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THE RELATIVE CONSISTENCY OF THE CLASS AXIOMS OF ABSTRACTION AND EXTENSIONALITY AND THE AXIOMS OF NBG IN A THREE-VALUED LOGIC

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This paper is an extension of a previous one entitled "The Consistency of the Axioms of Abstraction and Extensionality in Three-Valued Logic" [8]. This proof differs from the one in [8] in that the structure M_0 (below) contains a model of NBG and the method of generating the sequence of structures, $M_0 \leqslant M_1 \leqslant \ldots \leqslant M_\mu \leqslant \ldots$, is more complicated.

1. The formal system that we shall show to be relatively consistent to $Z - F$ is the following:

Primitives
1. u, v, w, x, y, z, etc. are variables over special classes, i.e., the classes of NBG.
2. U, V, W, X, Y, Z, etc. are variables over classes.
3. ε (is a member of); $\sim$, $\rightarrow$, A (connectives and quantifiers of Łukasiewicz three-valued logic).

Formation Rules
1. For variables x, y, X, Y, the following are atomic wffs: $x \varepsilon y$, $x \varepsilon X$, $X \varepsilon x$, $X \varepsilon Y$.
2. The propositional constants 1, 0, $\frac{1}{2}$ are atomic wffs.
3. If B and C are wffs and x and X are variables then $\sim B$, $B \rightarrow C$, $(Ax)B$, $(AX)B$ are wffs.

The three-valued logic concerned is that of Łukasiewicz and the connectives and quantifiers are represented as follows:

	$p \mathbin{\&} q$			$p \vee q$			$p \rightarrow q$			$p \leftrightarrow q$			$p \supset q$			$p \equiv q$		
p/q	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0	1	1	1	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	1	1
0	0	0	0	1	$\frac{1}{2}$	0	1	1	1	0	$\frac{1}{2}$	1	1	1	1	0	1	1

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p	$\sim p$	$\mathsf{T}p$	$\mathsf{F}p$	$\mathsf{P}p$	$\mathsf{C}p$
1	0	1	0	0	1
$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	0
0	1	0	1	0	1

$(AX)\,\phi(X)$ has the minimum value of the values of $\phi(X)$, $(SX)\,\phi(X)$ has the maximum value of the values of $\phi(X)$. Similarly for $(Ax)\,\phi(x)$ and $(Sx)\,\phi(x)$.

All the above connectives and quantifiers can be defined in terms of $\sim$, $\rightarrow$, A as follows:

$$p \vee q =_{df} (p \rightarrow q) \rightarrow q.$$
$$p \,\&\, q =_{df} \sim(\sim p \vee \sim q).$$
$$p \leftrightarrow q =_{df} (p \rightarrow q) \,\&\, (q \rightarrow p).$$
$$p \supset q =_{df} p \rightarrow (p \rightarrow q).$$
$$p \equiv q =_{df} (p \supset q) \,\&\, (q \supset p).$$
$$\mathsf{T}p =_{df} \sim(p \rightarrow \sim p).$$
$$\mathsf{F}p =_{df} \mathsf{T} \sim p.$$
$$\mathsf{P}p =_{df} \sim \mathsf{T}p \,\&\, \sim \mathsf{F}p.$$
$$\mathsf{C}p =_{df} \mathsf{T}p \vee \mathsf{F}p.$$
$$(Sx)A =_{df} \sim(Ax) \sim A.$$
$$(SX)A =_{df} \sim(AX) \sim A.$$

Definitions

$$X = Y =_{df} (AZ)\,(Z \,\varepsilon\, X \leftrightarrow Z \,\varepsilon\, Y).$$
$$x \stackrel{s}{=} y =_{df} (Az)\,(z \,\varepsilon\, x \equiv z \,\varepsilon\, y).$$

The definitions of NBG (cf. [6]).

$$(Ax')\,\phi(x') =_{df} (Ax)\,(M(x) \supset \phi(x)).$$
$$(Sx')\,\phi(x') =_{df} (Sx)\,(M(x) \,\&\, \phi(x)).$$
$$\mathsf{SCl}(X) =_{df} (Sx)\,(Az)\,(z \,\varepsilon\, x \leftrightarrow z \,\varepsilon\, X).$$

(X is a special class in that it has the same special class members as some special class but X may not lie in the range of the special class variables.)

Axioms

T. $x \stackrel{s}{=} y \supset (x \,\varepsilon\, z \equiv y \,\varepsilon\, z)$.

P. $(Ax')(Ay')(Sz')(Au')(u' \,\varepsilon\, z' \equiv u' = x' \vee u' = y')$.

N. $(Sx')(Ay')(\sim y' \,\varepsilon\, x')$.

B. $(Sz)(Ax_1') \ldots (Ax_n')(\langle x_1', \ldots, x_n' \rangle \,\varepsilon\, z \equiv \phi(x_1', \ldots, x_n', y_1, \ldots, y_m))$
where only set variables are quantified in ϕ.

U. $(Ax')(Sy')(Au')(u' \,\varepsilon\, y' \equiv (Sv')(u' \,\varepsilon\, v' \,\&\, v' \,\varepsilon\, x'))$.

W. $(Ax')(Sy')(Au')(u' \,\varepsilon\, y' \equiv u' \subseteq x')$.

S. $(Ax')(Ay)(Sz')(Au')(u' \,\varepsilon\, z' \equiv u' \,\varepsilon\, x' \,\&\, u' \,\varepsilon\, y)$.

R. $(Ax')(Un(x) \supset (Sy')(Au')(u' \,\varepsilon\, y' \equiv (Sv')(\langle v', u' \rangle \,\varepsilon\, x \,\&\, v' \,\varepsilon\, x')))$.

I. $(Sx')(O \,\varepsilon\, x' \,\&\, (Au')(u' \,\varepsilon\, x' \supset u' \cup \{u'\} \,\varepsilon\, x'))$.

A. $(SY)(AX)(X \,\varepsilon\, Y \leftrightarrow \phi(X, z_1, \ldots, z_m, Z_1, \ldots, Z_n))$, where ϕ is either a propositional constant or constructed from atomic wffs of forms, $U \varepsilon V$, $U \varepsilon v$, $u \varepsilon V$, $u \varepsilon v$, by using only $\sim$, &, A.

E. $X = Y \supset (Az)(X \varepsilon Z \leftrightarrow Y \varepsilon Z)$.

Extra Axioms

1. $(AX)\,\phi(X) \rightarrow (Ax)\,\phi(x)$.
2. $(Az)(z \varepsilon x \leftrightarrow z \varepsilon X) \supset (Aw)(x \varepsilon w \leftrightarrow X \varepsilon w)$.
3. $\mathsf{F}(\mathsf{SCl}(X)) \supset \mathsf{F}(X \varepsilon x)$.
4. $\mathsf{P}(\mathsf{SCl}(X)) \supset \mathsf{P}(X \varepsilon x)$.
5. $\mathsf{C}(x \varepsilon y)$.
6. $x \stackrel{s}{=} y \supset x = y$.

2. Take any model $\mathfrak{N}$ of NBG whose domain is a denumerable set. The domain will consist of special class constants and the membership between any two of these constants will be determined as true or false in $\mathfrak{N}$. To construct the model of the whole system, we need to extend the above wffs by adding special class constants of the above model of NBG, a, b, $c, \ldots$, and some terms to be defined. The domain of the model will consist of some of these terms as well as the special class constants. We give the formation rules for terms and wffs as follows:

1. If x and y are special class variables, a and b are special class constants, and X and Y are class variables, then $a \varepsilon b$, $a \varepsilon x$, $x \varepsilon a$, $a \varepsilon X$, $X \varepsilon a$, $x \varepsilon y$, $x \varepsilon X$, $X \varepsilon x$, $X \varepsilon Y$, are atomic wffs.
2. Any combination of wffs using $\sim, \rightarrow, A$ as in the Łukasiewicz three-valued logic is a wff.
3. A propositional constant (i.e., 1, $\frac{1}{2}$ or 0) is an atomic wff.
4. A propositional constant or a wff constructed from atomic wffs using only $\sim$, &, A is a standard wff.
5. If P is a standard wff and X is a class variable, then $\{X : P\}$ is a term.
6. If $\{X : P\}$ and $\{X : Q\}$ are terms, y is a special class variable, a is a special class constant and Y is a class variable, then $\{X : P\} \varepsilon a$, $a \varepsilon \{X : P\}$, $\{X : P\} \varepsilon y$, $y \varepsilon \{X : P\}$, $\{X : P\} \varepsilon Y$, $Y \varepsilon \{X : P\}$, $\{X : P\} \varepsilon \{X : Q\}$ are all atomic wffs.

