $(y)\,(y \in A \rightarrow (x)\,(x \in y \rightarrow x \in A))$.
(g) A is *supertransitive* if and only if $(y)\,(y \in A \rightarrow (x)\,(x \subseteq y \rightarrow x \in A))$.

Remark: In [5], p. 270, supertransitivity is defined for certain sets.

2. *Transitivity and Induction*. It is known that every theorem of ZF is a theorem of GB and that any theorem in GB about sets only is a theorem in ZF. (Cohen has these proofs in [2], p. 77, for example.) Below, we prove Montague's induction principle (**A**) in GB and show that Tarski's induction principle (**B**) is a special case of Montague's. Virtually the same proofs will suffice to prove these principles in ZF.

Lemma 1. (Montague's **A**) *If the formula* $\varphi(x)$ *does not contain the variable* y *and the relation* R *is well-founded then*

$$(y)\,[y \in \mathrm{Fld}\,R \wedge (x)\,(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)\,(y \in \mathrm{Fld}\,R \rightarrow \varphi(y))\,.$$

Proof: We use an indirect proof. Let us assume the hypotheses of **A** and that

(1) $(y)[y \in \mathsf{Fld}\, R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)]$
(2) $(\exists y)[\ y \in \mathsf{Fld}\, R \wedge \sim(\varphi(y))]$.

Let a_0 be such that $a_0 \in \mathsf{Fld} R$ and $\sim(\varphi(a_0))$. First, suppose that a_0 has no predecessor in $\mathsf{Fld} R$, that is, there is no $b \in \mathsf{Fld} R$ such that bRa_0. Then clearly

(3) $a_0 \in \mathsf{Fld} R \wedge (x)(xRa_0 \rightarrow \varphi(x))$.

By (1) we may deduce $\varphi(a_0)$, which contradicts our supposition that $\sim(\varphi(a_0))$. Thus we may suppose that a_0 has at least one predecessor in $\mathsf{Fld} R$. If φ holds for every precedessor of a_0 then (3) holds and we can use (1) to deduce $\varphi(a_0)$, again obtaining a contradiction. Consequently we have

(4) $(\exists y)(yRa_0 \wedge \sim(\varphi(y)))$.

Let a_1 be such that a_1Ra_0 and $\sim(\varphi(a_1))$. Then $a_1 \in \mathsf{Fld} R$. If a_1 has no predecessors in $\mathsf{Fld} R$ or if φ holds for every predecessor of a_1 we can again use (1) to derive the contradiction $\varphi(a_1)$. Consequently we have

(5) $(\exists y)(yRa_1 \wedge \sim(\varphi(y)))$.

By induction we obtain a sequence $\{a_n\}_{n<\omega}$ in $\mathsf{Fld} R$ such that $a_{n+1}Ra_n$ for every $n \geqq 0$. This contradicts the well-foundedness of R. Hence, Lemma 1 follows by contraposition.

The following lemma is needed to show that Tarski's induction principle **B** is a special case of Montague's principle **A**.

Lemma 2. *Let* $\{a_n\}_{n<\omega}$ *be a sequence of sets. It is false that* $a_{n+1} \in a_n$ *for every* $n \geqq 0$.

Proof: If $a_{n+1} \in a_n$ for every $n \geqq 0$, the class $\{a_n | n \geqq 0\}$ contradicts the axiom of regularity. Lemma 2 follows by contraposition.

Lemma 3. *Tarski's induction principle* **B** *is a special case of Montague's induction principle* **A**.

Proof: Let V be the class of all sets in GB. Let us assume that $\varphi(x)$ is a formula not containing the variable y. We define a relation R as follows:

(1) $(x)(y)(xRy \equiv x \in \mathsf{V} \wedge y \in \mathsf{V} \wedge x \in y)$.

By Lemma 2 the relation R is well-founded. Clearly $\mathsf{Fld} R = \mathsf{V}$. Montague's principle thus yields

(2) $(y)[y \in \mathsf{Fld} R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(y \in \mathsf{Fld} R \rightarrow \varphi(y))$.

But (2) is equivalent to

(3) $(y)[(x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(\varphi(y))$

where quantification is over sets. This is Tarski's induction principle as required.

We shall now discuss some results about transitive classes, develop a transitive decomposition formula for classes, and prove induction principle **C**.

Lemma 4. (Belding) *For every* A, *if* A *is a class, then the following statements are equivalent*: (i) A *is transitive*, (ii) $A \subseteq \mathcal{P}^i(A)$ *for every* $i \geqq 0$, *and* (iii) *there is a class* C *such that* $A = \bigcap_{i=0}^{\infty} \mathcal{P}^i(C)$.

Proof: Let A be a class. Then (i)$\Rightarrow$(ii). We prove this by induction. By Definition (e), we have

(1) $A \subseteq \mathcal{P}^0(A)$.

