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LOGICAL AND HISTORICAL REMARKS ON SACCHERI'S GEOMETRY

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Resumen

El autor se ocupa de la geometría contenida en el libro *Euclides ab omni naevo vindicatus*, 1733, de Girolamo Saccheri. Después de plantear el problema en §1, el autor analiza en §2 los paralogismos cometidos por Saccheri, cuando éste rechaza las geometrías del ángulo obtuso (elíptica) y del ángulo agudo (hiperbólica). En §3 expone el método de Saccheri, basado principalmente en la ley de Clavius, $(\sim p \supset p) \supset p$, y muestra su influencia y sus profundas consecuencias en la evolución del concepto de geometría en Saccheri, Lambert, Taurinus y Gauss. Finalmente el autor en §4 intenta hacer comprensibles los motivos que impulsaron a Saccheri a cometer sus paralogismos, siendo el principal motivo la situación histórica y filosófica contemporánea.

§1. *Introduction*: *Saccheri's Geometry*. In this paper*, we shall be concerned with the treatise *Euclides ab omni naevo vindicatus* (1733) by Girolamo Saccheri[1][17], which is a text on *real elementary plane geometry*, and as a matter of fact is the first one without assuming Euclid's fifth postulate. Therefore we shall consider only *plane geometry*, that is an aggregate or universe of points and (straight) lines, that are related by relations of incidence and order or any other equivalent. Also we consider only *elementary* geometry, that is Riemannian geometry of constant curvature or in a more historical way geometry in which there is a very specific relation of congruence among figures. Finally we consider only *real* geometry, that is assuming the postulates of continuity of Hilbert or the equivalent one of Dedekind.

We resort to the axiomatic method for a precise formulation. The axioms of elliptic, Euclidean and hyperbolic (real, elementary) plane geometries are well established and known, and we need not list them here.

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It is enough that we refer to the works of D. Hilbert [10] and H.S.M. Coxeter [5].[2] They are divided into five groups according to Hilbert:

1) Axioms of incidence: There is an unique line through any two different points. Any line contains two points. There are three non-collinear points. These axioms are the planar ones of the group I and are the same for the three types of geometry.

2) Axioms of order or separation: For Euclidean and hyperbolic geometries one may take those of group II by Hilbert, but for elliptic geometry they must be different and one may take those listed by Coxeter (Axioms 2.121-2.126).

3) Axioms of congruence: For Euclidean and hyperbolic geometries the group III by Hilbert for one plane and for elliptic geometry substantially the same, but with slight modifications due to the facts that one has to determine which of the two possible segments is relevant and that the elliptic plane is not orientable. They can be stated in such a form as to be the same for the three types of geometries.

4) Axioms of parallelism: there is one parallel, there are two parallels, there is no parallel.

5) Axioms of continuity: The group V of Hilbert or axiom 2.13 of Coxeter. The same for the three geometries.

Euclid's geometry, as distinct from Euclidean Geometry, is the geometry contained in the *Elements* [7]. We interpret this geometry as assuming the axioms of incidence and those of congruence (axiom IV, the common notions and proposition I,4), also axiom V of Euclid and for simplicity we suppose also that it assumes the axioms of continuity. We interpret axiom II as not requiring infinite length of the line (in spite of proposition I,12) not excluding that the line be closed; and also that axiom V (or proposition I,12) does not require the two sides of a line in a plane to be different; therefore, we interpret Euclid's geometry as assuming only the elliptic axioms of order. It follows that Euclid's geometry may be elliptic geometry as well as Euclidean geometry. The parallogism of Euclid, by which the elliptic geometry is excluded, is committed in the proof of proposition I,16 as is well known.[3]

In the same way Saccheri's geometry is the geometry contained in the First Book of the *Euclides ab omni naevo vindicatus* [17].

In order to be more precise we axiomatically interpret Saccheri's geometry as assuming the same axioms as Euclid's geometry, but without the fifth axiom.[4] This relative interpretation obviously seems to be correct. Of course for an absolute interpretation there is the same vagueness as in the interpretation of Euclid's geometry, but we have already given a precise form for the common interpretation of this geometry in order that the following pages be properly understood.