We construct a model for the axioms with domain the set D of all special class constants and all constant terms $\{X : P\}$, i.e., P is a standard wff and either has no free variables at all or has X as its only free variable. Let D^S denote the set of all special class constants and so $D - D^S$ is the set of all constant terms. We shall use constants A, B, C, etc. for members of D. Non-constant terms can be defined from these as follows. Associate with any term $\{X : P(X, z_1, \ldots, z_m, Z_1, \ldots, Z_n)\}$, for which $z_1, \ldots, z_m, Z_1, \ldots, Z_n$ are the only free variables, the function which for constants $a_1, \ldots, a_m$ of D^S and $A_1, \ldots, A_n$ of D takes as value the constant term $\{X : P(X, a_1, \ldots, a_m, A_1, \ldots, A_n)\}$ of D.

Let any specification of values, including the value assignments already given to members of D^S in the model $\mathfrak{N}$, for all the constant atomic wffs $A \varepsilon B$, where A and B are members of D, be called a *structure on D*. Let $V[M](P)$ denote the value of the constant wff P given by the structure M on

D. Also let $V[M](1) = 1$, $V[M](0) = 0$ and $V[M](\frac{1}{2}) = \frac{1}{2}$. Define $M_1 \leqslant M_2$ for two structures M_1 and M_2 on D as, for any constant atomic wff P, if $V[M_1](P) = 1$ then $V[M_2](P) = 1$ and if $V[M_1](P) = 0$ then $V[M_2](P) = 0$. Here, '$\leqslant$' defines a partial ordering on the set of structures, since (i) $M \leqslant M$, (ii) if $M_1 \leqslant M_2$ and $M_2 \leqslant M_3$ then $M_1 \leqslant M_3$ and (iii) if $M_1 \leqslant M_2$ and $M_2 \leqslant M_1$ then $M_1 = M_2$ (i.e., M_1 and M_2 are the same structure).

From now on, when mentioning values of wffs in a structure it is automatically assumed that the wffs are constant ones, i.e., they have no free variables.

Lemma 1 *Let M and M' be two structures on D, such that $M \leqslant M'$. Then, for any standard wff P, if $V[M](P) = 1$ then $V[M'](P) = 1$ and if $V[M](P) = 0$ then $V[M'](P) = 0$.*

Proof. By induction on the wff evaluation procedure. This means that we start at the values of all the constant atomic wffs obtained by substitution for free variables in P, and then build up the value of P from these values according to the connectives and quantifiers in the Łukasiewicz logic. If P is an atomic wff, the lemma holds.

(i) Let $V[M](\sim Q) = 1$. Then $V[M](Q) = 0$. By the induction hypothesis $V[M'](Q) = 0$. Hence $V[M'](\sim Q) = 1$. Let $V[M](\sim Q) = 0$. Then as above, $V[M'](\sim Q) = 0$.

(ii) Let $V[M](Q \,\&\, R) = 1$. Then $V[M](Q) = 1 = V[M](R)$. By the induction hypothesis, $V[M'](Q) = 1 = V[M'](R)$. Hence $V[M'](Q \,\&\, R) = 1$. Let $V[M](Q \,\&\, R) = 0$. Then as above, $V[M'](Q \,\&\, R) = 0$.

(iii) Let $V[M]((Ax)\,Q(x)) = 1$. Then $V[M](Q(x)) = 1$ for all $x \,\varepsilon\, D^S$. By the induction hypothesis $V[M'](Q(x)) = 1$ for all $x \,\varepsilon\, D^S$. Hence $V[M']((Ax)\,Q(x)) = 1$. Let $V[M]((Ax)\,Q(x)) = 0$. Then as above, $V[M']((Ax)\,Q(x)) = 0$.

(iv) The case for $(AX)\,Q(X)$ is similar to (iii).

Define the structure M_0 as follows: If $A \not\varepsilon D^S$ or $B \not\varepsilon D^S$, then $V[M_0](A \,\varepsilon\, B) = \frac{1}{2}$. If $A \,\varepsilon\, D^S$ and $B \,\varepsilon\, D^S$, then $V[M_0](A \,\varepsilon\, B) = 1$ if $A \,\varepsilon\, B$ is true in the model $\mathfrak{N}$ and $V[M_0](A \,\varepsilon\, B) = 0$ if $A \,\varepsilon\, B$ is false in the model $\mathfrak{N}$.

Hence M_0 with domain D^S is a model of NBG satisfying all the axioms. The model of the whole system will be the limit of a sequence of structures, $M_0 \leqslant M_1 \leqslant \ldots \leqslant M_\mu \leqslant \ldots$, on D.

Assuming M_μ defined for some ordinal μ, $M_{\mu+1}$ is defined as follows. For all standard wffs P, $V[M_{\mu+1}](A \,\varepsilon\, \{X : P(X)\}) = V[M_\mu](P(A))$. If $\sim z \,\varepsilon\, A \vee z \,\varepsilon\, a \,\&.\, \sim z \,\varepsilon\, a \vee z \,\varepsilon\, A$ is valid in M_μ for some a, then, for all b, $V[M_{\mu+1}](A \,\varepsilon\, b) = V[M_0](a \,\varepsilon\, b)$. If $(Ax)(Sz)(z \,\varepsilon\, A \,\&\, \sim z \,\varepsilon\, x \vee . z \,\varepsilon\, x \,\&\, \sim z \,\varepsilon\, A)$ has the value 1 in M_μ then, for all b, $V[M_{\mu+1}](A \,\varepsilon\, b) = 0$. If neither $(Sx)(Az)(\sim z \,\varepsilon\, A \vee z \,\varepsilon\, x \,\&.\, \sim z \,\varepsilon\, x \vee z \,\varepsilon\, A)$ nor $(Ax)(Sz)(z \,\varepsilon\, A \,\&\, \sim z \,\varepsilon\, x \vee . z \,\varepsilon\, x \,\&\, \sim z \,\varepsilon\, A)$ have the value 1 in M_μ then $V[M_{\mu+1}](A \,\varepsilon\, b) = \frac{1}{2}$.

For a limit ordinal μ, on the assumption that $M_\nu \leqslant M_\tau$ for all $\nu \leqslant \tau$, for all $\tau < \mu$, for all atomic wffs P, if $V[M_\nu](P) = 1$ for some $\nu < \mu$ then $V[M_\mu](P) = 1$, if $V[M_\nu](P) = 0$ for some $\nu < \mu$ then $V[M_\mu](P) = 0$, and if $V[M_\nu](P) = \frac{1}{2}$ for all $\nu < \mu$ then $V[M_\mu](P) = \frac{1}{2}$.

Lemma 2 *$M_\nu \leqslant M_\mu$, for all $\nu \leqslant \mu$.*

Proof. By transfinite induction on μ. The induction hypothesis is: $M_\nu \leqslant M_\tau$ for all $\nu \leqslant \tau$, for all $\tau < \mu$.

(i) $\mu = 0$: $M_0 \leqslant M_0$.

(ii) μ is a successor ordinal:

(A) Let $V[M_\nu](A\,\varepsilon\{X : P\}) = 1$. There is a $\eta < \nu$ such that $V[M_\eta](P(A)) = 1$ by the method of construction of the structures. Since $\eta \leqslant \mu - 1, M_\eta \leqslant M_{\mu-1}$ by the induction hypothesis. Hence $V[M_{\mu-1}](P(A)) = 1$. By the construction of M_μ, $V[M_\mu](A\,\varepsilon\{X : P\}) = 1$. Similarly, if $V[M_\nu](A\,\varepsilon\{X : P\}) = 0$ then $V[M_\mu](A\,\varepsilon\{X : P\}) = 0$.

(B) Let $V[M_\nu](A\,\varepsilon\, b) = 1$ (or 0). There is an $\eta < \nu$ such that $V[M_\eta]((Sx)(Az)(\sim z\,\varepsilon A \vee z\,\varepsilon x \,\&.\, \sim z\,\varepsilon x \vee z\,\varepsilon A)) = 1$ or $V[M_\eta]((Ax)(Sz)(z\,\varepsilon A \,\&\, \sim z\,\varepsilon x \vee. z\,\varepsilon x \,\&\, \sim z\,\varepsilon A)) = 1$.

(a) Let $V[M_\eta]((Sx)(Az)(\sim z\,\varepsilon A \vee z\,\varepsilon x \,\&.\, \sim z\,\varepsilon x \vee z\,\varepsilon A)) = 1$. Then $V[M_{\eta+1}](A\,\varepsilon\, b) = V[M_0](a\,\varepsilon\, b) = 1$ (or 0), for some a. Since $\eta \leqslant \mu - 1$, $M_\eta \leqslant M_{\mu-1}$, by the induction hypothesis. Hence $V[M_{\mu-1}]((Sx)(Az)(\sim z\,\varepsilon A \vee z\,\varepsilon x \,\&.\, \sim z\,\varepsilon x \vee z\,\varepsilon A)) = 1$ and $V[M_\mu](A\,\varepsilon\, b) = V[M_0](a\,\varepsilon\, b) = 1$ (or 0), for some a.