By (i) and Definition (f), we have that $A \subseteq \mathcal{P}(A)$. Suppose inductively that for a given i, $A \subseteq \mathcal{P}^i(A)$. Thus, it follows that $(x)(x \in A \rightarrow x \subseteq \mathcal{P}^i(A))$. Therefore, by the definitions of inclusion and power set, we have, $A \subseteq \mathcal{P}^{i+1}(A)$. We conclude

(2) For every $i > 0$, if $A \subseteq \mathcal{P}^i(A)$ then $A \subseteq \mathcal{P}^{i+1}(A)$.

Therefore, by (1), (2) and induction over i, we have

(3) $A \subseteq \mathcal{P}^i(A)$ for every $i \geqq 0$.

(ii)$\Rightarrow$(iii). Put $C = A$ and note that $\bigcap_{i=0}^{\infty} \mathcal{P}^i(A) \subseteq \mathcal{P}^0(A) = A$.

(iii)$\Rightarrow$(i). Assume (iii). Then there is a class C such that

(4) $A = \bigcap_{i=0}^{\infty} \mathcal{P}^i(C)$.

Therefore, by (4), we have

(5) $(y)(x)(y \in A \rightarrow (x \in y \rightarrow (i)(i \geqq 0 \rightarrow x \in \mathcal{P}^i(C))))$.

By (5) we have

(6) $(y)(x)\left(y \in A \rightarrow \left(x \in y \rightarrow x \in \bigcap_{i=0}^{\infty} \mathcal{P}^i(C)\right)\right)$.

Thus, by (4) and (6) we have

(7) $(y)(y \in A \rightarrow y \subset A)$.

Thus (iii)$\Rightarrow$(i) which concludes the proof of Lemma 4.

Lemma 5. *For every* A, *if* A *is a class, then there is a unique maximal transitive subclass of* A.

Proof: If A is a class, define B by:

(1) $B = \bigcap_{i=0}^{\infty} \mathcal{P}^i(A)$.

Then, by (1) and Lemma 4, B is a transitive subclass of A. Now let us assume that K is any transitive subclass of A. Then, by Lemma (4), for every $i \geqq 0$, $K \subseteq \mathcal{P}^i(K)$ and, since $K \subseteq A$, we have at once: for every $i \geqq 0$, $\mathcal{P}^i(K) \subseteq \mathcal{P}^i(A)$. Hence we have:

(2) for every $i \geqq 0$, $K \subseteq \mathcal{P}^i(A)$

which together with (1) gives $K \subseteq B$ as required, also proving uniqueness.

The following theorem yields a transitive decomposition formula for classes.

Theorem 6. (Belding) *If A is a class, B the maximal transitive subclass of A and $A_n = \{x \mid x \in \mathcal{P}^i(A)$ for $0 \leqq i \leqq n$ and $x \notin \mathcal{P}^{n+1}(A)\}$, then*: (i) $A = \bigcup_{n=0}^{\infty} A_n \cup B$, (ii) *for every $n \geqq 0$, $B \cap A_n = \emptyset$, and* (iii) *for every m and n such that $m \neq 0 \neq n$ and $m \neq n$, $A_m \cap A_n = \emptyset$.*

Proof: Let A be a class. Then clearly

(1) $\bigcup_{n=0}^{\infty} A_n \cup B \subseteq A$.

For every $x \in A$, either (a) for every $i \geqq 0$, $x \in \mathcal{P}^i(A)$ or (b) there is a smallest n such that $x \in \mathcal{P}^n(A)$ and $x \notin \mathcal{P}^{n+1}(A)$. If point (a) holds, then, clearly, $x \in B$. If (b) holds, by definition of A_n, $x \in A_n$. Therefore, we have:

(2) $A \subseteq \bigcup_{n=0}^{\infty} A_n \cup B$.

Thus, by (1) and (2) we obtain $A = \bigcup_{n=0}^{\infty} A_n \cup B$, and (i) is proved.

By definition of A_n, for every x, if $x \in A_n$, then $x \notin \mathcal{P}^{n+1}(A)$. Thus, for every n, $A_n \cap B = \emptyset$, and (ii) is proved.

Suppose $m \neq n$ and $m < n$. Then for every x, $x \in A_m$ implies $x \notin \mathcal{P}^{m+1}(A)$ while $A_n \subseteq \mathcal{P}^{m+1}(A)$. Thus $A_m \cap A_n = \emptyset$ and (iii) is proved.

Example 1. Let $a = \{\emptyset\}$, $A = \{a, \{a\}\}$. Then $a \in A$ but $a \notin \mathcal{P}(A)$. Also $\{a\} \in A \cap \mathcal{P}(A)$ but $\{a\} \notin \mathcal{P}^2(A)$. Thus $A = A_0 \cup A_1$ and the maximal transitive subset of A is $\emptyset$.

Example 2. Let a be as in Example 1 and $A = \{a, \{\{a\}\}\}$. Here $\{\{a\}\} \notin \mathcal{P}(A)$ so $A = A_0$.

