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THE CARROLLIAN MATRIX

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Introduction An important task of logic is the testing of arguments; that is, the assertion that a certain statement (called the conclusion) will follow from other statements (called premises). Following [3], p. 42, an argument is said to be valid if and only if the conjunction of the premises implies the conclusion. Therefore, the logician is concerned with two ingredients: A set of premises, and the set of valid conclusions which can be reached. The former is usually given, whereas the set of valid conclusions is not obvious. For example, "If it rains, then the ground will be wet," and, "It rains," have eight valid conclusions. They are:

It will rain *or* it will not rain.
It will rain *or* the ground will be wet.
It will not rain *or* the ground will be wet.
It will rain *and* the ground will be wet.
It will rain *or* the ground will not be wet.
It will rain *iff* the ground will be wet.
It will rain.
The ground will be wet.

Also, there are eight invalid conclusions from this particular set of premises. They are:

It will rain *and* it will not rain.
It will not rain *or* the ground will not be wet.
It will not rain *and* the ground will be wet.
It will not rain *iff* the ground will be wet.
It will not rain *and* the ground will not be wet.
It will rain *and* the ground will not be wet.
It will not rain.
The ground will not be wet.

And, as the number of premises increase, so do the number of valid and and invalid conclusions.

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Purpose of paper We establish here a theorem which determines, enumerates and lists, all of the valid conclusions for any given finite set of premises. In addition, we offer two corollaries; the first defines the invalid conclusions for any given finite set of premises in the same manner as the theorem, and the second shows that all the conclusions of contradictory premises are valid.

Methodology of paper Our tool is the algebra of symbolic logic developed by George Boole,[1] as refined by Edward V. Huntington, [2] and adapted by this writer. Huntington's fundamental concepts are: (1) a class (K) of elements ($a, b, c, \ldots$); (2) two binary operations denoted by: (a) '$\oplus$'[2] called the logical sum (read as "plus"),[2] (b) '$\odot$' called the logical product (read as "times"); and (3) a dyadic relation[3] denoted by '$\otimes$' called inclusion (read "is a subset of"); and fulfilling the following set of postulates:

- Ia) $a \oplus b$ is in the class whenever a and b are in the class.
- Ib) $a \odot b$ is in the class whenever a and b are in the class.
- IIa) There is an element $\emptyset$ such that $a \oplus \emptyset = a$ for every element a.
- IIb) There is an element I such that $a \odot \mathsf{I} = a$ for every element a.
- IIIa) $a \oplus b = b \oplus a$ whenever a, b, $a \oplus b$, and $b \oplus a$ are in the class.
- IIIb) $a \odot b = b \odot a$ whenever a, b, $a \odot b$, and $b \odot a$ are in the class.
- IVa) $a \oplus (b \odot c) = (a \oplus b) \odot (a \oplus c)$ whenever a, b, c, $a \oplus b$, $a \oplus c$, $b \odot c$, $a \oplus (b \odot c)$, and $(a \oplus b) \odot (a \oplus c)$ are in the class.
- IVb) $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ whenever a, b, c, $a \odot b$, $a \odot c$, $b \oplus c$, $a \odot (b \oplus c)$, and $(a \odot b) \oplus (a \odot c)$ are in the class.
- V) If the elements $\emptyset$ and I in postulates IIa) and IIb) exist and are unique, then for every element a there is an element $\overline{a}$ such that $a \oplus \overline{a} = \mathsf{I}$ and $a \odot \overline{a} = \emptyset$.
- VI) There are at least two elements x and y in the class such that $x \neq y$.[4]

A Boolean Algebra "susceptible only of the values 0 and 1 can be used to show consistency of the postulates."[5] Here, the system $(K, \oplus, \odot)$ is defined as follows:

$K = \{0, 1\}$, with $\oplus$ and $\odot$ defined by the tables[6]

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

For the purpose of proving our theorem, we adapt this Boolean Algebra of two elements (0 and 1) to the notion of a matrix.

The proof Let C be an ordered set of $2n$ quantities a_{ij} arranged in a rectangular array of 2 rows and n columns. We call C a Carrollian Matrix (after Lewis Carroll[7]), hereafter referred to as the matrix.

Definitions

D1. $C = [a]_n^2 = \begin{bmatrix} a_{11}\, a_{12} \ldots a_{1n} \\ a_{21}\, a_{22} \ldots a_{2n} \end{bmatrix}$, $a_{ij} \in C$ such that $a_{ij} = 0 \vee a_{ij} = 1$, $0 \neq 1$

D2. These matrices, a form of truth table, are derived as follows:

(a) For two variables, p and q,

p	q
1	1
1	0
0	1
0	0

in matrix form as

$$p = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$q = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) For three variables, p, q, and r,

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

in matrix form as

$$p = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$r = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) For m variables, $p, q, r, \ldots, m$

m
a_{11}
a_{21}
a_{12}
a_{22}
.
.
.
a_{1n}
a_{2n}

in matrix form as

$$m = \begin{bmatrix} a_{11}\, a_{12} \ldots a_1 \\ a_{21}\, a_{22} \ldots a_2 \end{bmatrix}$$

Where the number of elements, $a = 2^m$; and the number of columns, $n = 2^{m-1}$.