(b) Let $V[M_\eta]((Ax)(Sz)(z\,\varepsilon A \,\&\, \sim z\,\varepsilon x \vee. z\,\varepsilon x \,\&\, \sim z\,\varepsilon A)) = 1$. If $V[M_\nu](A\,\varepsilon\, b) = 1$, this does not apply. Let $V[M_\nu](A\,\varepsilon\, b) = 0$. Since $\eta \leqslant \mu - 1, M_\eta \leqslant M_{\mu-1}$, by the induction hypothesis. Hence $V[M_{\mu-1}]((Ax)(Sz)(z\,\varepsilon A \,\&\, \sim z\,\varepsilon x \vee. z\,\varepsilon x \,\&\, \sim z\,\varepsilon A)) = 1$ and $V[M_\mu](A\,\varepsilon\, b) = 0$.

(iii) μ is a limit ordinal: Let $\nu < \mu$. Let $V[M_\nu](A\,\varepsilon B) = 1$. Then $V[M_\mu](A\,\varepsilon B) = 1$ by definition of M_μ. Similarly when $V[M_\nu](A\,\varepsilon B) = 0$ then $V[M_\mu](A\,\varepsilon B) = 0$. If $\nu = \mu, M_\nu \leqslant M_\mu$.

Lemma 3 *There is an ordinal* λ *of the second number class such that* $M_\lambda = M_{\lambda+1}$.

Proof. The increasing chain of structures $M_0 \leqslant M_1 \leqslant \ldots \leqslant M_\mu \leqslant \ldots$ can be regarded as two increasing chains of subsets of the denumerable set of all atomic wffs of the form $A\,\varepsilon B$. One chain is of those atomic wffs taking the value 1 and the other is of those taking the value 0. If $M_\nu = M_{\nu+1}$ then $M_\nu = M_\mu$ for all ordinals μ, $\nu \leqslant \mu$, since, by the method of construction, there is no way of changing the values of any atomic wffs. There is a denumerable set of ordinals μ such that $M_\mu \neq M_{\mu+1}$. But the set of all ordinals of the second number class is non-denumerable, and hence for some λ in this class, $M_\lambda = M_{\lambda+1}$.

3. Now it is necessary to show that M_λ is the required model.

Theorem 1 *All the axioms of* NBG *are valid in* M_λ.

Proof. By the definitions of M_0 and the domain D^S, M_0 with D^S as domain is a model of NBG. By lemma 2, if $V[M_0](A\,\varepsilon B) = 1$ (or 0) then $V[M_\lambda](A\,\varepsilon B) = 1$ (or 0). Hence M_λ with domain D^S is a model of NBG.

Theorem 2 $Y\,\varepsilon\{X : P\} \leftrightarrow P(Y)$ *is valid in* M_λ.

Proof. Let $V[M_\lambda](A\,\varepsilon\{X : P\}) = 1$. Let ν be the least ordinal such that $V[M_\nu](A\,\varepsilon\{X : P\}) = 1$. ν is a successor ordinal. Hence $V[M_{\nu-1}](P(A)) = 1$. Since $\nu - 1 \leqslant \lambda$, $M_{\nu-1} \leqslant M_\lambda$, by lemma 2. Since P is a standard wff, by lemma 1, $V[M_\lambda](P(A)) = 1$. Similarly, if $V[M_\lambda](A\,\varepsilon\{X : P\}) = 0$, then $V[M_\lambda](P(A)) = 0$.

Let $V[M_\lambda](P(A)) = 1$. Then $V[M_{\lambda+1}](A\,\varepsilon\{X : P\}) = 1$. Since $M_\lambda = M_{\lambda+1}$, $V[M_\lambda](A\,\varepsilon\{X : P\}) = 1$. Similarly, if $V[M_\lambda](P(A)) = 0$, then $V[M_\lambda](A\,\varepsilon\{X : P\}) = 0$.

Theorem 3 *The Abstraction Axiom* (A) *is valid in* M_λ.

Proof. By theorem 2, for any standard wff P, $Y\varepsilon\{X : P\}\leftrightarrow P(Y)$ is valid in M_λ. Therefore $(SZ)(AX)(X\,\varepsilon\, Z\leftrightarrow P(X, y_1, \ldots, y_m, Y_1, \ldots, Y_n))$ is valid in M_λ, for all wffs P which are either propositional constants or constructed from atomic wffs of forms, $U\varepsilon V$, $U\varepsilon v$, $u\varepsilon V$, $u\varepsilon v$ by using $\sim$, &, A; since all wffs of this sort are standard wffs.

Let P be a standard wff such that $V[M_\lambda](P) = 1$ or 0. Let $\nu(P)$ be the least ordinal such that $V[M_{\nu(P)}](P) = 1$ or $V[M_{\nu(P)}](P)=0$. Form the set of all constant atomic wffs of P (i.e., atomic wffs of P with all substitutions made for any variables that occur in them) which take the value 1 or 0 in $M_{\nu(P)}$. Call this *the dependent set of* P, $\mathsf{D}(P)$.

Lemma 4 *Let* $P(A)$ *be a standard wff such that* $V[M_\lambda](P(A)) = 1$ *or* 0. *If, for each* $Q(A)\varepsilon\,\mathsf{D}(P(A))$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$, *then* $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

Proof. By induction on the wff evaluation procedure. Let $P(A)$ be an atomic wff such that $V[M_\lambda](P(A)) = 1$ or 0. Then $\mathsf{D}(P(A)) = \{P(A)\}$. Hence $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(i) Let $P(A)$ be $\sim R(A)$. Since $\mathsf{D}(\sim R(A)) = \mathsf{D}(R(A))$, for each $Q(A)\varepsilon\,\mathsf{D}(R(A))$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$. By the induction hypothesis, $V[M_\lambda](R(B)) = V[M_\lambda](R(A))$. Hence $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(ii) Let $P(A)$ be $(R(A)\ \&\ S(A))$ and $V[M_\lambda](R(A)\ \&\ S(A)) = 1$. Then $V[M_\lambda](R(A)) = 1$ and $V[M_\lambda](S(A)) = 1$. Since $\nu(R(A)) \leqslant \nu(R(A)\ \&\ S(A))$, $\mathsf{D}(R(A)) \subseteq \mathsf{D}(R(A)\ \&\ S(A))$. Hence, for each $Q(A)\,\varepsilon\,\mathsf{D}(R(A))$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$. By the induction hypothesis, $V[M_\lambda](R(B)) = V[M_\lambda]R(A)$. Similarly, $V[M_\lambda](S(B)) = V[M_\lambda](S(A))$. Hence $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(iii) Let $P(A)$ be $(R(A)\ \&\ S(A))$ and $V[M_\lambda](R(A)\ \&\ S(A)) = 0$. Then as above, $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(iv) Let $P(A)$ be $(AZ)\,R(A,Z)$ and $V[M_\lambda]((AZ)\,R(A,Z)) = 1$. Then $V[M_\lambda](R(A,Z)) = 1$ for all Z. Since $\nu(R(A,Z)) \leqslant \nu((AZ)\,R(A,Z)$ for all Z, then $\mathsf{D}(R(A,Z))\subseteq \mathsf{D}((AZ)\,R(A,Z))$ for all Z. Hence, for each $Q(A)\,\varepsilon\,\mathsf{D}(R(A,Z))$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q, (A))$. By the induction hypothesis, $V[M_\lambda](R(B,Z)) = V[M_\lambda](R(A,Z))$. Since this holds for all Z, $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(v) Let $P(A)$ be $(AZ)\,R(A,Z)$ and $V[M_\lambda]((AZ)\,R(A,Z)) = 0$. Then as above, $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(vi) Let $P(A)$ be $(Az)\,R(A,z)$ and $V[M_\lambda]((Az)\,R(A,z)) = 1$. Similarly to (iv), $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(vii) Let $P(A)$ be $(Az)\,R(A,z)$ and $V[M_\lambda]((Az)\,R(A,z)) = 0$. Similarly to (v), $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

Let P be an atomic wff (not 1 or 0) of the form $A\,\varepsilon\{X : Q(X)\}$ such that $V[M_\lambda](P) = 1$ or 0. Define *the corresponding standard wff of* P, $\mathsf{C}(P)$ as

$Q(A)$. Let P be a standard wff such that $V[M_\lambda](P) = 1$ or 0. Let P have dependent set, $\mathsf{D}(P)$. We define a *general dependent set of P*, $\mathsf{GD}(P)$ as follows:

(i) The dependent set $\mathsf{D}(P)$ of P is a $\mathsf{GD}(P)$.

(ii) If $V[M_\lambda](R) = \underline{1}$ or 0, R is an atomic wff (not 1 or 0) and $\mathsf{C}(R)$ is defined for R, then $\mathsf{D}(\mathsf{C}(R))$ is a $\mathsf{GD}(R)$.

(iii) Let D' be a $\mathsf{GD}(P)$. Let $S \subseteq D'$. If $Q \varepsilon S$, then $(D' \cap \overline{S}) \cup \bigcup_{Q\varepsilon S} Q^*$ is a $\mathsf{GD}(P)$, where Q^* is a $\mathsf{GD}(Q)$.

This assumes $V[M_\lambda](Q) = 1$ or 0, for all $Q \varepsilon S$. Note that lemma 5 (below) should be coupled with the definition of a general dependent set so that the assumption can be made before the construction of the general dependent sets $\mathsf{GD}(Q)$.