Consider the quadrilateral of Saccheri $ABCD$, that has right angles at A and B and $AD = BC$. Saccheri, after proving that angle C = angle D, begins by making the only three possible and mutually exclusive hypothesis: that the angle C be obtuse, be right or be acute.

Proposition XI of Saccheri proves, under the hypothesis of the right angle, the fifth postulate of Euclid. Therefore, this proposition proves the axioms of Euclidean geometry without the fifth axiom, or more exactly, proves that the axioms of Saccheri's geometry together with the hypothesis of the right angle are equivalent to the axioms of Euclidean geometry.

Proposition XII proves, under the hypothesis of the obtuse angle, that any two lines always meet at a finite distance.[5] Therefore, this proposition establishes that the axioms of Saccheri's geometry together with the hypothesis of the obtuse angle are equivalent to the axioms of elliptic geometry.

Proposition XXV and more explicitly proposition XXXII prove rigorously, under the hypothesis of the acute angle, that through a point A outside of a straight line BX there are always two parallels (asymptotic straight lines) to BX. Therefore these propositions establish that the axioms of Saccheri's geometry together with the hypothesis of the acute angle are equivalent to the axioms of hyperbolic geometry.

Remark. One may state the relation of congruence of lines and angles as equality of measure. Consider the Saccheri quadrilateral $ABCD$ and assume that we give arbitrary positive measures to the segment AB, to the segment AD and to the angle ADC, this last one such that $0 <$ measure of angle $ADC < \pi$. In the Saccheri quadrilateral $ABCD$ it is understood that AD and BC do not intersect; and that for any $ADC > \pi/2$ the length of the straight line must be greater than $2\,\overline{AD} = 2h$ and greater than $\overline{AB} = 2p$. Then Saccheri's quadrilateral defines one and only one real elementary plane geometry either Euclidean or elliptic or hyperbolic according to the angle ADC being right, obtuse or acute, except that there is no geometry if ang $ADC - \pi/2 \geqslant \pi/2 \cdot p/h$. This statement may be easily verified by means of the trigonometrical formulae for right triangles. We see that Saccheri's axioms may lead to elliptic or Euclidean or hyperbolic geometry. But he claims to have proved that only the Euclidean geometry is true. Therefore there must be in his *Euclides* some paralogisms.

Although there already exists a vast bibliography on the theory of parallels in general and on Saccheri in particular[6], I want to point out more clearly the structure of the paralogisms committed by Saccheri (in §2); the method initiated by Saccheri and its consequences (in §3); and the possible motivations and reasons that led Saccheri to the commission of the paralogisms (in §4).

§2. *Saccheri's paralogisms.* (a) The first paralogism, that which excludes the elliptic geometry, is more Euclid's paralogism than Saccheri's.

The exclusion of the hypothesis of the obtuse angle is achieved by Saccheri in the proposition XIV. All of the fourteen proofs given by Saccheri are correct and the first thirteen propositions are true. The proof of proposition XIV is also logically correct, but the proposition is false, because it leans on a proposition of Euclid's *Elements*, I,16, that is false. We quote from *Euclides*:

> *Proposition XIII.* If the straight *XA* (of designated length however great) meeting two straights *AD, XL*, makes with them toward the same parts internal angles *XAD, AXL* less than two right angles: I say, these two (even if neither of those angles be a right angle) at length will mutually meet in some point on the side toward those angles, and indeed at a finite, or terminated distance, if either hypothesis holds, of right angle or of obtuse angle.

This proposition only puts together the preceding propositions XI and XII and is obviously intended to prepare for the next proposition.

> *Proposition XIV.* The hypothesis of obtuse angle is absolutely false, because it destroys itself.[7]

This is the first false proposition of Saccheri. The pseudo-proof is very simple. Proposition XIII proves the fifth postulate and from this follows, according to Euclid, that the Saccheri quadrilateral must be a rectangle. Therefore the hypothesis of the obtuse angle is false. Therefore Saccheri avows the same paralogism that is contained in the proof of proposition I,16 of the *Elements*.

There is a question about the validity of the proofs of Saccheri's propositions from III to XIII, because in several of them Saccheri uses propositions I,16 and I,17 of the *Elements*, and these two propositions in the hypothesis of the obtuse angle are not generally valid.

