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MAXIMAL FULL MATRICES

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A language $\mathcal{L}_0$ of order zero is determined by a countably infinite set $\{p, q, r, \ldots\}$ of propositional variables, a non-empty finite set Ω^* of propositional connectives of finite positive arity and the set Γ of all wffs defined in the usual way. A wff α is called *variable-like* iff no propositional variable occurs more than once in α, otherwise it is called *non-variable-like*. A *structure* for $\mathcal{L}_0$ is a system $\mathfrak{M} = \langle A, D, \Omega\rangle$, where A, called the truth-values of $\mathfrak{M}$, is a non-empty set of elements; D, called the designated elements of $\mathfrak{M}$, is a subset of A; Ω is a non-empty finite set of operations of finite positive arity defined in A and there is an onto and arity-preserving mapping from Ω^* to Ω. The fact that we do not require the mapping from Ω^* to Ω in the above definition to be one-to-one poses no limitation on the usefulness of these structures for the purposes they are used. A structure $\mathfrak{M} = \langle A, D, \Omega\rangle$ is called *full* iff for each $\omega \in \Omega$, the range of ω is A, i.e. ω is onto as a function. Validity of wffs of $\mathcal{L}_0$ in any structure of it is defined in the usual way. Two structures of $\mathcal{L}_0$ are called *equivalent* iff they have the same class of valid wffs. Let Δ be a subset of Γ such that Δ is closed under the usual rule of substitution of wffs for propositional variables and let $\mathfrak{M}$ be a structure of $\mathcal{L}_0$. Then $\mathfrak{M}$ is called a *matrix* of Δ iff every wff in Δ is valid in $\mathfrak{M}$. A matrix $\mathfrak{M}$ of Δ is called a *characteristic matrix* of Δ iff every wff valid in $\mathfrak{M}$ is in Δ. The important role played by full matrices in the study of a large class of extensions of propositional calculi is established in [2] and [3] and some of their general properties are studied in [1]. The purpose of the present note is to prove the following:

Theorem. *The set of all non-variable-like wffs of any language of order zero that has at least one propositional connective of arity greater than one is characterizable by a countably infinite full matrix but by no finite matrix.*

Proof: Let $\mathcal{L}_0$ be a language of order zero that has at least one propositional connective of arity greater than one. By the definition of non-variable-like wffs, the set of Δ of all non-variable-like wffs of $\mathcal{L}_0$ is closed under substitution. Let A be the set of all non-negative integers. For each positive integer $n \geq 1$ we define an n-ary operation ω_n in A as follows: for all $a, b \in A$

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$$\omega_1\,(a) = a$$

$$\omega_2\,(a,\,b) = \begin{cases} i, & \text{if } i \in A,\ a = (2i+1) \text{ and } b = (2i+2). \\ 0, & \text{otherwise.} \end{cases}$$

For each $n \geq 2$ and all $a_1, a_2, \ldots, a_n, a_{n+1} \in A$ we set

$$\omega_{n+1}\,(a_1, a_2, \ldots, a_n, a_{n+1}) = \omega_2\,(\omega_n(a_1, a_2, \ldots, a_n), a_{n+1})$$

For each $j \geq 1$, we let ω_j correspond to all the j-ary propositional connectives (if any) of $\mathcal{L}_0$. Now let Ω be the finite set of all ω_j obtained in the above way and consider the structure $\mathfrak{M} = \langle A, \{0\}, \Omega\rangle$.

$\mathfrak{M}$ is easily seen to be full. For example, $5 = \omega_1(5) = \omega_2(11, 12) = \omega_2(\omega_2(23, 24), 12) = \omega_3(23, 24, 12) = \omega_2(\omega_2(\omega_2(47, 48,) 24), 12) = \omega_4(47, 48, 24, 12)$ etc. In general, for each $i \in A$ and $n \geq 2$,

$$i = \omega_n(2^{n-2}\,(2i+2) - 1,\ 2^{n-2}(2i+2),\ 2^{n-3}(2i+2),\ \ldots,\ 2^1(2i+2),\ 2i+2).$$

We note that ω_1 is the identity operation in A and that for each $j \geq 3$, ω_j is defined by using ω_2. It follows from the definition of ω_2 that for any $a \in A$, $\omega_2(a, a) = \omega_2(0, a) = \omega_2(a, 0) = 0$. Therefore, no non-zero integer in A has a factorization in $\mathfrak{M}$ in terms of the ω_j and the elements of A in which a non-negative integer occurs more than once. Let now α be any non-variable-like wff. Then some propositional variable occurs more than once in α. Consequently α must assume the value 0 for any assignment of values in $\mathfrak{M}$ and hence is a valid wff of $\mathfrak{M}$. Thus, $\mathfrak{M}$ is a matrix of Δ. To show that $\mathfrak{M}$ is a characteristic matrix of Δ it is sufficient to show that every variable-like wff α fails to be valid in $\mathfrak{M}$. We know that $\mathfrak{M}$ is full. Let α be variable-like. By induction on the number of propositional connectives in α it is easily shown that α can be given any desired value in A and hence is not valid in $\mathfrak{M}$. (See Lemma I in [2] or the remark following the proof of Theorem I in [1].) Thus $\mathfrak{M}$ is a full characteristic matrix of the set Δ of all non-variable-like wffs of $\mathcal{L}_0$. It remains to be shown that no finite matrix of Δ is equivalent to $\mathfrak{M}$.

Let $\mathfrak{M}_1 = \langle A_1, D_1, \Omega_1\rangle$ by any matrix of Δ such that A_1 has a finite number, say m, of elements. Since $\mathcal{L}_0$ has at least one propositional connective of arity greater than one, it has a variable-like wff α in which at least $m+1$ distinct propositional variables occur. If such an α fails to be valid in $\mathfrak{M}_1$, then there is an assignment in $\mathfrak{M}_1$ to the variables of α such that α takes an undesignated value in $\mathfrak{M}_1$. But A_1 has only m elements and α has at least $m+1$ variables. Thus by using this invalidating assignment for α in $\mathfrak{M}_1$ we can construct a substitution instance α^* of α such that α^* is non-variable-like and also fails to be valid in $\mathfrak{M}_1$. This shows that $\mathfrak{M}$ and $\mathfrak{M}_1$ cannot be equivalent.

Remark: If $\mathcal{L}_0$ has only connectives of arity one, then all its wffs are variable-like and Δ is empty—and hence the trivial full-matrix with one truth-value and no designated element characterizes Δ. In a full-matrix all variable-like wffs are invalid, unless all truth-values are designated. Therefore, under the hypothesis of the theorem any full matrix of Δ that is not equivalent to the infinite full-matrix $\mathfrak{M} = \langle A, \{0\}, \Omega\rangle$ constructed above must be equivalent to the trivial full-matrix with one truth-value which is also designated. This shows the maximality of the constructed infinite matrix. The writer has not found any important use for this infinite matrix. However, it seems to have some intrinsic appeal.

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