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THE NAME SOLID AS PRIMITIVE IN PROJECTIVE GEOMETRY

THEODORE F. SULLIVAN

Leśniewski, in a logical system he called Mereology ([3], [4]) has given axioms to characterize the relation A is part of B where A and B are understood to be names, e.g. arm is part of body. These axioms, which characterize the collective notion of class, lead to the idea of using the name solid as primitive in geometry as opposed to point. In [6], Tarski has shown that axioms for Euclidean Geometry may be given in Mereology (with the name solid added) if solid were interpreted as an open (solid) sphere. A similar result was obtained for Affine Geometry (see [5]) where solid is interpreted as parallelepiped. We shall now consider Projective Geometry.

Let $\mathfrak{P}^*$ be an axiom system for Plane Projective Geometry which includes axioms of separation (order) and the axiom of Pappus. Such an axiom system may be obtained from [1] and [2] where the primitives are: point, a quaternary relation $\mathsf{T}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ which holds if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are vertices of a non-degenerate quadrilateral, and a quaternary relation $\mathsf{D}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ which holds if points $\mathbf{a}$ and $\mathbf{b}$ separate points $\mathbf{c}$ and $\mathbf{d}$ ($\overleftrightarrow{\mathbf{acbd}}$). By defining the primitives of $\mathfrak{P}^*$ within Mereology (extended by adding the name solid) we may form an axiom system $\mathfrak{M}^*$ whose primitive is solid. If the name solid is interpreted in $\mathfrak{P}^*$ as a convex quadilateral (formal definition below) then $\mathfrak{P}^*$ will be equivalent to $\mathfrak{M}^*$ and the proofs go roughly as in [5] if it is recalled that convex subsets of projective spaces are affine spaces ([7]).

We now present the formal definitions for interpreting solid and defining the primitives of $\mathfrak{P}^*$ and $\mathfrak{M}^*$. Our notation is that of Peano-Russell with dots separating conjuncts instead of parentheses. The following example should explain the notation : $[A, B] :: \mathsf{EXT}(A, B) .\dot{:}. \equiv .\dot{:}. \mathscr{P}(A) . \mathscr{P}(B) : [C] : \mathscr{P}(C) . C \,\varepsilon\, \mathbf{el}(A) . \supset . \sim (C \,\varepsilon\, \mathbf{el}(B))$. This is read "For all A and B, A and B are EXTernal iff A is a solid ($\mathscr{P}(A)$) and B is a solid and for all C if C is a solid and C is part of (or equal to) A ("ε **el**") then not C is part of (or equal to) B.

Solid is defined in $\mathscr{P}^*$ as follows (Fig. 1):

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$[Q, \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}] \therefore \mathcal{S}(Q, \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}) . \equiv : \mathsf{e} =$ line $\mathsf{ab} \cap$ line cd. $\mathsf{f} =$ line $\mathsf{ad} \cap$ line bc: $Q = \{\mathsf{x} | \mathrm{D}(\mathsf{e}, \mathsf{x}; \mathsf{a}, \mathsf{b}) . \vee . \mathrm{D}(\mathsf{e}, \mathsf{x}; \mathsf{c}, \mathsf{d}) . \vee . \mathrm{D}(\mathsf{f}, \mathsf{x}; \mathsf{a}, \mathsf{d}) . \vee . \mathrm{D}(\mathsf{f}, \mathsf{x}; \mathsf{b}, \mathsf{c}) . \vee . [\exists \mathsf{uv}] . \mathrm{D}(\mathsf{f}, \mathsf{u}; \mathsf{c}, \mathsf{b}) . \mathrm{D}(\mathsf{f}, \mathsf{v}; \mathsf{a}, \mathsf{d}) . \mathrm{D}(\mathsf{e}, \mathsf{x}; \mathsf{u}, \mathsf{v})\} \cup \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$

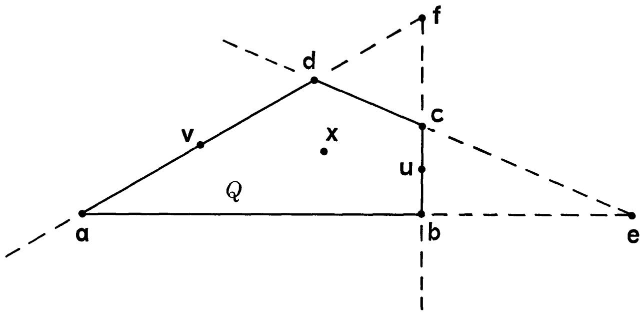

Fig. 1

The primitives of $\mathcal{P}^*$ are defined as follows:

1) point—a is a point ($\mathrm{pt}(a)$) is defined as an equivalence class of solids all having the same center. The formal development is given in [5] pp. 2-5 but with the addition of the following conjunct to *DM3* : $[\exists C] . A \varepsilon \mathbf{el}(C) . B \varepsilon \mathbf{el}(C)$

2) $[a, b, c, d] :: \mathrm{T}(a, b, c, d) . \equiv :\cdot: \mathrm{pt}(a) . \mathrm{pt}(b) . \mathrm{pt}(c) . \mathrm{pt}(d) :: [\exists U] :: \mathcal{P}(U) \therefore [x, y, z] \therefore x = a . \vee . x = b . \vee . x = c . \vee . x = d : y = a . \vee . y = b . \vee . y = c . \vee . y = d : z = a . \vee . z = b . \vee . z = c . \vee . z = d : x \neq y . x \neq z : y \neq z : \supset . [\exists V, X, Y, Z] . \mathcal{P}(V) . X \varepsilon x . Y \varepsilon y . Z \varepsilon z . X \varepsilon \mathbf{el}(V) . Y \varepsilon \mathbf{el}(V) . V \varepsilon \mathbf{el}(U) . Z \varepsilon \mathbf{el}(U) . \mathrm{EXT}(V, Z)$

3) $[a, b, c, d] :\cdot: \mathrm{D}(a, b; c, d) . \equiv :: \mathrm{pt}(a) . \mathrm{pt}(b) . \mathrm{pt}(c) . \mathrm{pt}(d) . [\exists RTABCD] . A \varepsilon a . B \varepsilon b . C \varepsilon c . D \varepsilon d . \mathcal{P}(R) . \mathcal{P}(T) . A \varepsilon \mathbf{el}(R) . B \varepsilon \mathbf{el}(R) . C \varepsilon \mathbf{el}(R) . A \varepsilon \mathbf{el}(T) . C \varepsilon \mathbf{el}(T) . D \varepsilon \mathbf{el}(T) . \mathrm{EXT}(D, R) . \mathrm{EXT}(B, T) :: [U] :: \mathcal{P}(U) . [A', C'] . A' \varepsilon a . C' \varepsilon c . A' \varepsilon \mathbf{el}(U) . C' \varepsilon \mathbf{el}(U) : \supset \therefore [V] : V \varepsilon b . \supset . \sim \mathrm{EXT}(V, U) : \vee : [V] : V \varepsilon d . \supset . \sim \mathrm{EXT}(V, U)$

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University of South Carolina
Columbia, South Carolina