The first time that Saccheri uses one of these two propositions, namely I,16, is in the proof of proposition III. At this point P. Staeckel remarks in a footnote that consequently, in the hypothesis of the obtuse angle, the proof is insufficient (ungenuegend),[8] since it is clear that in the hypothesis of the obtuse angle, the straight line cannot be assumed as having infinite length. On the other hand C. Segre remarks, in footnote 4, that nevertheless the proofs of Saccheri are fully rigorous (pienamente esatti).[9]

I recall that before proposition I,29 (in which for the first time Euclid uses the fifth postulate), the propositions, whose proofs depend on the infinite length of the straight, are I,16-17, I,21, I,26-28. The first of them, I,16, states that the exterior angle Γ of a triangle $AB\Gamma$ is greater than any of the interior angles A or B. And the other five propositions depend on I,16. I remark that Saccheri never uses I,21 or I,26.[10]

It seems to me that a key for a clarification of Saccheri's mind is given by the interesting and puzzling statement of Saccheri's Preface: "I will never use from those prior propositions of Euclid's First Book, not merely the 27th or 28th, but not even the 16th or 17th, except where it is clearly a question of a triangle every way restricted" (omni ex parte circumscripto).[11]

Why does Saccheri establish a distinction between propositions I,16-17 on the one hand and I,27-28 on the other hand? Because, I think, the first pair is valid for a restricted class of triangles, while the second pair, which establish the existence of parallel straight lines, are not. The second pair, I,27-28, necessarily require that the straight be of infinite length; but I,16-17 are valid, provided only that the segments, interior to the triangle and going from one vertex to the midpoint of the opposite side, be smaller than the half of the length of the straight line. In modern terms we would say that propositions I,16-17 are locally valid; whereas I,27-28 are in the large and necessarily false in the hypothesis of the obtuse angle.

Now, the local character of the controverted propositions in the *Euclides* is obvious from the very hypotheses. Actually, let $ABCD$ be a Saccheri quadrilateral with right angles at A and B. In the hypothesis of the obtuse angle, if $AD = BC$ and is greater than the half of the straight line, then the angles at C and D are acute, and the hypothesis does not make any sense unless one assumes that A, D, $AD = BC$ are taken such that AD and BC do not intersect, which seems should be granted at least in virtue of the figures.

One is inclined to think that Saccheri had in mind the fact that, in the hypothesis of the obtuse angle, two perpendiculars AD, BC on AB should intersect each other, which is (almost) explicitly contained in proposition XII (see note 5); and that for this reason he says that he will never use propositions I,16-17, "except where it is clearly a question of a triangle every way restricted."

Saccheri's proposition XII is in the large, but he sufficiently proves and states, explicitly and even redundantly, that in the hypothesis of the obtuse angle the two straight lines will meet "and indeed at a finite, that is, terminated distance" (et quidem ad finitam, seu terminatam distantiam).

Therefore, so it seems to me, the proofs of propositions I-XIII are rigorous. Also we would have here probably the first case, where a non-trivial distinction between "locally" and "in the large" is implied.

(b) The second paralogism of Saccheri is contained in proposition XXXIII, which is the second, and the last, false proposition in the First Part of the First Book of *Euclides*.

In proposition XXIII he proves that two straight lines in a plane either have a common perpendicular, or meet at a finite distance or approach mutually more to each other. A further step is given as follows.

> *Proposition XXV.* If two straights (Fig. 30) AX, BX existing in the same plane (standing upon AB, one indeed at an acute angle in the point A, and the other perpendicular at the point B) always so mutually approach more to each other, toward the parts of the point X, that nevertheless their distance is always greater than a certain assigned length, the hypothesis of acute angle is destroyed.[12]

For the proof assume R to be given and $AB > NK > \ldots > DK > HK > LK > \ldots > R$. Let $BK = \ldots = KK = R = TK = \overline{SK}$. The sum of the angles of a quadrilateral $HKKL$ is smaller than four rights, but increases and tends to four rights when LK moves away from AB. But the same sum must be smaller than the sum of the angles of the Saccheri quadrilateral $TKKS$, which is also smaller than two rights and fixed. So we get a contradiction.

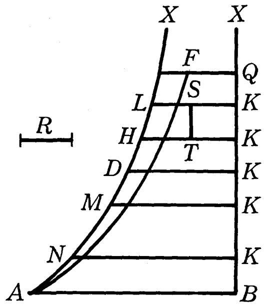

Figure 30.