Lemma 5 *Let P be a standard wff such that* $V[M_\lambda](P) = 1$ *or* 0. *If* D' *is a general dependent set of P then, for each* $Q \varepsilon D'$, $V[M_\lambda](Q) = 1$ *or* 0.

Proof. By induction on the stages of construction of general dependent sets of all standard wffs P such that $V[M_\lambda](P) = 1$ or 0.

(i) By definition of $\mathsf{D}(P)$, if $Q \varepsilon \mathsf{D}(P)$ then $V[M_\lambda](Q) = 1$ or 0 and hence the lemma holds for $\mathsf{D}(P)$.

(ii) If $Q \varepsilon \mathsf{D}(\mathsf{C}(R))$, then $V[M_\lambda](Q) = 1$ or 0 and the lemma holds for $\mathsf{D}(\mathsf{C}(R))$.

(iii) Let D' be a general dependent set of P. Let $S \subseteq D'$. If $Q \varepsilon S$, then, by the induction hypothesis for D', $V[M_\lambda](Q) = 1$ or 0 and so, we let Q^* be a general dependent set of Q. Now let $T \varepsilon (D' \cap \overline{S}) \cup \bigcup_{Q\varepsilon S} Q^*$. If $T \varepsilon Q^*$ for some $Q \varepsilon S$, then, we have by the induction hypothesis for Q^*, $V[M_\lambda](T) = 1$ or 0. If $T \varepsilon D' \cap \overline{S}$, then, by the induction hypothesis for D', $V[M_\lambda](T) = 1$ or 0. Hence, if $T \varepsilon (D' \cap \overline{S}) \cup \bigcup_{Q\varepsilon S} Q^*$, then $V[M_\lambda](T) = 1$ or 0. Hence the lemma holds.

Lemma 6 *Let P be an atomic wff such that* $V[M_\lambda](P) = 1$ *or* 0 *and such that* $\mathsf{C}(P)$ *is defined. If* D' *is a general dependent set of P which is not* $\mathsf{D}(P)$ *then, for each* $Q \varepsilon D'$, $V[M_{\nu(P)-1}](Q) = 1$ *or* 0.

Proof. By transfinite induction on the ordinals $\nu(P)$. $\nu(P)$ is a successor ordinal. The induction hypothesis is that the lemma holds for all atomic wffs Q such that $\nu(Q) < \nu(P)$.

(i) $\nu(P) = 1$: If $Q \varepsilon \mathsf{D}(\mathsf{C}(\overline{P}))$, then $V[M_0](Q) = 1$ or 0. No further members of general dependent sets of P can be obtained.

(ii) $\nu(P)$ is a successor ordinal (>1): Use induction on the stages of construction of general dependent sets of P.

(I) $\mathsf{D}(P)$ is not used as a general dependent set in this lemma.

(II) Let $Q \varepsilon \mathsf{D}(\mathsf{C}(R))$. In constructing general dependent sets of P, the only R to consider is where R is P or where R is a member of a general dependent set of P (which is not $\mathsf{D}(P)$). If R is P, then $V[M_{\nu(P)-1}](Q) = 1$ or 0. If R is a member of a general dependent set of P (not $\mathsf{D}(P)$), then, by induction hypothesis, $V[M_{\nu(P)-1}](R) = 1$ or 0 and hence $V[M_{\nu(P)-1}](Q) = 1$ or 0.

(III) Let D' be a general dependent set of P for which the lemma holds.

Let $S \subseteq D'$. By the induction hypothesis for D', $V[M_{\nu(P)-1}](Q) = 1$ or 0, for all $Q \varepsilon S$. By the induction hypothesis for the ordinals, the lemma holds for any general dependent set Q^* of Q, except for $\mathsf{D}(Q)$. Let $T \varepsilon (D' \cap \overline{S}) \cup \bigcup_{Q\varepsilon S} Q^*$. If $T \varepsilon Q^*$ (where $Q^* \neq \mathsf{D}(Q)$), for some $Q \varepsilon S$, then $V[M_{\nu(P-1)}](T) = 1$ or 0. If $T \varepsilon Q^*$, where now Q^* is $\mathsf{D}(Q)$, for some $Q \varepsilon S$, then, since $\mathsf{D}(Q)$ is $\{Q\}$, $T \varepsilon D'$. By induction hypothesis for D', $V[M_{\nu(P)-1}](T) = 1$ or 0. If $T \varepsilon D' \cap \overline{S}$, then, by induction hypothesis for D', $V[M_{\nu(P)-1}](T) = 1$ or 0. Hence the lemma holds.

Lemma 7 *Let $P(A)$ be a standard wff such that $V[M_\lambda](P) = 1$ or 0. Consider any general dependent set D' of $P(A)$, such that, in the process of construction (ii) is not applied to any atomic wff of the form $C \varepsilon A$. If, for all $Q(A) \varepsilon D'$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$, then $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.*

Proof. By induction on the stages of construction of general dependent sets of *all* standard wffs $P(A)$ such that $V[M_\lambda](P(A)) = 1$ or 0, such that (ii) is not applied to any atomic wffs of form $C \varepsilon A$.

(i) Let a $\mathsf{GD}(P(A))$ be $\mathsf{D}(P(A))$. Now by the lemma condition for each $Q(A) \varepsilon \mathsf{D}(P(A))$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$. Hence $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$, by lemma 4.

(ii) Let a $\mathsf{GD}(P(A))$ be $\mathsf{D}(\mathsf{C}(P(A)))$, where $P(A)$ is an atomic wff (not 1 or 0) of the form $A \varepsilon \{X : Q(X)\}$. Then $\mathsf{C}(P(A))$ is $Q(A)$. $V[M_\lambda](Q(A)) = 1$ or 0. By the lemma condition, if $R(A) \varepsilon \mathsf{D}(\mathsf{C}(P(A)))$ then $V[M_\lambda](R(B)) = V[M_\lambda](R(A))$. Hence by lemma 4, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$. Therefore, $V[M_\lambda](B \varepsilon \{X : Q(X)\}) = V[M_\lambda](A \varepsilon \{X : Q(X)\})$ and $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(iii) Let D' be a general dependent set of $P(A)$ and let $S \subseteq D'$. For each $Q(A) \varepsilon S$, let the lemma hold for D' and the $(Q(A))^*$, by the induction hypothesis. By the condition of the lemma for all $T(A) \varepsilon (D' \cap \overline{S}) \cup \bigcup_{Q(A)\varepsilon S} (Q(A))^*$, $V[M_\lambda](T(B)) = V[M_\lambda](T(A))$. But since $(Q(A))^* \subseteq (D' \cap \overline{S}) \cup \bigcup_{Q(A)\varepsilon S} (Q(A))^*$ for all $Q(A) \varepsilon S$, by induction hypothesis, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$, for all $Q(A) \varepsilon S$. Also, for all $T(A) \varepsilon D' \cap \overline{S}$, $V[M_\lambda](T(B)) = V[M_\lambda](T(A))$. Hence, if $U(A) \varepsilon D'$, $V[M_\lambda](U(B)) = V[M_\lambda](U(A))$. By induction hypothesis for D', $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

Lemma 8 *If $V[M_\lambda](A \varepsilon C) = 1$ or 0 then $A \varepsilon C$ has a general dependent set without any wffs of the form $A \varepsilon B$ for any B, except for A and for $B \varepsilon D^S$. The general dependent sets so constructed are such that (ii) is not applied to any atomic wffs of form $A' \varepsilon A$.*

Proof. Let the wff $A \varepsilon C$ be W. The proof is by transfinite induction on $\nu(W)$, which is 0 or a successor ordinal. The induction hypothesis is that the lemma holds for all wffs $A \varepsilon C'$ (call it X) such that $\nu(X) < \nu(W)$.

(i) $\nu(W) = 0$: $A \varepsilon D^S$ and $C \varepsilon D^S$. Let the general dependent set be $\mathsf{D}(W)$, i.e., $\{W\}$. $\mathsf{C}(W)$ is not defined.

(ii) $\nu(W)$ is a successor ordinal: If $C \varepsilon D^S$, let the general dependent set be $\mathsf{D}(W)$. Otherwise, $V[M_{\nu(W)-1}](Z(A)) = 1$ or 0, where $Z(A)$ is $\mathsf{C}(W)$. Hence $\mathsf{D}(Z(A))$ is a general dependent set of W. It has a subset S of all atomic wffs of the form $A \varepsilon B$, except where B is A or where $B \varepsilon D^S$. For all Q, if $Q \varepsilon S$, then $V[M_{\nu(W)-1}](Q) = 1$ or 0. Hence, by the induction hypothesis, all these wffs $Q \varepsilon S$ have general dependent sets Q^* without wffs of the above form. Form the set $(\mathrm{D}(Z(A)) \cap \overline{S}) \cup \bigcup_{Q \varepsilon S} Q^*$, which has no atomic wffs of the above form. This is a general dependent set of W which satisfies the lemma.