D3. Two matrices having the same number of elements are said to be *compatible*. It will be assumed that operations will be performed on compatible matrices.

D4. $[0]^2_n = \emptyset$ (read as "the empty set"), and
$[1]^2_n = \mathbf{I}$ (read as "the universe")

D5. *Equality* (=)

$[a]^2_n = [b]^2_n$ if and only if $a_{ij} = b_{ij}$

D6. *Inclusion* ($\circledcirc$)

$[a]_n^2 \circledcirc [b]_n^2$ if and only if $a_{ij} \leq b_{ij}$[8]

To illustrate: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \circledcirc \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ whereas $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \not\circledcirc \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

D7. Binary operation, *Logical Sum* ($\oplus$)

$\oplus$	0	1
0	0	1
1	1	1

To illustrate: $p \oplus q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

D8. Binary operation, *Logical Product* ($\odot$)[9]

$\odot$	0	1
0	0	0
1	0	1

To illustrate: $p \odot q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

D9. Unary operation, *Negation* (-) (read as "the complement of")[10]

$$\left\{\begin{matrix} a_{ij} = 0 \text{ if and only if } \overline{a}_{ij} = 1 \\ a_{ij} = 1 \text{ if and only if } \overline{a}_{ij} = 0 \end{matrix}\right\}$$

To illustrate: $\overline{p} = \overline{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

D10. Binary operation, *Logical Implication* ($\Longrightarrow$) (read as "implies")[11]

$\Longrightarrow$	0	1
0	1	1
1	0	1

$$\left\{\begin{matrix} (a_{ij}, b_{ij}) \text{ such that } a_{ij} = 1 \wedge b_{ij} = 0 \text{ if and only if } a_{ij} \Longrightarrow b_{ij} = 0 \\ \text{otherwise} \qquad a_{ij} \Longrightarrow b_{ij} = 1 \end{matrix}\right\}$$

To illustrate: $p \Longrightarrow q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

D11. Binary operation, *Logical Equivalence* ($\Longleftrightarrow$) (read as "is equivalent to")[12] (also known as "if and only if")

$\Longleftrightarrow$	0	1
0	1	0
1	0	1

$$\left\{\begin{matrix} a_{ij} = b_{ij} \text{ if and only if } a_{ij} \Longleftrightarrow b_{ij} = 1 \\ \text{otherwise} \qquad a_{ij} \Longleftrightarrow b_{ij} = 0 \end{matrix}\right\}$$

To illustrate: $p \Longleftrightarrow q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

D12. A matrix will be called *reduced* when all possible operations have been performed. It will be assumed that all matrices under discussion have been reduced.

D13. Given any number of premises, a logical conclusion will be said to be *valid* "if and only if the conjunction of the premises implies the conclusion." That is, a logical conclusion is said to be *valid* if and only if the logical product of the matrices of the premises is a subset of the matrix of the conclusion.

To illustrate: If the premises are $p \Rightarrow q$ and q, then p is not a valid conclusion.

$$(p \Rightarrow q) \odot q = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \not\mathbin{\textcircled{$<$}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = p$$

Whereas if the premises are $p \Rightarrow q$ and p, then q is a valid conclusion.

$$(p \Rightarrow q) \odot p = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \odot \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbin{\textcircled{$<$}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = q$$

In fact, all of the supersets of $(p \Rightarrow q) \odot p$ are also valid conclusions, as follows:

$$(p \Rightarrow q) \odot p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbin{\textcircled{$<$}} \left\{\begin{array}{l}
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \mathsf{I} = p \oplus \bar{p} \\
\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = p \oplus q \\
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \bar{p} \oplus q = p \Rightarrow q \\
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = p \odot q = (p \Rightarrow q) \odot p \\
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = p \oplus \bar{q} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = p \Leftrightarrow q = \bar{p} \Leftrightarrow \bar{q} \\
\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = p \\
\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = q
\end{array}\right.$$

D14. The *power set* is the set of all the supersets of a given matrix.

Lemma *The number of elements in a power set equals* 2^z *where "z" is the number of "0's" in the given matrix.*

To illustrate: Given the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the number of elements in this power set equals $2^3 = 8$ (as is seen above).

