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A NOTE ON THE INTUITIONIST FAN THEOREM

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The purpose of this note is to point out that the Intuitionist Fan Theorem as stated in the literature mentioned is classically false. A classical counterexample to the Theorem is given. It is pointed out that the modified Fan Theorem does not give rise to a classical contradiction mentioned in Heyting [5].

The usual statement of the Fan Theorem* is, following [5],

If S is an Intuitionist fan and φ an integer valued function defined for every element δ of S then a natural number N can be computed for $\langle S, \varphi\rangle$ such that for any element δ of S, $\varphi(\delta)$ is determined by the first N components of δ.

We refer to this as the weak theorem. The counter example we presently introduce leads us to modify the above statement to the strong theorem,

If S is an Intuitionist fan and φ an integer valued function defined for every element δ of S such that $\varphi(\delta)$ is determined by a finite number of components of δ, then a natural number N can be computed such that for any element δ of S, $\varphi(\delta)$ is determined by the first N components of δ.

Consider the fan S whose elements are infinitely proceeding sequences (ips) $\{d_n\}$. The spread law SL is as follows:

(i) 0 *is an admissible* 1-*sequence*,
(ii) 0, 1 *are admissible* n-*components for any* $n \geq 2$, *that is, given an admissible* $(n-1)$-*sequence* $d_1, \ldots, d_{n-1}$, *we may choose* $d_n = 0$ *or* $d_n = 1$ *subject to*
(iii) *if* $d_{n-1} = 1$ *then* $d_n = 1$ *for any* $n \geq 2$.

The complementary law CL, assigns to any admissible n-sequence $d_1, d_2, \ldots, d_n$, the number d_n. Clearly S is a fan and may be represented by the following tree:

*See: [1] p. 430; [2], p. 462; [3], p. 143; [4], p. 15; and [5], p. 42.

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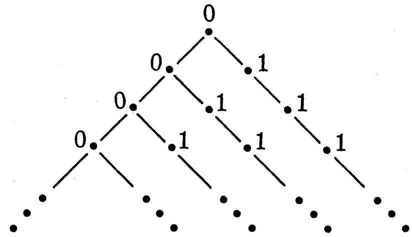

Any ips $\{d_n\}$ of S is obtained by tracing a descending path in the tree. Define the integer valued function f on S as follows:

$$f(\{d_n\}) = \begin{cases} 1, \text{ if } \{d_n\} = \{0\} \\ m + 1, \textit{ otherwise}, \end{cases}$$

where m is the number of zero components in $\{d_n\}$. Clearly f is defined for every element of S but a number N cannot be stated such that $f(\{d_n\})$ is determined by the first N components of $\{d_n\}$.

This example $\langle S, f\rangle$ is a counterexample to the weak statement of the Fan Theorem in [1-5]. (In this case the proof in [5] cannot assert the F-sequences 0; 0, 0; 0, 0, 0; etc. to be K-barred by C.) The proof in [5] is a proof of the strong statement by invoking *Brouwer's principle*: $\varphi(\delta)$ must be effectively determined by a finite number of components of δ.

Heyting [5], pp. 104-105, wishes to show that

$$(\forall x) \sim\sim p(x) \rightarrow \sim\sim (\forall x)\, p(x)$$

is not an intuitionist thesis by exhibiting the following counterexample to the thesis. Consider the fan and function $\langle S, f\rangle$ described above. Let x be any element of S and $p(x)$ the proposition "f assigns a number to x." The thesis $(\forall x) \sim\sim p(x)$ is easily established by assuming $(\exists x) \sim p(x)$, deriving a contradiction, hence $\sim(\exists x) \sim p(x)$ and consequently $(\forall x) \sim\sim p(x)$, because

$$\sim(\exists x) \sim p(x) \rightarrow (\forall x) \sim\sim p(x)$$

is an Intuitionist thesis (see [5], p. 103). In order to show that $\sim\sim(\forall x)\, p(x)$ does not hold he derives $\sim(\forall x)\, p(x)$ by assuming $(\forall x)\, p(x)$, and then applying the Fan Theorem and deriving the contradiction we have already pointed out. However, the Fan Theorem cannot be invoked on the weak assumption $(\forall x)\, p(x)$. If $q(x)$ is the proposition "f assigns a number to x on the basis of a finite number of components of x", then $(\forall x)\, q(x)$ is sufficient to invoke the Fan Theorem to prove $\sim(\forall x)\, q(x)$; but then $(\forall x) \sim\sim q(x)$ does not hold as $\sim q(\{0\})$ is true. Consequently neither of the (classical) contradictions

$$(\forall x) \sim\sim p(x)\ \&\ \sim(\forall x)\, p(x) \text{ or } (\forall x) \sim\sim q(x)\ \&\ \sim(\forall x)\, q(x)$$

arise.

I am indebted to Dr. Konrad Schultz of Berlin, Germany, for kindly telling me of my mistake in not taking Brouwer's principle into account in an earlier version of this note.

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