Lemma 9 *If* $Y \varepsilon A \leftrightarrow Y \varepsilon B$ *is valid in* M_λ, *then* $A \varepsilon A \leftrightarrow B \varepsilon B$ *has value* 1 in M_λ.

Proof. Call $A \varepsilon A$, W. Let $V[M_\lambda](W) = 1$ or 0. By lemma 8, W has a general dependent set D' without atomic wffs of certain forms and constructed in a certain way. For the sake of lemma 8, the right hand A of $A \varepsilon A$ is regarded as different from the left hand A. So (ii) is applied in forming a general dependent set of $A \varepsilon A$, but apart from this one instance all the usual conditions apply. $\nu(W)$ is either 0 or a successor ordinal.

(i) $\nu(W) = 0$: Then $A \varepsilon D^S$.

(A) Let $B \varepsilon D^S$. Then $z \varepsilon A \leftrightarrow z \varepsilon B$ is valid in M_λ and hence in M_0. Because the Extensionality Axiom holds in NBG, $A \varepsilon y \leftrightarrow B \varepsilon y$ is valid in M_0. Hence, $A \varepsilon A \leftrightarrow B \varepsilon A$ and, since $B \varepsilon A \leftrightarrow B \varepsilon B$, $A \varepsilon A \leftrightarrow B \varepsilon B$ is valid in M_0 and hence in M_λ.

(B) Let $B \not\varepsilon D^S$. Then $z \varepsilon A \leftrightarrow z \varepsilon B$ is valid in M_λ. Hence by Łukasiewicz logic, $\sim z \varepsilon A \vee z \varepsilon B \&. z \varepsilon A \vee \sim z \varepsilon B$ is valid in M_λ. Hence, by construction of $M_{\lambda+1}$, $V[M_{\lambda+1}](B \varepsilon A) = V[M_0](A \varepsilon A)$ and then, since $V[M_\lambda](B \varepsilon B) = V[M_\lambda](B \varepsilon A)$, $V[M_\lambda](B \varepsilon B) = V[M_\lambda](A \varepsilon A)$.

(ii) $\nu(W)$ is a successor ordinal: Then $A \not\varepsilon D^S$. By lemma 6, all members of D' have the value 1 or 0 in $M_{\nu(W)-1}$. Hence W is not a member of D'. Hence D' has atomic wffs containing A, only of the forms $A' \varepsilon A$ ($A' \neq A$) and $A \varepsilon B'$, where $B' \varepsilon D^S$. Consider the atomic wff $A \varepsilon B'$.

(A) Let $\sim z \varepsilon A \vee z \varepsilon a \&. \sim z \varepsilon a \vee z \varepsilon A$ be valid in M_λ for some a. Hence $V[M_{\lambda+1}](A \varepsilon B') = V[M_0](a \varepsilon B')$. By the condition of the lemma, $\sim z \varepsilon B \vee z \varepsilon a \&. \sim z \varepsilon a \vee z \varepsilon B$ is valid in M_λ. Hence $V[M_{\lambda+1}](B \varepsilon B') = V[M_0](a \varepsilon B')$ and $V[M_\lambda](B \varepsilon B') = V[M_\lambda](A \varepsilon B')$.

(B) Let $(Ax)(Sz)(z \varepsilon A \& \sim z \varepsilon x \vee. z \varepsilon x \& \sim z \varepsilon A)$ have the value 1 in M_λ. Hence $V[M_{\lambda+1}](A \varepsilon B') = 0$. (That is, the case (B) is not a possibility if $V[M_\lambda](A \varepsilon B') = 1$). By the lemma condition, $(Ax)(Sz)(z \varepsilon B \& \sim z \varepsilon x \vee. z \varepsilon x \& \sim z \varepsilon B)$ has the value 1 in M_λ. Hence $V[M_{\lambda+1}](B \varepsilon B') = 0$ and $V[M_\lambda](B \varepsilon B') = V[M_\lambda](A \varepsilon B')$. Hence, if $Q(A) \varepsilon D'$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$. By lemma 7, $V[M_\lambda](B \varepsilon A) = V[M_\lambda](A \varepsilon A)$. Note that the substitution of B for A is only for the left hand A because (ii) was applied to $A \varepsilon A$. By the condition of the lemma, $V[M_\lambda](B \varepsilon B) = V[M_\lambda](B \varepsilon A)$ and hence $V[M_\lambda](B \varepsilon B) = V[M_\lambda](A \varepsilon A)$ (similarly for the case when $V[M_\lambda](B \varepsilon B)$ is 1 or 0, by setting W as $B \varepsilon B$). Hence the lemma holds.

Theorem 4 *The Extensionality Axiom* (E) *is valid in* M_λ.

Proof. We will prove: If $V \varepsilon A \leftrightarrow V \varepsilon B$ is valid in M_λ then $A \varepsilon Z \leftrightarrow B \varepsilon Z$ is valid in M_λ. Let $V[M_\lambda](A \varepsilon C) = 1$ or 0. By lemma 8, $A \varepsilon C$ has a general dependent set D' without any wffs of the form $A \varepsilon B'$ for any B' except for A

and cases where $B' \varepsilon D^S$. Hence the only occurrences of A in D' are of the forms: $A' \varepsilon A$ $(A' \neq A)$, $A \varepsilon A$ and $A \varepsilon B'$ (where $B' \varepsilon D^S$). Consider the atomic wff $A \varepsilon B'$.

(A) Let $A \varepsilon D^S$.

(i) Let $B \varepsilon D^S$. Then $z \varepsilon A \leftrightarrow z \varepsilon B$ is valid in M_0. By the Extensionality Axiom of NBG, $V[M_0](B \varepsilon B') = V[M_0](A \varepsilon B')$ and hence $V[M_\lambda](B \varepsilon B') = V[M_\lambda](A \varepsilon B')$.

(ii) Let $B \not\varepsilon D^S$. Then $z \varepsilon A \leftrightarrow z \varepsilon B$ is valid in M_λ. By the Łukasiewicz logic, $\sim z \varepsilon A \vee z \varepsilon B \,\&.\, \sim z \varepsilon B \vee z \varepsilon A$ is valid in M_λ. Hence $V[M_{\lambda+1}](B \varepsilon B') = V[M_0](A \varepsilon B')$ and $V[M_\lambda](B \varepsilon B') = V[M_\lambda](A \varepsilon B')$.

(B) Let $A \not\varepsilon D^S$:

(i) Let $\sim z \varepsilon A \vee z \varepsilon a \,\&.\, \sim z \varepsilon a \vee z \varepsilon A$ be valid in M_λ for some a. Hence $V[M_{\lambda+1}](A \varepsilon B') = V[M_0](a \varepsilon B')$. By the condition of the theorem, $\sim z \varepsilon B \vee z \varepsilon a \,\&.\, \sim z \varepsilon a \vee z \varepsilon B$ is valid in M_λ. Hence $V[M_{\lambda+1}](B \varepsilon B') = V[M_0](a \varepsilon B')$ and $V[M_\lambda](B \varepsilon B') = V[M_\lambda](A \varepsilon B')$.

(ii) Let $(Ax)(Sz)(z \varepsilon A \,\&\, \sim z \varepsilon x \vee. z \varepsilon x \,\&\, \sim z \varepsilon A)$ have the value 1 in M_λ. Hence $V[M_{\lambda+1}](A \varepsilon B') = 0$. (That is, the case (ii) is not a possibility if $V[M_\lambda](A \varepsilon B') = 1$). By the condition $(Ax)(Sz)(z \varepsilon B \,\&\, \sim z \varepsilon x \vee. z \varepsilon x \,\&\, \sim z \varepsilon B)$ has the value 1 in M_λ. Hence $V[M_{\lambda+1}](B \varepsilon B') = 0$ and $V[M_\lambda](B \varepsilon B') = V[M_\lambda](A \varepsilon B')$. Hence, in all cases, $V[M_\lambda](B \varepsilon B') = V[M_\lambda](A \varepsilon B')$. By lemma 9, $V[M_\lambda](B \varepsilon B) = V[M_\lambda](A \varepsilon A)$. By the condition of the theorem, $V[M_\lambda](A' \varepsilon B) = V[M_\lambda](A' \varepsilon A)$. Hence, if $Q(A) \varepsilon D'$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$. By lemma 7, $V[M_\lambda](B \varepsilon C) = V[M_\lambda](A \varepsilon C)$. Similarly, if $V[M_\lambda](B \varepsilon C) = 1$ or 0, then $V[M_\lambda](A \varepsilon C) = V[M_\lambda](B \varepsilon C)$. Hence the theorem holds.

Theorem 5 $(Az)(z \varepsilon x \leftrightarrow z \varepsilon X) \supset (Aw)(x \varepsilon w \leftrightarrow X \varepsilon w)$ *is valid in* M_λ.

