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THE GÖDEL-HERBRAND THEOREMS

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1. *Introduction.* The past several years have seen a renewal of interest in the results contained in Herbrand's thesis [1]. This is due mainly to the work of Dreben and his colleagues. The relationship between Herbrand's results and Gödel's Completeness Theorem for elementary quantification theory has been discussed by Dreben [2]. In particular, he shows that Gödel's theorem may be derived by combining the "finitistic" results of Herbrand for provability with the "set-theoretic" half of Gödel's proof, i.e., that part of Gödel's proof which deals with disprovability. In this way Dreben derives Gödel's theorem rather easily from Herbrand's.

To the best of my knowledge no one has observed that finitistic results, very similar to Herbrand's, are obtainable from the proof of Gödel's theorem, as e.g., proved in [3]. In this paper, we show that this is indeed the case. We do this by using the proof of the completeness theorem of [3] as a basis for reformalizing quantification theory. We accomplish the reformalization in three steps. Our first system, E_1, formalizes the logically valid closed formulas in Skolem normal form; system E_2: the logically valid closed prenex normal form formulas; and finally E_3: the logically valid formulas of elementary quantification theory. Each system yields a normal form for proofs.

2. *The System* E_1. The formulas of E_1 are those of the usual first-order predicate calculus built up from atomic formulas (predicates with argument places occupied by individual variables) in customary fashion by means of the usual elementary connectives and existential and universal quantifiers, E and A respectively. Capital letters R, S, T, . . . are used to represent predicates. The individual variables are symbolized by x_0, y_0, x_1, y_1, etc. Formulas are indicated by F, G, . . . , M, N, with or without subscripts.

It is assumed that the set of k-tuples formed from the individual variables $x_0, x_1, \ldots$ are ordered in the standard way, according to increasing index sums and lexicographically for tuples with the same index sum. The i-th k-tuple $0 \leq i$, shall be indicated by $(x_{i_1}, \ldots, x_{i_k})$.

Definition 1. If F and F' are exactly alike except that each occurrence

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of a variable in F is replaced by an occurrence of another (possibly the same) variable in F', then F and F' are called *wild variants* of one another. If each variable occurring in F is replaced in each of its occurrences in F by the same variable in F', then F and F' are called *variants* of one another. If $F(x_1, \ldots, x_k; y_1, \ldots, y_l)$, the indicated variables being the only variables occurring in F, has the variant $F'(x_{i_1}, \ldots, x_{i_k}; x_{(i-1)l+1}, x_{(i-1)l+2}, \ldots, x_{il})$, the replacement having been made in the manner indicated, then F' is called the *i-th k-l-variant of F*.

Definition 2. Let $M = M_0 \vee M_1 \vee \ldots \vee M_n$ be a quantifier-free tautology whose disjuncts are wild variants of one-another and such that the variables occurring in M_0 are $x_0, \ldots x_l$. If there is some k, $k \leq n$ such that for all i, $1 \leq i \leq k$ we have:

(i) Some occurrences of x_0 in M_0 are replaced by x_1 in M_i, the other occurrences of x_0 in M_0 remaining unaltered in M_i,
(ii) each occurrence of x_0 in M_0 is replaced by an occurrence of x_1 in one and only one M_i,

then M is called a *pre-G-disjunction*. The number k is called the E-*length* of M and l is called the A-*length* of M.

Definition 3. Let M be a pre-G-disjunction, we define the *matrix derived from M* to be the formula F constructed as follows:

(i) F_0 is M_0, with x_i, $1 \leq i \leq l$, replaced by y_i in all of its occurrences;
(ii) for $1 \leq i \leq k$, F_i is F_{i-1} with those occurrences of x_0 in F_{i-1} which are replaced by x_1 in M_i, replaced by $x_{k-(i-1)}$;

and finally, F is defined to be F_k.

Definition 4. Let M be a pre-G-disjunction and let $F(x_1, \ldots, x_k; y_1, \ldots, y_l)$ be the matrix derived from M. If for each i, $0 \leq i \leq n$, M_i is the i-th k-l-variant of F, then M is called a *G-disjunction*.

Remark: We note that it is obvious from the definitions above that there are effective procedures for determining whether or not a quantifier-free tautology is a pre-G-disjunction, for obtaining the matrix derived from the disjunction and for deciding whether or not the disjunction is a G-disjunction.

