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A NOTE ON IMMUNE SETS

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In [1] the following sequence of theorems were proved, which were clearly inspired by the fact that an infinite set is immune iff it has no infinite recursively enumerable subset.* (Throughout this paper "set" means "subset of $\mathfrak{N}$" where $\mathfrak{N}$ denotes the set of non-negative integers.)

Theorem 1: *Let A be an infinite set. Then*:

(a) *A is hyperimmune iff it has no strongly finitely almost recursively enumerable infinite subset*
(b) *A is hyperhyperimmune iff it has no finitely almost recursively enumerable infinite subset*
(c) *A is strongly hyperhyperimmune iff it has no almost recursively enumerable infinite subset.*

Definitions of these concepts may all be found in [1]; however we recall the most important of them here. An injective total function on $\mathfrak{N}$ is called *almost recursive* if its inverse has a partial recursive extension. A set is called almost recursively enumerable (hereafter abbreviated a.r.e.) if it is finite or the range of an almost recursive function; and is called almost recursive (abbreviated a.r.) if it is finite or the range of a strictly increasing almost recursive function. If $f(x)$ is a recursive function, the sequence of sets

$$\omega_{f(0)}, \omega_{f(1)}, \ldots$$

is called a disjoint array if $\bigwedge_i \bigwedge_j [i \neq j \rightarrow \omega_{f(i)} \cap \omega_{f(j)} = \phi]$ and $\bigwedge_i [\omega_{f(i)} \neq \phi]$. It is called a finite disjoint array if in addition: $\bigwedge_i [\omega_{f(i)}$ is a finite set].

We shall use an effective enumeration of all finite sets denoted by

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$\mathrm{D}(x)$: $\mathrm{D}(0) = \phi$, and if $x_1 > x_2 > \ldots > x_n$ and $x = 2^{x_1} + 2^{x_2} + \ldots + 2^{x_n}$ then $\mathrm{D}(x) = \{x_1, \ldots x_n\}$.

If $f(x)$ is a recursive function then the sequence of finite sets

$$\mathrm{D}(f(0)),\ \mathrm{D}(f(1)),\ \ldots$$

is called a discrete array if $\bigwedge_i \bigwedge_j [i \neq j \rightarrow \mathrm{D}(f(i)) \cap \mathrm{D}(f(j)) = \phi]$ and $\bigwedge_i [\mathrm{D}(f(i)) \neq \phi]$.

In [1] we proved *that an infinite set A is* a.r.e. *iff there is a disjoint array* $\{\omega_{f(x)}\}$ *such that*

$$A \subset \bigcup_{x \in \mathfrak{N}} \omega_{f(x)} \tag{1}$$

and

$$\bigwedge_x [\,|\omega_{f(x)} \cap A| = 1] \tag{2}$$

(where for any set B, $|B|$ denotes the cardinality of B). By analogy we define a set A to be

i) finitely almost recursively enumerable if it is finite or if there is a finite disjoint array satisfying (1) and (2)

ii) strongly finitely almost recursively enumerable if it is finite or if there is a discrete array $\{\mathrm{D}(f(x))\}$ satisfying

$$A \subset \bigcup_{x \in \mathfrak{N}} \mathrm{D}(f(x)) \tag{3}$$

and

$$\bigwedge_x [\,|\mathrm{D}(f(x)) \cap A| = 1] \tag{4}$$

Note that a set A is recursively enumerable iff it is finite or if there is a discrete array $\{\mathrm{D}(f(x))\}$ satisfying (3), (4) and also $\bigwedge_x [\,|\mathrm{D}(f(x))| = 1]$. So we have here a sort of hierarchy of constructivity.

An examination of Theorem 1 suggests that it might be profitable to attempt to isolate a smaller subclass of the immune sets beyond strong hyperhyperimmunity by attempting to copy the transition from immunity to hyperimmunity, replacing recursive functions by almost recursive functions. The purpose of this paper is to show that such an attempt is futile: the class of infinite sets whose principal function is majorized by no almost recursive function turns out to be so small that it is empty!