Proof. (i) Let $A \not\varepsilon D^S$. Let $z \varepsilon a \leftrightarrow z \varepsilon A$ be valid in M_λ. Hence $\sim z \varepsilon a \vee z \varepsilon A \,\&.\, \sim z \varepsilon A \vee z \varepsilon a$ is valid in M_λ. Hence $V[M_{\lambda+1}](A \varepsilon c) = V[M_0](a \varepsilon c)$ and $V[M_\lambda](A \varepsilon c \leftrightarrow a \varepsilon c) = 1$, for any $c \varepsilon D^S$.

(ii) Let $A \varepsilon D^S$. Then, by the Extensionality Axiom for NBG, the theorem holds.

Theorem 6 *Each of the following are valid in* M_λ.

(*i*) $\mathsf{C}(x \varepsilon y)$

(*ii*) $\mathsf{F}(\mathsf{SCl}(X)) \supset \mathsf{F}(X \varepsilon x)$

(*iii*) $\mathsf{P}(\mathsf{SCl}(X)) \supset \mathsf{P}(X \varepsilon x)$.

Proof. (*i*) is valid by definition of M_0. Let $V[M_\lambda](\mathsf{F}(\mathsf{SCl}(A))) = 1$. Hence $V[M_\lambda]((Ax)(Sz)(z \varepsilon x \,\&\, \sim z \varepsilon A \vee. z \varepsilon A \,\&\, \sim z \varepsilon x)) = 1$ and $V[M_{\lambda+1}](A \varepsilon b) = 0$, for any b. Hence $\mathsf{F}(A \varepsilon x)$ is valid in M_λ. Let $V[M_\lambda](A \varepsilon b) = 1$ or 0. Then either $A \varepsilon D^S$, $z \varepsilon a \leftrightarrow z \varepsilon A$ is valid in M_λ for some a, or $(Ax)(Sz) \sim (z \varepsilon x \leftrightarrow z \varepsilon A)$ is valid in M_λ. Hence $\mathsf{SCl}(A)$ has the value 1 or 0 in M_λ. Hence if $V[M_\lambda](\mathsf{SCl}(A)) = \frac{1}{2}$ then $V[M_\lambda](A \varepsilon b) = \frac{1}{2}$.

Theorem 7 $(AX)A(X) \to (Ax)A(x)$ *is valid in* M_λ.

Proof. Let $V[M_\lambda]((AX)A(X)) = 1$. Then $V[M_\lambda](A(X)) = 1$, for all $X \varepsilon D$. Hence $V[M_\lambda](A(x)) = 1$, for all $x \varepsilon D^S$, since $D^S \subseteq D$. Therefore $V[M_\lambda]((Ax)A(x)) = 1$.

Let $V[M_\lambda]((AX)A(X)) = \frac{1}{2}$. Then $V[M_\lambda](A(X)) = \frac{1}{2}$ or 1, for all X. Hence $V[M_\lambda](A(x)) = \frac{1}{2}$ or 1, for all x. Therefore $V[M_\lambda]((Ax)A(x)) = \frac{1}{2}$ or 1.

Theorem 8. $(Az)(z \varepsilon x \equiv z \varepsilon y) \supset (AZ)(Z \varepsilon x \leftrightarrow Z \varepsilon y)$ *is valid in* M_λ.

Proof. Let $z \varepsilon a \leftrightarrow z \varepsilon b$ be valid in M_λ. If $A \varepsilon D^S$, then $V[M_\lambda](A \varepsilon a \leftrightarrow A \varepsilon b) = 1$. Let $A \not\varepsilon D^S$.

(i) If $V[M_\lambda](\mathsf{SCl}(A)) = 1$, then, for some c, $z \varepsilon c \leftrightarrow z \varepsilon A$ is valid in M_λ. Hence $V[M_{\lambda+1}](A \varepsilon a) = V[M_0](c \varepsilon a)$ and $V[M_{\lambda+1}](A \varepsilon b) = V[M_0](c \varepsilon b)$. But $V[M_0](c \varepsilon a) = V[M_0](c \varepsilon b)$ and so $V[M_\lambda](A \varepsilon a \leftrightarrow A \varepsilon b) = 1$.

(ii) If $V[M_\lambda](\mathsf{SCl}(A)) = 0$, then $V[M_\lambda](A \varepsilon a) = 0$ and $V[M_\lambda](A \varepsilon b) = 0$. Hence $V[M_\lambda](A \varepsilon a \leftrightarrow A \varepsilon b) = 1$.

(iii) If $V[M_\lambda](\mathsf{SCl}(A)) = \frac{1}{2}$, then $V[M_\lambda](A \varepsilon a) = \frac{1}{2}$ and $V[M_\lambda](A \varepsilon b) = \frac{1}{2}$. Hence $V[M_\lambda](A \varepsilon a \leftrightarrow A \varepsilon b) = 1$.

4. The above method can be used to extend any set or class theory with a two-valued model with an axiom of extensionality to a three-valued class theory satisfying the axioms of abstraction and extensionality. By using appropriate models of NBG, the consistency and independence of the axiom of choice, the generalized continuum hypothesis and the axiom of constructibility can be shown. Also the connectives and quantifiers of the three-valued logic used to define standard wffs can be extended to include any which satisfy the following property.

(i) *For connectives* $\Gamma(p_1, \ldots, p_n)$.
Let $V[M](\Gamma(p_1, \ldots, p_n)) = 1$ (or 0). Let X_0 (X_1) be the set of indices i such that $V[M](p_i) = 0$ (=1). For some structure M', let X_0' (X_1') be the set of indices i such that $V[M'](q_i) = 0$ (=1). If $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$, then $V[M'](\Gamma(q_1, \ldots, q_n)) = 1$ (or 0).

(ii) *For quantifiers* $(QX)A(X)$.
Let $V[M]((QX)A(X)) = 1$ (or 0). Let X_0 (X_1) be the set of all X in D such that $V[M](A(X)) = 0$ (=1). For some structure M', let X_0' (X_1') be the set of all X in D such that $V[M'](B(X)) = 0$ (=1). If $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$, then $V[M']((QX)B(X)) = 1$ (or 0).

(iii) *For quantifiers* $(Qx)A(x)$.
Similar to (ii) except D^S for D.

Proposition *Any quantifier or connective defined in terms of quantifiers and connectives satisfying the above property also satisfies the above property.*

Proof. (i) *Connectives.* Let $V[M](\Gamma(\Delta_1(q_1, \ldots, q_n), \ldots, \Delta_m(q_1, \ldots, q_n))) = 1$ (or 0), where $\Gamma, \Delta_1, \ldots, \Delta_m$ satisfy the property. Let X_0 (X_1) be the set of indices i such that $V[M](q_i) = 0$ (=1). For some structure M', let X_0' (X_1') be the set of indices i such that $V[M'](r_i) = 0$ (=1). Let $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$. Let Y_0(Y_1) be the set of indices i such that $V[M](\Delta_i(q_1, \ldots, q_n)) = 0$ (=1). Let $i \varepsilon Y_0 \cup Y_1$. Because $\Delta_i(q_1, \ldots, q_n)$ satisfies the property, and $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$, $V[M'](\Delta_i(r_1, \ldots, r_n)) = V[M](\Delta_i(q_1, \ldots, q_n))$. Hence, if Y_0' (Y_1') is the set of indices i such that $V[M'](\Delta_i(r_1, \ldots, r_n)) = 0$ (=1), then $Y_1 \subseteq Y_1'$ and $Y_0 \subseteq Y_0'$. Since $\Gamma(\Delta_1, \ldots, \Delta_m)$ satisfies the property, $V[M'](\Gamma(\Delta_1, \ldots, \Delta_m)) = 1$ (or 0).

(ii) *Quantifiers*. Let $V[M](\Gamma((QX)\Delta(A(X)))) = 1$ (or 0), where Γ, Δ and (QX) satisfy the property. Let X_0 (X_1) be the set of all X from D such that $V[M](A(X)) = 0$ $(=1)$. For some structure M', let X_0' (X_1') be the set of all X from D such that $V[M'](B(X)) = 0$ $(=1)$. Let $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$. Because Δ satisfies the property, if $V[M](\Delta(A(X))) = 1$ or 0 then $V[M'](\Delta(B(X))) = V[M](\Delta(A(X)))$, for any $X \varepsilon D$. If Y_0 (Y_1) is the set of all X in D such that $V[M](\Delta(A(X))) = 0$ $(=1)$, Y_0' (Y_1') is the set of all X in D such that $V[M'](\Delta(B(X))) = 0$ $(=1)$, then $Y_0 \subseteq Y_0'$ and $Y_1 \subseteq Y_1'$. Because (QX) satisfies the property, if $V[M]((QX)\Delta(A(X))) = 1$ or 0 then $V[M']((QX)\Delta(B(X))) = V[M]((QX)\Delta(A(X)))$. Since Γ satisfies the property, $V[M'](\Gamma((QX)\Delta(B(X)))) = 1$ (or 0). Similarly for quantifiers, (Qx).

Some examples of connectives satisfying the property are the following.