Example 3. Let a be as in Example 1 and $A = \{a, \{a\}, \{\{a\}\}, \ldots\}$. Then $a \in A$ but $a \notin \mathcal{P}(A)$, so $a \in A_0$. $\{a\} \in A \cap \mathcal{P}(A)$ but $\{a\} \notin \mathcal{P}^2(A)$, so $\{a\} \in A_1$. Similarly each $\{\ldots\{a\}_1 \ldots\}_n \in A_n$. Hence $A = \bigcup_{n=0}^{\infty} A_n$, each A_n having only one member. The maximal transitive subset of A is $\emptyset$.

Example 4. Let $A = \omega$ or $A = \mathsf{On}$ (the class of all ordinal numbers) or $A = \mathsf{V}$ (the class of all sets). In each case, A is transitive and thus is its own maximal transitive subclass.

Example 5. Let $A = \{\emptyset, \{\{\emptyset\}\}\}$. Here, $\{\{\emptyset\}\} \notin \mathcal{P}(A)$, so $\{\{\emptyset\}\} \in A_0$. But $\{\emptyset\}$ is transitive and, since A is not transitive, $\{\emptyset\}$ is the maximal transitive subset of A. Thus $A = A_0 \cup B$, where $A_0 = \{\{\{\emptyset\}\}\}$ and $B = \{\emptyset\}$.

Lemma 7. *Let A be a class and $\bigcup_{n=0}^{\infty} A_n \cup B$ the transitive decomposition of A. The following statements are equivalent*:

(i) *A is transitive,*

(ii) $A_0 = \emptyset$,

(iii) $A = B$.

Proof: (i)$\Longrightarrow$(ii). If A is transitive then $A \subseteq \mathcal{P}(A)$, i.e.: $A - \mathcal{P}(A) = \emptyset$. But, by definitions of A_n and $\mathcal{P}^0(A)$, we have $A_0 \subset A - \mathcal{P}(A)$. Hence we have that $A_0 = \emptyset$ and (i)$\Longrightarrow$(ii) is proved.

(ii)$\Longrightarrow$(iii). If (ii) holds, then, clearly, $A \subseteq \mathcal{P}(A)$ which by induction gives:

(1) for every $i \geqq 0$, $\mathcal{P}^i(A) \subseteq \mathcal{P}^{i+1}(A)$.

Thus, due to (1) we can conclude that for every $i \geqq 0, A \subseteq \mathcal{P}^i(A)$; i.e., that A is the maximal transitive subclass of A. Therefore, this fact and the hypotheses of the lemma imply that $A = B$ and thus (ii)$\Longrightarrow$(iii) is proved.

(iii)$\Longrightarrow$(i). Assume (iii). Then, clearly, from the definition of B, and Lemma 4, it follows that A is transitive. Thus we have shown that (iii)$\Longrightarrow$(i) and the proof of Lemma 7 is complete.

From the following theorem we can easily deduce our induction principle **C** mentioned earlier.

Theorem 8. (Poss). *Let A be a class. A necessary and sufficient condition for the following induction principle to hold is that A be transitive.*

Let $\varphi(x)$ be a formula which does not contain the variable y; then

$$(y)\,[y \in A \wedge (x)\,(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)\,(y \in A \rightarrow \varphi(y)) .$$

Proof: We give two proofs of the necessity part of the theorem, the second being an application of Theorem 6. Let us assume that formula $\varphi(x)$ does not contain the variable y and that A is a class. Now assume also that

(1) $(y)\,[y \in A \wedge (x)\,(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)\,(y \in A \rightarrow \varphi(y))$.

We define the formula $\varphi(x)$, which satisfies the assumptions, as follows:

(2) $(x)\,(\varphi(x) \equiv x \in A \wedge x \subseteq A)$.

Then, clearly, by (2),

(3) $(y)\,(y \in A \wedge (x)\,(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y))$.

Hence, by (1), (3) and (2), we have $(y)\,(y \in A \rightarrow y \subseteq A)$, i.e., A is transitive.

Alternative indirect proof. Under the same assumptions including (1) suppose that A is not transitive. Then, by Theorem 6 there is the transitive decomposition of A:

(4) $A = \bigcup_{n=0}^{\infty} A_n \cup B.$

Now, define a formula $\varphi(x)$, which satisfies the assumptions as follows:

(5) $(x)\left(\varphi(x) \equiv (\exists m)\left(x \in A - \bigcup_{n=0}^{m} A_n\right)\right)$

By (4), the supposition, and Lemma 7, $A_0 \neq \emptyset$. Hence there is a b such that $b \in A_0$ which, together with the fact that $b \in A$ and (5), yields that $\sim(\varphi(b))$. Therefore, we have $\sim((y)\,(y \in A \rightarrow \varphi(y)))$. Whence, by (1), there is a z such that $z \in A$, $(x)\,(x \in z \rightarrow \varphi(x))$, and $\sim(\varphi(z))$. By (5), $z \subset A$. Therefore $z \in \mathcal{P}(A)$.