Theorem 2. *Let A be any infinite set and let its principal function be denoted by* p_A. *Then*:

i) p_A *is majorized by a strictly increasing almost recursive function*
ii) *these is an almost recursive function f such that*

$$\bigwedge_x [\mathrm{D}(f(x)) \cap A \neq \phi] \tag{5}$$

and

$$\bigwedge_x \bigwedge_y [x \neq y \rightarrow \mathrm{D}(f(x)) \cap \mathrm{D}(f(y)) = \phi] \tag{6}$$

iii) *there is an almost recursive function g such that*

$$\bigwedge_x [\omega_{g(x)} \textit{ is a finite set and } \omega_{g(x)} \cap A \neq \phi] \tag{7}$$

and

$$\bigwedge_x \bigwedge_y [x \neq y \rightarrow \omega_{g(x)} \cap \omega_{g(y)} = \phi] \tag{8}$$

Proof: We first show that (i) $\rightarrow$ (ii) $\rightarrow$ (iii), and then that any set A has property (i).

((i) $\rightarrow$ (ii)): Suppose that p_A is majorized by a strictly increasing a.r. function h. We can assume without loss of generality that $\mathrm{p}_A(h(n)+1) \leq h(n+1)$ for all n. For if not we can replace h by h', defined by $h'(0) = h(0)$ and $h'(n+1) = h(h'(n)+1)$, which has this property, and is clearly almost recursive if h is. Now consider the array of disjoint finite sets A_n defined by $A_0 = \{0, 1, \ldots, h(0)\}$ and $A_{n+1} = \{h(n)+1, h(n)+2, \ldots, h(n+1)\}$. Define a function f by the condition that $A_n = \mathrm{D}(f(n))$. We shall show that f is almost recursive and satisfies (ii). First of all, we clearly have $n \neq m \rightarrow A_n \cap A_m = \phi$. Since $h(0) \geq \mathrm{p}_A(0)$ it follows that $A \cap A_0 \neq \phi$. The largest element of A_{n+1} is $h(n+1)$ and $h(n+1) \geq \mathrm{p}_A(h(n)+1)$. Therefore at least $h(n)+2$ elements of A are $\leq h(n+1)$. But $A_0 \cup A_1 \cup \ldots \cup A_n$ contains only $h(n)+1$ elements in all. Hence $A \cap A_{n+1} \neq \phi$. To see that f is almost recursive, let s be a recursive function such that $s(x)$ = the largest element of $\mathrm{D}(x)$. Let K be a partial recursive extension of h^{-1}. Define a partial recursive function G by $\mathrm{G}(y) \simeq \mathrm{K}(s(y))$. G is clearly a partial recursive extension of f^{-1}.

((ii) $\rightarrow$ (iii)): Let f be an almost recursive function satisfying (ii). Let h be a recursive function such that $\mathrm{D}(x) = \omega_{h(x)}$ for every x. Define g by $g(x) = h(f(x))$. It is clear that g is an almost recursive function which satisfies (iii).

Now we show that any infinite set A has property (i). Let j denote the standard pairing function $\mathrm{j}(x, y) = \frac{1}{2}[(x+y)^2 + 3y + x]$ and let $k(x)$, $l(x)$ be recursive functions which satisfy $x = \mathrm{j}(k(x), l(x))$ for all x, and $k(\mathrm{j}(x, y)) = x$, $l(\mathrm{j}(x, y)) = y$ for all x and y. Define a function g by setting $g(x) = \mathrm{j}(\mathrm{p}_A(x), x)$. Then g is an almost recursive function which majorizes p_A, since $\mathrm{j}(x, y) \geq x$ for all x. Also $l(\mathrm{j}(\mathrm{p}_A(x), x)) = x$ for all x so l is a partial recursive extension of g^{-1}. We can convert g to a strictly increasing almost recursive function by setting (as in the first part of the proof) $h(0) = g(0)$ and $h(n+1) = g(h(n)+1)$. Then h is clearly a strictly increasing almost recursive function which majorizes p_A.

REFERENCES

[1] Berry, John W., *Almost Recursively Enumerable Sets* (To appear).

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