We now describe the systems E_1, E_2, and E_3 and prove (or indicate the proofs of) some theorems about them. Note that the axioms and rules of inference of our systems are recursive in each case.

Axioms of E_1:
The axioms of E_1 are the quantifier free tautologies.

Rule of inference of E_1:
(R1) If M is a G-disjunction, and $F(x_1, \ldots, x_k; y_1, \ldots, y_l)$ is the matrix derived from M, then $\vdash \mathrm{E}x_1 \ldots \mathrm{E}x_k \mathrm{A}y_1 \ldots \mathrm{A}y_l F(x_1 \ldots x_k; y_1, \ldots, y_l)$.

Completeness Theorem 1. *Every logically valid closed* E-A *formula is derivable from an axiom by one use of the rule of inference* **(R1)**.

Proof: In [3] it is shown that every logically valid closed E-A formula has a matrix which expands, by a certain substitution procedure on its variables, into a G-disjunction. Our rule **(R1)** simply allows the immediate inference from the G-disjunction of a formula to the formula.

3. *The System* E_2. The system E_2 yields the prenex normal forms of the logically valid formulas of the predicate calculus. We obtain E_2 by adding the following rule of inference to E_1:

(R2) If $\vdash G$ where G has been derived from an axiom by means of **(R1)**, then $\vdash H$ where H is any prenex formula of which G is its Skolem normal form.

Completeness Theorem 2. *Every logically valid closed prenex formula of the first order predicate calculus is provable in* E_2.

Proof: It is well known that there is an effective process for reducing a formula to its Skolem normal form and that a formula is logically valid if and only if its Skolem normal form is. The result thus follows from completeness theorem 1.

4. *The system* E_3. The system E_3 yields the logically valid closed formulas of the predicate calculus. To obtain E_3 we add the following group of rules to E_2.

(R3) These rules are just the usual rules for moving quantifiers in and out of formulas.

Completeness Theorem 3. *Every logically valid closed formula of the first order predicate calculus is derivable in* E_3.

Proof: Every logically valid formula is equivalent to its prenex normal form which is derivable in E_3 by completeness theorem 2. The rules of **(R3)** yield A from the prenex form of A.

5. *Concluding Remarks.*

(1) The preceding systems possess certain features which differ from the usual formulations of the first order predicate calculus. It has long been a tradition, in formulating axiom systems, to utilize axioms and rules which can be called "simple" or "elementary" in some sense. The preceding systems do not possess these characteristics. These systems are not put forward, however, as having any practical value, but solely for the purpose of demonstrating that Gödel's proof of completeness is even more akin to Herbrand's work than had previously been thought. That the relationship between their theorems is not immediately obvious stems from (a) their differing conceptions of what constituted a meaningful metamathematical question (as pointed out by Dreben [2]), (b) the differences in the technique employed by each to prove his theorem, and (c) the difficulty, which persisted until only recently, that logicians encountered in attempting to follow Herbrand's arguments.

(2) Certain other features of the systems E_1 - E_3 should be pointed out, for we can draw even stronger conclusions than have been indicated

above. First of all we have the result that each system possesses a normal form for proofs. In fact, any proof in the systems E_1 and E_2 must be given in this normal form. This feature, combined with the reversibility of the rules of inference yields results similar to the Herbrand-Gentzen results on the eliminability of "modus-ponens" and "cut" rules, thus yielding a proof-search procedure.

(3) We could, of course, have considered the system whose axioms are the quantifier-free tautologies and which possesses the one rule of inference:

(R4) If C is a closed formula whose Skolem normal form "expands" to a G-disjunction A, then $\vdash C$.

Here "expands" refers to the method of expansion employed in the proof of Gödel's theorem in [3]. In this system every logically valid closed formula of the predicate calculus is derivable from an axiom by means of a proof which employs the single rule of inference **(R4)** exactly once.

(4) There is some slight similarity here between this treatment of the predicate calculus and the treatments of the propositional and many-valued propositional calculi of [4] and [5].

(5) The systems of this paper have been constructed by condensing what would ordinarily be long sequences of steps into one step. This kind of reduction often simplifies metamathematical considerations. Thus the simplicity of the above systems enables us to see them as similar to Herbrand's and thus the possibility of the further reformalization of quantification theory on the basis of Gödel's proof of completeness to yield Herbrand's systems.

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