Hence $z \notin A - \mathcal{P}(A)$; i.e., $z \in A - A_0$ which, together with (5) yields $\varphi(z)$. Since this contradicts $\sim(\varphi(z))$, our supposition is false and, therefore, we now have that A is transitive, which completes the indirect proof.

We now show that it is sufficient that A be transitive for the induction principle to hold. Assume that formula $\varphi(x)$ does not contain the variable y and that A is a class, and that

(6) A is transitive

(7) $(y)(y \in A \wedge (x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y))$

and, in addition, suppose that

(8) $\sim((y)(y \in A \rightarrow \varphi(y)))$.

By (8), there is an a_0 such that

(9) $a_0 \in A$ and $\sim(\varphi(a_0))$.

By (7) and (9) we may conclude that there is an a_1 such that

(10) $a_1 \in a_0$ and $\sim(\varphi(a_1))$.

It follows from (6), (9), and (10) that $a_1 \in A$. Continuing this process, we obtain a sequence $\{a_n\}_{n<\omega}$ of elements of A such that, for $n \geqq 0$, $a_{n+1} \in a_n$. But this is impossible. (See Lemma 2). Thus (8) is false and the desired result holds. This completes the proof of Theorem 8.

Corollary 8.1 (**C**). *If the formula $\varphi(x)$ does not contain the variable y and A is a transitive class, then*:

$$(y)[y \in A \wedge (x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(y \in A \rightarrow \varphi(y)).$$

We now discuss the relationship that **C** has to **A** and **B**. For our first result we need to use a theorem due to Mostowski in which the following definition occurs. Given a set S the ϵ-relation limited to S, ϵ_S, is defined as follows:

$$(x)(y)(x \epsilon_S y \equiv x \in S \wedge y \in S \wedge x \in y)$$

Mostowski, in [4], Theorem 3, has proved the following:

For every well-founded and internal relation R whose field is a set, there is a set S such that ϵ_S is an internal relation, R is isomorphic with ϵ_S, and $\mathrm{Fld}\, \epsilon_S$ *is transitive.*

In particular, there is a bijection f between $\mathrm{Fld} R$ and $\mathrm{Fld}\, \epsilon_S$ such that $f'aRf'b$ if and only if $a \in b$, for a, b in $\mathrm{Fld}\, \epsilon_S$.

Lemma 9. *If we consider only well-founded relations R which are internal and whose fields are sets, then* **A** *is a special case of* **C**.

Proof: Let f be the bijection given by Mostowski's theorem between $\mathrm{Fld} R$ and $\mathrm{Fld}\, \epsilon_S$. By the definition of ϵ_S, $\mathrm{Fld}\, \epsilon_S \subseteq S$ and, hence is a set. By Mostowski's theorem, $\mathrm{Fld}\, \epsilon_S$ is transitive, thus we may use our **C**. Hence,

(1) $(y)(y \in \mathsf{Fld}\ \epsilon_s \wedge (x)(x \in y \to \varphi(x)) \to \varphi(y)) \to (y)(y \in \mathsf{Fld}\ \epsilon_s \to \varphi(y))$

for formulas $\varphi(x)$ not containing the variable y. By the properties of f and $\mathsf{Fld}\ \epsilon_s$, we have that $y \in \mathsf{Fld}\ \epsilon_s \equiv f'y \in \mathsf{Fld}\, R$, and $y \in \mathsf{Fld}\ \epsilon_s \wedge x \in y \to x \in \mathsf{Fld}\ \epsilon_s$. Hence by (1) and the fact that f is an isomorphism, we have

(2) $(u)(u \in \mathsf{Fld}\, R \wedge (v)(vRu \to \varphi(v)) \to \varphi(u)) \to (u)(u \in \mathsf{Fld}\, R \to \varphi(u))$.

Thus **A** holds for the relation R with the given properties.

The following result shows that **C** is a special case of Montague's **A**. (Note that **A** does not imply that it is *necessary* that the class A be transitive for the induction principle to hold on it).

Lemma 10 (Poss). *The induction principle* **C** *is a special case of Montague's induction principle* **A**.

Proof: It is sufficient to show that, for any transitive class A, there is a relation R such that Montague's principle for R is the same as principle **C** for R. Let A be a transitive class. Define the relation R as follows:

(1) $(x)(y)(xRy \equiv x \in y \wedge y \in A)$.

By (1), it is clear that R is well-founded and we have that $\mathsf{Fld}R = A$. Hence we can apply Montague's principle **A** and we obtain for formulas $\varphi(x)$ not containing the variable y:

(2) $(y)(y \in \mathsf{Fld}R \wedge (x)(xRy \to \varphi(x)) \to \varphi(y)) \to (y)(y \in \mathsf{Fld}R \to \varphi(y))$.

By (1) and (2), we have

(3) $(y)(y \in A \wedge (x)(x \in y \wedge y \in A \to \varphi(x)) \to \varphi(y)) \to (y)(y \in A \to \varphi(y))$.

But (3) is the same as

(4) $(y)(y \in A \wedge (x)(x \in y \to \varphi(x)) \to \varphi(y)) \to (y)(y \in A \to \varphi(y))$.