	1	0	$\frac{1}{2}$
1	1, 0 or $\frac{1}{2}$	1, 0 or $\frac{1}{2}$	$\frac{1}{2}$
0	1, 0 or $\frac{1}{2}$	1, 0 or $\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

	1	0	$\frac{1}{2}$
1	1	1, 0 or $\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1 or $\frac{1}{2}$
$\frac{1}{2}$	1 or $\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

1	1
0	1
$\frac{1}{2}$	1 or $\frac{1}{2}$

1	1, 0 or $\frac{1}{2}$
0	1, 0 or $\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$

The Łukasiewicz (AX), (Ax), (SZ), (Sx) are examples satisfying the quantifier property. The following connectives are *not* examples:

$\rightarrow$	1	0	$\frac{1}{2}$
1	1	0	$\frac{1}{2}$
0	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1

$\leftrightarrow$	1	0	$\frac{1}{2}$
1	1	0	$\frac{1}{2}$
0	0	1	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

$\supset$	1	0	$\frac{1}{2}$
1	1	0	$\frac{1}{2}$
0	1	1	1
$\frac{1}{2}$	1	1	1

T	
1	1
0	0
$\frac{1}{2}$	0

C	
1	1
0	1
$\frac{1}{2}$	0

To show that any of the connectives or quantifiers satisfying the property above can be used to define standard wffs and hence be substituted into the Abstraction Axiom, it is only necessary to examine lemmas 1 and 4 in the proof. Lemma 1 is obvious from the definition of the property. In Lemma 4, leave out the original steps for the connectives and quantifiers and replace it by the following.

(i) Let $P(A)$ be $\Gamma(R_1(A), \dots, R_n(A))$ and let $V[M_\lambda](\Gamma(R_1(A), \dots, R_n(A))) = 1$ or 0. Let $\Gamma(R_1(A), \dots, R_n(A))$ be W. Then $V[M_{\nu(W)}](W) = 1$ or 0. Let X_0 (X_1) be the set of indices i such that $V[M_{\nu(W)}](R_i(A)) = 0$ $(=1)$. Since $\nu(R_i(A)) \leqslant \nu(W)$, for all $i \varepsilon X_0 \cup X_1$, $\mathsf{D}(R_i(A)) \subseteq \mathsf{D}(W)$, for all $i \varepsilon X_0 \cup X_1$. By the lemma condition, for each $Q(A) \varepsilon \mathsf{D}(R_i(A))$ where $i \varepsilon X_0 \cup X_1$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$. By the induction hypothesis, $V[M_\lambda](R_i(B)) = V[M_\lambda](R_i(A))$, for all $i \varepsilon X_0 \cup X_1$. Let X_0' (X_1') be the set of indices i such that $V[M_\lambda](R_i(B)) =$

0 (=1). Hence $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$. By the property of Γ, $V[M_\lambda](\Gamma(R_1(B),\ldots, R_n(B))) = V[M_{\nu(W)}](\Gamma(R_1(A),\ldots, R_n(A)))$ and $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(ii) Let $P(A)$ be $(QZ)R(A,Z)$ and $V[M_\lambda]((QZ)R(A,Z)) = 1$ or 0. Let $(QZ)R(A,Z)$ be W. Let $X_0\,(X_1)$ be the set of all Z in D such that $V[M_{\nu(W)}](R(A,Z)) = 0$ (=1). Since $\nu(R(A,Z)) \leqslant \nu(W)$ for all $Z \varepsilon X_0 \cup X_1$, $\mathrm{D}(R(A,Z)) \subseteq \mathrm{D}(W)$, for all $Z\varepsilon X_0 \cup X_1$. By the lemma condition, for each $Q(A)\,\varepsilon\,\mathrm{D}(R(A,Z))$, $V[M_\lambda](Q(B)) = V[M_\lambda](Q(A))$, where $z\,\varepsilon X_0 \cup X_1$. By the induction hypothesis, $V[M_\lambda](R(B,Z)) = V[M_\lambda](R(A,Z))$, for all $Z\,\varepsilon X_0 \cup X_1$. Let $X_0'\,(X_1')$ be the set of all Z in D such that $V[M_\lambda](R(B,Z)) = 0$ (=1). Hence $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$. By the property of (QZ), $V[M_\lambda]((QZ)R(B,Z)) = V[M_{\nu(W)}]((QZ)R(A,Z))$. Hence $V[M_\lambda](P(B)) = V[M_\lambda](P(A))$.

(iii) Let $P(A)$ be $(Qz)R(A,z)$. This case follows as for (ii) except that D^S replaces D.

5. There is a further generalization which allows any set or class theory using a many-valued (finite or infinite) logic L and with a model in which an axiom of extensionality is satisfied to be extended to a class theory, using a logic L' of one more value and with a model in which the axioms of extensionality and abstraction are satisfied.

The many-valued logic L must contain a quantifier S such that $(SZ)A(Z)$ takes the value m (where m is some designated value) if and only if at least one of the $A(Z)$ are designated and takes the value n (where n is some undesignated value) if and only if all of the $A(Z)$ are undesignated (similarly for $(Sz)A(z)$); a quantifier A such that $(AZ)A(Z)$ takes the value m (same as above) if and only if all of the $A(Z)$ are designated and takes the value n (same as above) if and only if at least one of the $A(Z)$ is undesignated (similarly for $(Az)A(z)$); an equivalence connective $\leftrightarrow$ such that $p \leftrightarrow q$ is designated if and only if p and q take the same value; and an implication connective $\supset$ such that $p \supset q$ is designated if and only if q is designated or p is undesignated.

The many-valued logic L', which has an extra value (call it pd) added to L, must contain appropriate extensions of S, A, $\leftrightarrow$ and $\supset$. The value pd is undesignated. $p \supset q$ is defined so that it is designated if and only if q is designated or p is undesignated. $p \leftrightarrow q$ is defined so that if p and q take values in L, then $p \leftrightarrow q$ takes the value in L, if p does not take the value pd and q takes the value pd or if p takes the value pd and q does not then $p \leftrightarrow q$ takes the value pd, and if p and q both take the value pd then $p \leftrightarrow q$ is designated. The quantifier S is defined in L' as follows. If $A(Z)$ has a designated value for some Z, then $(SZ)A(Z)$ has the value m. If $A(Z)$ has an undesignated value, not pd, for all Z then $(SZ)A(Z)$ has the value n. Otherwise $(SZ)A(Z)$ has the value pd. The quantifier A is defined in L' as follows. If $A(Z)$ has a designated value for all Z, then $(AZ)A(Z)$ has the value m. If $A(Z)$ has an undesignated value, not pd, for some Z, then $(AZ)A(Z)$ has the value n. Otherwise $(AZ)A(Z)$ has the value pd. Similarly for $(Az)A(z)$ and $(Sz)A(z)$.

The Extensionality Axiom can now be stated as $(AZ)(Z\,\varepsilon\,X \leftrightarrow Z\,\varepsilon\,Y) \supset (AZ)(X\,\varepsilon\,Z \leftrightarrow Y\,\varepsilon\,Z)$. The Abstraction Axiom can be stated as $(SY)(AX)$

$(X \varepsilon Y \leftrightarrow \phi(X, z_1, \ldots, x_m, Z_1, \ldots, Z_n))$, where ϕ is constructed from atomic wffs $U \varepsilon V$, $U \varepsilon v$, $u \varepsilon V$, $u \varepsilon v$, using the connectives and quantifiers used in forming standard wffs. The Extensionality Axiom for special classes can be stated as $(Az)(z \varepsilon x \leftrightarrow z \varepsilon y) \supset (Az)(x \varepsilon z \leftrightarrow y \varepsilon z)$. $\mathsf{Cl}(X)$ is defined as $(Sx)(Az)(z \varepsilon x \leftrightarrow z \varepsilon X)$.

The propositional constants are left out from the atomic wffs and if atomic wffs with some of these values are wanted then perhaps an atomic wff of the form $a \varepsilon b$ can be used. The connectives and quantifiers used in forming standard wffs are ones which satisfy the following property S.

(i) *For connectives* $\Gamma(p_1, \ldots, p_n)$. Let $V[M]\ \Gamma(p_1, \ldots, p_n)) = k$, some value of L. Let X_m be the set of indices i such that $V[M](p_i) = m$, for each value m of L. For some structure M', let X_m' be the set of indices i such that $V[M'](q_i) = m$, for each value m of L. If $X_m \subseteq X_m'$, for all m of L, then $V[M'](q_1, \ldots, q_n) = k$.

(ii) *For quantifiers* $(QX)A(X)$. Let $V[M]((QX)A(X)) = k$, some value of L. Let X_m be the set of all X in D such that $V[M](A(X)) = m$, for each value m of L. For some structure M', let X_m' be the set of all X in D such that $V[M'](B(X)) = m$, for each m in L. If $X_m \subseteq X_m'$, for all m of L, then $V[M']((QX)B(X)) = k$.

(iii) *For quantifiers* $(Qx)A(x)$. Similar to (ii) except D^S for D.