Proof: By inclusion definition,

$$[a]_n^2 \circledcirc [b]_n^2 \text{ if and only if } a_{ij} \leq b_{ij}$$

Inasmuch as for each $a_{ij} = 0$, there are two choices for each b_{ij} (i.e., either $b_{ij} = 0$ or $b_{ij} = 1$), for z "0's" there are 2^z choices.[13]

Theorem *The number of valid conclusions, stemming from any given set of premises, is equal to the number of elements in the power set of the matrix which is determined by the logical product of the matrices of the premises. In fact, there are a finite number (from a finite number of premises) and they can be identified.*

To illustrate: The premises $p \Rightarrow q$ and p have the logical product $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. As illustrated above, there are 8 elements in the power set (i.e., supersets). Therefore, these premises can produce 8 valid conclusions. These 8 conclusions are found above.

Proof: By definition 13 (valid conclusions) and the Lemma.

Corollary A *The number of invalid conclusions, stemming from any given set of premises is equal to* $2^m - 2^z$ *(where* $2^m = a$*, the number of elements in a given matrix, see definition* 2(c)*; and* 2^z *is the number of elements in the power set of the given matrix). These, too, are finite for a finite number of premises and can be identified.*

To illustrate: The premises $p \Rightarrow q$ and p have the logical product $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and have $2^4 - 2^3 = 8$ invalid conclusions. They are:

$$(p \Rightarrow q) \odot p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \not\circledcirc \left\{ \begin{array}{l} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \emptyset = p \odot \bar{p} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \bar{p} \oplus \bar{q} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \bar{p} \odot q \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \bar{p} \Leftrightarrow q = p \Leftrightarrow \bar{q} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \bar{p} \odot \bar{q} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = p \odot \bar{q} \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \bar{p} \\ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \bar{q} \end{array} \right.$$

Corollary B *If the premises are contradictory (i.e.,* $[a]_n^2 \odot [b]_n^2 \odot \ldots \odot [x]_n^2 = [0]_n^2 = \emptyset$), *then all of the* $2^m = 2^z$ *conclusions are valid.*

NOTES

1. "George Boole was really the second founder (to Augustus DeMorgan) of symbolic logic," p. 1864 of [4].

2. Attributed to Leibniz by Huntington, [2], p. 292.

3. "A dyadic relation, R, in the given class, is determined when, if any two elements a and b are given in a definite order, we can decide whether a stands in the relation R to b or not; . . ." Huntington, [2], p. 289.

4. Huntington, [2], pp. 292-293, except that Huntington's symbols $\wedge$ and $\vee$ (attributed to Peano) are replaced by $\emptyset$ and I, respectively.

5. George Boole, as quoted by Clarence Irving Lewis and Cooper Harold Langford ("History of Symbolic Logic") in Newman, [4], p. 1867. Hereafter this is noted as *Lewis.*

6. Huntington, [2], p. 293. Note that Lewis, *loc. cit.*, states that $1 \oplus 1 = 1$ is "The distinctive law of the system. . ."

7. Although matrices using only two elements, 0 and 1, are usually called Boolean Matrices, we feel that the matrix used here functions in an unique manner. Lewis Carroll (the Reverend Charles Lutwidge Dodgson) had his Alice saying to Humpty Dumpty:

   "That's a great deal to make one word mean,"
   Alice said in a thoughtful tone.
   "When I make a word do a lot of work like that,"
   said Humpty Dumpty, "I always pay it extra."

   We pay our matrix extra by calling it Carrollian.

8. Inclusion may be defined in terms of logical sum ($\oplus$), logical product ($\odot$), $\emptyset$, and I as follows: "If $a \oplus b = b$; or, if $a \odot b = a$; or, if $\overline{a} \oplus b = \mathsf{I}$; or, if $a \odot \overline{b} = \emptyset$; then we write $a \otimes b$ (or $b \otimes a$))." Huntington, [2], p. 294. Note that if $b \otimes a$, we call b a *superset* of a.

9. Note that $\oplus$ and $\odot$ may be defined in terms of each other. That is, $a \oplus b = -(\overline{a} \odot \overline{b})$ and $a \odot b = -(\overline{a} \oplus \overline{b})$.

10. It is interesting to note that Huntington calls the element $\overline{a}$, ". . . non-a, or the supplement of a." Huntington, [2].

11. Although $\Longrightarrow$ can be defined in terms of $\oplus$ and $-$, we prefer it to be used here as a binary operation; that is, $p \Longrightarrow q = \overline{p} \oplus q$.

12. Although $\Longleftrightarrow$ can be defined in terms of $\Longrightarrow$ and $\odot$, we prefer it to be used here as a binary operation; that is, $p \Longleftrightarrow q = (p \Longrightarrow q) \odot (q \Longrightarrow p)$.

13. The principle of Sequential Counting. See p. 38 of [1].

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