But since A is transitive, (7) gives us **C**; thus Lemma 10 has been proved.

Theorem 11. (Poss) *In the field of* GB *set theory, Tarski's induction principle* **B** *is a special case of* **C**.

Proof: We need only find a transitive class A such that, when **C** is applied to A, we obtain **B**. As mentioned above in Example 4, V, the class of all sets, is transitive. Applying **C** to V, we obtain:

(1) $(y)[y \in \mathsf{V} \wedge (x)(x \in y \to \varphi(x)) \to \varphi(y)] \to (y)(y \in \mathsf{V} \to \varphi(y))$.

Since, in **B**, the quantification is over sets, (1) is precisely the same as **B**.

Theorem 12. *In the field of* GB *set theory, transfinite induction is a special case of* **C**.

Proof: As in the proof of Theorem 11, we need only find a transitive class A such that when **C** is applied to A, we obtain the principle of transfinite induction. As mentioned above, in Example 4, On, the class of all ordinal numbers, is transitive. Applying **C** to On, we obtain:

(1) If $\varphi(x)$ is a formula not containing the variable y, then $(y)[y \in \mathrm{On} \wedge (x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(y \in \mathrm{On} \rightarrow \varphi(y))$.

But, for ordinal numbers the $\in$-relation is the same as the $<$ relation. Thus (1) is the same as:

(2) If $\varphi(x)$ is a formula not containing the variable y, then $(y)[y \in \mathrm{On} \wedge (x)(x < y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(y \in \mathrm{On} \rightarrow \varphi(y))$.

But (2) is itself the principle of transfinite induction and our theorem is proved.

3. *Transitivity and Supertransitivity.* Putting $0 = \emptyset$, $1 = \{0\}$, $2 = 1 \cup \{1\}$ and so on, we have that the only supertransitive integers are 0, 1, and 2. By a simple check each of these integers is supertransitive. Since $2 \in 3$ and $\{\{\emptyset\}\} \subset 2$ but $\{\{\emptyset\}\} \notin 3$ we see that 3 is not supertransitive. If n is any integer such that $n \geqq 3$ then $2 \in n$ and $\{\{\emptyset\}\} \subset 2$, but $\{\{\emptyset\}\} \in n$ as $\{\{\emptyset\}\}$ is not an integer. So n is not supertransitive.

Lemma 13. *For every A, if A is a class then $\boldsymbol{P}(A)$ is a supertransitive class.*

Proof: Let $a \in \boldsymbol{P}(A)$ and $b \subseteq a$. Then a is a set, $a \subseteq A$ and $b \subseteq a$. Since b also is a set, $b \in \boldsymbol{P}(A)$. Thus $\boldsymbol{P}(A)$ is a supertransitive class.

Note that $1 = \boldsymbol{P}(0)$ and $2 = \boldsymbol{P}(1)$. From Lemma 13 and the previous discussion we conclude that 1 and 2 are the only integers which are power sets of other sets. From its definition we know that an integer is a transitive set each of whose elements are transitive. Since 3 is not supertransitive we have that transitivity does not imply supertransitivity. The following example shows that supertransitivity does not imply transitivity. Let $a \neq \emptyset$ and put $A = \boldsymbol{P}(\{a, \{a\}\})$. By Lemma 13, A is supertransitive. Since $\{a\} \in A$ but $a \notin A$ we see that A is not transitive. Hence supertransitivity and transitivity are independent properties in GB (and ZF).

Lemma 14. *If A is a supertransitive class and b is a member of A such that there is no b' in A such that $b \subset b'$ then $A - \{b\}$ is a supertransitive class.*

Proof: If $c \in A - \{b\}$ and $d \subseteq c$ then $d \neq b$ by hypothesis, so $d \in A - \{b\}$ by the supertransitivity of A.

In particular, since every set is a class, if A is a set then $\boldsymbol{P}(A) - \{A\}$ is a supertransitive set. If $\mathrm{card}(A) = n > 1$ then $\mathrm{card}(\boldsymbol{P}(A) - \{A\}) = 2^n - 1$, thus showing that not all supertransitive sets are power sets.

For any ordinal $m \leqq \omega$ we can construct a set which is supertransitive, transitive and has cardinality $\overline{\overline{m}}$. Define $a_0 = \emptyset$ and inductively define $a_{n+1} = \{a_n\}$. Then the set $\{a_i | 0 \leqq i < \omega\}$ has cardinality $\aleph_0$ and the set $\{a_i | 0 \leqq i \leqq m - 1\}$ has cardinality $\overline{\overline{m}}$. It is easy to check that these sets are transitive and supertransitive.

Lemma 15. *If there exists a set of cardinality* ρ *then there exists a supertransitive set of cardinality* ρ.

Proof: By the above comment we only need consider the case $\rho > \aleph_0$. Suppose A is a set such that $\mathrm{card}(A) = \rho$. Then $B = \{\{a\} \mid a \in A\} \cup \{\emptyset\}$ is supertransitive and has cardinality ρ. For, if $b \in B$ and $c \subseteq b$ then $c = \emptyset$ or $c = b$. In either case $c \in B$.