Note that the quantifiers S and A in L' satisfy the property S. For the definition of $M_1 \leqslant M_2$, for two structures M_1 and M_2 the generalization is as follows. $M_1 \leqslant M_2$ if and only if, for any atomic wff P, if $V[M_1](P) = m$, for some value m in L, then $V[M_2](P) = m$.

Lemma 1 follows by the generalized property for connectives and quantifiers used in forming standard wffs. It takes the form:

Let $M \leqslant M'$, where M and M' are two structures on D. Then, for any standard wff P, if $V[M](P) = m$, for some value m in L, then $V[M'](P) = m$.

Define the structure M_0 as follows. If $A \not\varepsilon D^S$ or $B \not\varepsilon D^S$, then $V[M_0](A \varepsilon B) =$ pd. If $A \varepsilon D^S$ and $B \varepsilon D^S$, then $V[M_0](A \varepsilon B) =$ the value of L given to $A \varepsilon B$ in the model of the special class theory.

Assuming M_μ defined for some ordinal μ, $M_{\mu+1}$ is defined as follows. For all standard wffs P, $V[M_{\mu+1}](A \varepsilon \{X : P(X)\}) = V[M_\mu](P(A))$. If $V[M_\mu](z \varepsilon A) = V[M_\mu](z \varepsilon a)$ for all $z \varepsilon D^S$, for some $a \varepsilon D^S$, then $V[M_{\mu+1}](A \varepsilon b) = V[M_0](a \varepsilon b)$. If there is no $a \varepsilon D^S$ such that for all $z \varepsilon D^S$, $V[M_\mu](z \varepsilon A) = V[M_\mu](z \varepsilon a)$, then $V[M_{\mu+1}](A \varepsilon b) = V[M_\mu](\mathsf{Cl}(A))$.

Note that $\mathsf{Cl}(X)$ has the property S, because $z \varepsilon x$ only takes values in L. Also $V[M_\mu](z \varepsilon A) = V[M_\mu](z \varepsilon a)$ for all $z \varepsilon D^S$, for some $a \varepsilon D^S$ if and only if $V[M_\mu](\mathsf{Cl}(A))$ is designated, and there is no $a \varepsilon D^S$ such that for all $z \varepsilon D^S$, $V[M_\mu](z \varepsilon A) = V[M_\mu](z \varepsilon a)$ if and only if $V[M_\mu](\mathsf{Cl}(A))$ is undesignated.

If μ is a limit ordinal, on the assumption that $M_\nu \leqslant M_\tau$ for all $\nu \leqslant \tau$, for all $\tau < \mu$, for all atomic wffs P, if $V[M_\nu](P) = k$, for some value k in L, for some $\nu < \mu$, then $V[M_\mu](P) = k$, and if $V[M_\nu](P) =$ pd for all $\nu < \mu$, then $V[M_\mu](P) =$ pd.

Lemma 2 follows similarly to the previous proof. In case (B), let $V[M_\nu](A \varepsilon b) = k$, for some value k in L. Then there is an ordinal $\eta < \nu$ such that $V[M_\eta]($ SCI$(A) = l$, for some value l in L. Since $\eta \leqslant \mu - 1$, $M_\eta \leqslant M_{\mu-1}$. Hence $V[M_{\mu-1}]($ SCI$(A)) = l$. If l is undesignated, $l = k$ and $V[M_\mu](A \varepsilon b) = k$. If l is designated, then there is an $a \varepsilon D^S$ such that $V[M_0](z \varepsilon a) = V[M_\eta](z \varepsilon A)$ for all $z \varepsilon D^S$. Then $V[M_0](a \varepsilon b) = k$. Hence $V[M_{\mu-1}](z \varepsilon A) = V[M_0](z \varepsilon a)$, for all $z \varepsilon D^S$, and $V[M_\mu](A \varepsilon b) = k$.

Lemma 3 follows as before except that there is one increasing chain of subsets of the denumerable set of all atomic wffs for every value in L. Theorems 1, 2 and 3 follow similarly to their previous proofs. The definitions of $\nu(P)$ and dependent set D(P) are the same except that all values of L must be put in place of values 1 and 0.

Lemma 4 can be shown for connectives and quantifiers satisfying the property S by a simple generalization using X_k, where k runs over the values of L, instead of X_0 and X_1.

Corresponding standard wffs and general dependent sets are defined as before. Lemmas 5, 6, 7 and 8 follow as before with the values of L in place of 1 and 0.

In lemma 9, (ii) (A) becomes: Let SCI(A) be valid in M_λ. Then $V[M_\lambda](z \varepsilon A) = V[M_\lambda](z \varepsilon a)$, for all $z \varepsilon D^S$, for some $a \varepsilon D^S$. Hence $V[M_{\lambda+1}](A \varepsilon B') = V[M_0](a \varepsilon B')$. By the condition of the lemma, $V[M_\lambda](z \varepsilon B) = V[M_\lambda](z \varepsilon a)$, for all $z \varepsilon D^S$. Hence $V[M_{\lambda+1}](B \varepsilon B') = V[M_0](a \varepsilon B')$. Hence $V[M_\lambda](a \varepsilon B') = V[M_\lambda](B \varepsilon B')$. (ii) (B) becomes: Let SCI(A) be invalid in M_λ. $V[M_\lambda]($ SCI$(A)) \neq$ pd because $V[M_\lambda](A \varepsilon B')$ is a value of L. Hence $V[M_{\lambda+1}](A \varepsilon B') = V[M_\lambda]($ SCI$(A))$. By the lemma condition, $V[M_\lambda]($ SCI$(A)) = V[M_\lambda]($ SCI$(B))$. Hence $V[M_{\lambda+1}](B \varepsilon B') = V[M_\lambda]($ SCI$(A))$ and $V[M_\lambda](B \varepsilon B') = V[M_\lambda](A \varepsilon B')$. The rest of lemma 9 follows as before.

In Theorem 4, (B) (i) and (B) (ii) are similar to (ii) A and (ii) B respectively of lemma 9. Otherwise the theorem follows as before. Theorem 5 follows as before.

Theorem 6 needs two monadic operators: C such that Cp is designated if and only if p takes a value in L, and U such that Up is designated if and only if p is undesignated. Theorem 6 becomes: (i) C$(x \varepsilon y)$ and (ii) U(SCI$(X)) \supset$. SCI$(X) \leftrightarrow X \varepsilon x$. are valid in M_λ, both of which are obvious.

Theorem 7 becomes: $(AX)A(X) \supset (Ax)A(x)$ *is valid in* M_λ, which is obvious.

Theorem 8 becomes: $(Az)(z \varepsilon x \leftrightarrow z \varepsilon y) \supset (AZ)(Z \varepsilon x \leftrightarrow Z \varepsilon y)$ *is valid in* M_λ.

Proof. Let $A \notin D^S$. Let $z \varepsilon a \leftrightarrow z \varepsilon b$ be valid in M_λ.

(i) If SCI(A) is valid in M_λ, $V[M_\lambda](z \varepsilon A) = V[M_0](z \varepsilon c)$, for all $z \varepsilon D^S$, for some $c \varepsilon D^S$. Hence $V[M_{\lambda+1}](A \varepsilon a) = V[M_0](c \varepsilon a)$ and $V[M_{\lambda+1}](A \varepsilon b) = V[M_0](c \varepsilon b)$. But $V[M_0](c \varepsilon a) = V[M_0](c \varepsilon b)$ and so $A \varepsilon b \leftrightarrow A \varepsilon a$ is valid in M_λ.

(ii) If SCI(A) is invalid in M_λ, $V[M_{\lambda+1}](A \varepsilon a) = V[M_{\lambda+1}](A \varepsilon b) = V[M_\lambda]($ SCI$(A))$. Hence $A \varepsilon b \leftrightarrow A \varepsilon a$ is valid in M_λ.

6. The above method of avoiding the class paradoxes has certain advantages. It allows each predicate to generate a class and separates the "paradoxical" class membership statements from the "non-paradoxical" ones using a criterion of circularity of definition. For, in order for a membership statement to take the value $\frac{1}{2}$ (or pd), there must be some circularity involved, in the sense that its value is dependent on itself or on the values of membership statements whose values are dependent on themselves. If there is no such circularity then there is a chain of dependent membership statements leading from the membership statement in question right back to membership statements of NBG (or other model) and propositional constants ($\neq \frac{1}{2}$ (or pd)). This is represented in the proof by the general dependent sets of a membership statement. If there is such a chain of dependent membership statements then the membership statement in question takes a value $\neq \frac{1}{2}$ (or pd). Basing the system on NBG allows the whole of mathematics to be deduced in the usual two-valued logic. One can make true and false statements about the universal class and Russell class which cannot normally be made in other attempts to avoid the class paradoxes. One can make true and false statements about classes of *proper* classes which cannot be made in NBG.

By defining all classes which can be generated by a predicate we can get a broader picture and see how the paradoxes arise; indeed, it shows that it is the circularity of definition of certain membership statements that leads *more directly* to the paradoxes than just the inclusion of a certain range of classes because true and false statements can be made about these classes and further, by using some general criterion for the rejection of classes one may well reject classes which lead to no paradoxes at all.

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