It is clear from the proof of Lemma 15 that if A is a class such that $\emptyset \in A$ and the cardinality of each element in $A - \{\emptyset\}$ is 1 then A is supertransitive. While this condition is sufficient for supertransitivity it is not necessary as some power sets have elements of cardinality greater than 1 and power sets are supertransitive.

It is already known that if A is a class then there is a smallest transitive class containing A and this latter class is called the transitive closure of A, (cf. [1], p. 136). We prove a similar result for supertransitive classes.

Lemma 16. *For any class* A *there is a smallest supertransitive class containing* A, *the supertransitive closure of* A.

Proof: If A is supertransitive then A is the supertransitive closure of A. If A is not supertransitive define $C_0 = A$, and define inductively

$$C_{n+1} = C_n \cup \bigcup_{a \in C_n} \{x \mid x \subset a\}.$$

Put $C = \bigcup_{n=0}^{\infty} C_n$. Then C is the smallest supertransitive class containing A. For, suppose $a \in C$ and $b \subseteq a$. Then $a \in C_n$ for some n and $b \in C_{n+1}$ by definition of C_{n+1}. Thus $b \in C$ and C is supertransitive. Now suppose that B is some supertransitive set containing A. We wish to show that $C \subseteq B$. Let $c \in C$. If $c \in A = C_0$ then $c \in B$. Otherwise there is an integer n such that $c \in C_n - C_{n-1}$. By definition of the C_i there is a finite sequence $\{c_i \mid 0 \leqq i \leqq n-1\}$ such that $c_i \in C_i$ and $c \subset c_{n-1} \subset c_{n-2} \subset \ldots \subset c_0$. Thus $c \subset c_0$ while $c_0 \in B$ as $B \supseteq A = C_0$. Since B is supertransitive we have $c \in B$ and hence $C \subseteq B$ as required.

4. *Supertransitivity and Induction.* We now present some results about supertransitivity which are analagous to those in section 2 about transitivity.

Lemma 17. (Belding) *For every* A, *if* A *is a class then the following statements are equivalent*:

(i) *A is supertransitive*,
(ii) $(x)(x \in A \to \mathcal{P}(x) \in \mathcal{P}(A))$,
(iii) $(x)(x \in A \to \mathcal{P}^i(x) \in \mathcal{P}^i(A)$ *for every* $i \geqq 0)$,
(iv) *there is a class* C *such that* $A = C \cap \{x \mid \mathcal{P}(x) \in \mathcal{P}(C)\}$.

Proof: Let A be a class. Then

(i)$\Rightarrow$(ii). Noting that $\mathcal{P}(x) \subset A$ means $\mathcal{P}(x) \in \mathcal{P}(A)$, (ii) is simply a restatement of (i).

(ii)$\Rightarrow$(iii). Given (ii) suppose inductively that for a given i, $(x)(x \in A \rightarrow \boldsymbol{P}^i(x) \in \boldsymbol{P}^i(A))$. From this we can deduce consecutively

(1) $(x)(x \in A \rightarrow \boldsymbol{P}^i(x) \subseteq \boldsymbol{P}^{i-1}(A))$
(2) $(x)(x \in A \rightarrow \boldsymbol{P}^{i+1}(x) \subseteq \boldsymbol{P}^i(A))$
(3) $(x)(x \in A \rightarrow \boldsymbol{P}^{i+1}(x) \in \boldsymbol{P}^{i+1}(A))$.

(2) follows from (1) by taking power sets of $\boldsymbol{P}^i(x)$ and $\boldsymbol{P}^{i-1}(A)$. (iii) follows by induction over i.

(iii)$\Rightarrow$(iv). Given (iii) we certainly have $(x)(x \in A \rightarrow \boldsymbol{P}(x) \in \boldsymbol{P}(A))$ and $A = A \cap \{x | \boldsymbol{P}(x) \in \boldsymbol{P}(A)\}$ which follows directly from $A = A$. (iv) is immediate.

(iv)$\Rightarrow$(i). Assume (iv), then $(x)(x \in A \rightarrow x \in C \wedge \boldsymbol{P}(x) \subseteq C)$ and $(x)(x \in A \rightarrow (y)(y \subseteq x \rightarrow y \in C))$. Since the relation $\subseteq$ is transitive we have,

(4) $(x)(x \in A \rightarrow (y)(y \subseteq x \rightarrow y \in C \wedge \boldsymbol{P}(y) \subseteq C))$
(5) $(x)(x \in A \rightarrow (y)(y \subseteq x \rightarrow y \in A))$.

(5) means that A is supertransitive as required. Lemma 17 is proved.

In the following decomposition results we shall use B^s and A_n^s to denote supertransitive component classes to distinguish this case from the transitive case.

Lemma 18. *For every A, if A is a class then there is a unique maximal transitive subclass of A.*

Proof: If A is a class define B^s by

(1) $B^s = A \cap \{x | \boldsymbol{P}(x) \in \boldsymbol{P}(A)\}$.

By Lemma 17, B^s is supertransitive. Suppose now that K is a supertransitive subclass of A. We show that $K \subseteq B^s$. Since $K \subseteq A$ and K is supertransitive we deduce $\boldsymbol{P}(K) \subseteq \boldsymbol{P}(A)$, $(x)(x \in K \rightarrow \boldsymbol{P}(x) \in \boldsymbol{P}(K))$, and

(2) $(x)(x \in K \rightarrow \boldsymbol{P}(x) \in \boldsymbol{P}(A) \wedge x \in A)$.

(2) means that $K \subseteq B^s$ as required, also proving uniqueness.

Theorem 19. *If A is a class, B^s the maximal supertransitive subclass of A and $A_n^s = \{x | \boldsymbol{P}^i(x) \in \boldsymbol{P}^i(A)$ for $0 \leqq i \leqq n$ and $\boldsymbol{P}^{n+1}(x) \notin \boldsymbol{P}^{n+1}(A)\}$, then:*

(i) $A = \bigcup_{n=0}^{\infty} A_n^s \cup B^s$

(ii) *for every $n \geqq 0$, $B^s \cap A_n^s = \emptyset$*

(iii) *for every m and n such that $m \neq 0 \neq n$ and $m > n$, $A_m \cap A_n = \emptyset$.*

Proof: Let A be a class. Clearly

(1) $\bigcup_{n=0}^{\infty} A_n^s \cap B^s \subseteq A$.

For every $x \in A$, either (a) for every $i \geqq 0$, $\boldsymbol{P}^i(x) \in \boldsymbol{P}^i(A)$ or (b) there is a smallest integer n such that $\boldsymbol{P}^i(x) \in \boldsymbol{P}^i(A)$ for $0 \leqq i \leqq n$ and $\boldsymbol{P}^{n+1}(x) \notin \boldsymbol{P}^{n+1}(A)$. If point (a) holds then $x \in B^s$ and if (b) holds then $x \in A_n^s$. Thus

(2) $A \subseteq \bigcup_{n=0}^{\infty} A_n^s \cup B^s$.

(1) and (2) yield $A = \bigcup_{n=0}^{\infty} A_n^s \cup B^s$ and (i) is proved.

By definition of A_n^s, for every x, if $x \in A_n^s$ then $\mathscr{P}^{n+1}(x) \notin \mathscr{P}^{n+1}(A)$, and $x \notin B^s$. Thus for every $n \geqq 0$, $A_n^s \cap B^s = \emptyset$. Hence (ii) is proved.

Suppose $m \neq n$ and $m > n > 0$. Then for every x, $x \in A_n^s$ implies $\mathscr{P}^{n+1}(x) \notin \mathscr{P}^{n+1}(A)$. Hence $A_m^s \cap A_n^s = \emptyset$ and (iii) is proved.

Lemma 20. *Let A be a class and $\bigcup_{n=0}^{\infty} A_n^s \cup B^s$ the supertransitive decomposition of A. The following statements are equivalent*:

(i) *A is supertransitive,*
(ii) $A_0^s = \emptyset$,
(iii) $A = B^s$.

Proof: Let A be a class. Then

(i)$\Longrightarrow$(ii). Suppose (i). In particular by Lemma 17 we have $(x)(x \in A \rightarrow \mathscr{P}(x) \in \mathscr{P}(A))$ and

(1) $A_0^s = A - \{x \mid \mathscr{P}(x) \in \mathscr{P}(A)\} = \emptyset$.

Thus (ii) holds.

(ii)$\Longrightarrow$(iii). Suppose (ii). In particular (1) holds so that $(x)(x \in A \rightarrow \mathscr{P}(x) \in \mathscr{P}(A))$. By Lemma 17 and by definition of B^s we deduce consecutively,

(2) $(x)(x \in A \rightarrow \mathscr{P}^i(x) \in \mathscr{P}^i(A)$ for every $i \geqq 0)$
(3) $(x)(x \in A \rightarrow x \in B^s)$.

Since $B^s \subseteq A$ be definition, (3) yields $B^s = A$ and (iii) is proved.

(iii)$\Longrightarrow$(i). Assume (iii), so $A = B^s$. By Lemma 18, A is supertransitive so (i) holds. Lemma 20 is proved.

From the following theorem we shall deduce induction principle **D**, mentioned in the introduction, as a corollary.

Theorem 21. (Belding) *Let A be a class and $\bigcup_{n=0}^{\infty} A_n^s \cup B^s$ the supertransitive decomposition of A. A sufficient condition for the following induction statement to hold is that A be supertransitive ($A_0^s = \emptyset$). If* $\mathrm{card}(A_0^s)$ *is finite then the supertransitivity of A ($A_0^s = \emptyset$) is a necessary condition.*

Let $\varphi(x)$ be a formula not containing the variable y. Let φ be such that for every sequence $\{a_n\}_{n<\omega}$ in A such that $a_{n+1} \subset a_n$ for every $n \geqq 0$, there is at least one m such that $\varphi(a_m)$ holds. Then

$$(y)[y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(y \in A \rightarrow \varphi(y)).$$

Proof: Assume that A is a class and let $\bigcup_{n=0}^{\infty} A_n^s \cup B^s$ be the supertransitive decomposition of A.

Sufficiency. Assume that A is supertransitive and

(1) $\varphi(x)$ is a formula not containing the variable y
(2) for every sequence $\{a_n\}_{n<\omega}$ in A such that $a_{n+1} \subset a_n$ for every $n \geqq 0$, there is at least one m such that $\varphi(a_m)$ holds.

We now proceed by an indirect method. Suppose the induction formula fails for this A and φ. Thus

(3) $(y)(y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow \varphi(y))$
(4) $(\exists y)(y \in A \wedge \sim\varphi(y))$.

Let a_0 be such that $a_0 \in A$ and $\sim\varphi(a_0)$. If $a_0 = \emptyset$ then there is no b such that $b \subset a_0$. Thus $(x)(x \subset a_0 \rightarrow \varphi(x))$. From (3) we deduce $\varphi(a_0)$, a contradiction. So $a_0 \neq \emptyset$. From (3) and the supertransitivity of A we now deduce respectively $(\exists x)(x \subset a_0 \wedge \sim\varphi(x))$ and

(5) $(\exists x)(x \subset a_0 \wedge \sim\varphi(x) \wedge x \in A)$.

Let a_1 be such that $a_1 \subset a_0$, $\sim\varphi(a_1)$ and $a_1 \in A$. Again we may deduce that $a_1 \neq \emptyset$ and

(6) $(\exists x)(x \subset a_1 \wedge \sim\varphi(x) \wedge x \in A)$.

Continuing in this manner we obtain a sequence $\{a_n\}_{n<\omega}$ in A such that $a_{n+1} \subset a_n$ and $\sim\varphi(a_n)$ for every $n \geqq 0$. This contradicts (2). The following statement now follows by contraposition on (3) and (4)

(7) $(y)[y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(y \in A \rightarrow \varphi(y))$.

Thus proving the sufficiency part of the theorem.

Necessity. We assume the induction principle and that A_0^s is a finite class. That is, $\operatorname{card}(A_0^s) < \aleph_0$ and

(8) $[(1) \text{ and } (2)] \Rightarrow (7)$

Under these hypotheses we shall construct a formula φ satisfying (1) and (2) for which (8) fails if $A_0^s \neq \emptyset$ and by Lemma 20 deduce that A is supertransitive. Define φ as

(9) $\varphi(x) \equiv x \in A - A_0^s$.

Clearly (1) holds. Let $\{a_n\}_{n<\omega}$ be a sequence in A such that $a_{n+1} \subset a_n$ for every $n \geqq 0$. Since $\operatorname{card}(A_0^s)$ is finite there is some m such that $a_m \notin A_0^s$. Thus $a_m \in A - A_0^s$ and $\varphi(a_m)$ holds. So (2) holds. Now suppose $A_0^s \neq \emptyset$. Then for some b, $b \in A_0^s$ and consequently $\sim\varphi(b)$. Thus (4) holds. It remains to show that (3) holds, for then we will have contradicted (8). Suppose $y \in A \wedge (x)(x \subset y \rightarrow \varphi(x))$. Then by the definitions of φ and A_0^s we have consecutively

$$
\begin{aligned}
& y \in A \wedge (x)(x \subseteq y \rightarrow x \in A) \\
& y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow \boldsymbol{P}(y) \subseteq A \\
& y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow y \notin A_0^s \\
& y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow y \in A - A_0^s \\
& y \in A \wedge (x)(x \subset y \rightarrow \varphi(x)) \rightarrow \varphi(y).
\end{aligned}
$$

(3) follows by generalization as required. The necessity part of the theorem is now proved.

Remark: The condition that φ hold on at least one member of any sequence $\{a_n\}_{n<\omega}$ in A such that $a_{n+1} \subset a_n$ for every $n \geqq 0$ is equivalent to the condition that φ hold for all but a finite number of elements in any such sequence in A.

Corollary 21.1. (**D**) *Let A be a supertransitive class and let $\varphi(x)$ be a formula not containing the variable y. If φ has the property that for every sequence of sets $\{a_n\}_{n<\omega}$ in A such that $a_{n+1} \subset a_n$ for every $n \geqq 0$, there is at least one integer m such that $\varphi(a_m)$ holds then*

$$(y)\,(y \epsilon A \wedge (x)\,(x \subset y \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)\,(y \epsilon A \rightarrow \varphi(y)).$$

Proof: The corollary follows from the sufficiency part of Theorem 21.

In a future paper Belding will present further results for fields of relations; including decomposition formulas, metrics, extended metrics, and another sufficient condition for Montague's induction principle.

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