It follows that if two straights *AX* and *BX* mutually approach more to each other, then they must be asymptotic, that is, *LK* becomes smaller than any assigned length when *LK* moves away from *AB*. Moreover, in this case, the obtuse angles *MDK, DHK, HLK* are decreasing and tend to a right angle (Proposition XXVIII).

The deepest theorem in *Euclides*, which has been called *Saccheri's theorem* by P. Mansion,[13] is contained in proposition XXXII. We may state it as follows.

In the hypothesis of the acute angle, two straights either intersect as *AC* and *BX* (Figure 1); or have a common perpendicular *MN*, from which they diverge, as *AD* and *BX*; or are asymptotic, as *AX* and *AX'* are asymptotic to *XX'*.

We remark that the word "asymptotic" is not used by Saccheri. We have used it for simplicity.

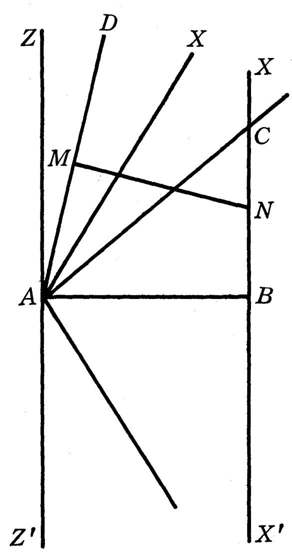

Figure 1

Proposition XXXIII, in which Saccheri commits the second paralogism and terminates the First Part, is as follows:

> *Proposition XXXIII.* The hypothesis of the acute angle is absolutely false; because repugnant to the nature of the straight line.[14]

The proof containing five lemmas and two corollaries is very long. The paralogism is committed in the first paragraph as follows. The straight AX asymptotic to BX, (Figure 1) is the limit of the intersecting straights AB, AC in the rotation around A, when the point C moves on BX away from B. Also the same asymptotic straight AX is the limit of the non-intersecting straights AZ, AD in the rotation around A, when the point N of the common perpendicular MN moves on BX away from B. Therefore, Saccheri argues, we reach the conclusion that there are "two straights AX, BX, which produced in infinitum toward the parts of the points X, must run together at length into one and the same straight line, truly receiving, at one and the same infinitely distant point a common perpendicular in the same plane with them." (ut agnoscere debeamus duas in eodem plano existentes rectas AX, BX, quae in infinitum protractae versus eas partes punctorum X in unam tandem eandemque rectam lineam coire debeant, nimirum recipiendo, in uno eodemque infinite dissito puncto X, commune in eodem cum ipsis plano perpendiculum.)[15]

After the paralogism, Saccheri correctly proves with five valid lemmas that such a conclusion is against the axioms or properties or "nature" of the straight line and therefore the hypothesis of the acute angle is absolutely false.

The paralogism of Saccheri lies in the assumption of the existence of two limits without proof. Certainly the asymptotic straight AX exists because of the axiom of continuity; but neither the point X as limit of C exists, nor the common normal at infinity as limit of MN. The point X occuring in several figures of *Euclides* has been introduced only nominally for sake of clarity and can be introduced really as a "point of infinity", but Saccheri fails to realize that it cannot be introduced as an *ordinary* point of plane geometry. Saccheri produces the straight up to and including the point X and further, as it were, to ideal points, but considers all these points as ordinary elements of the plane geometry, which cannot be done. Saccheri is here a victim, in spite of his warnings, of the confusion between nominal and real definitions in the case of a limiting process at infinity. He fails to realize that the point X and the common perpendicular at infinity, precisely because they do lead to a contradiction with the admitted axioms, cannot be introduced as ordinary points of the plane; and that, on the other side, the geometry of the acute angle can be perfectly developed without them.

In fact, the Beltrami model of hyperbolic plane geometry proves that the introduction of points at infinity and of ideal points beyond them (in order that the axioms of order be satisfied in the plane) can be carried out coherently, so that a new plane geometry is defined (geometry of incidence, order and continuity), that actually satisfies all of the axioms of Saccheri's geometry with the only exception the congruence axioms. Therefore, Saccheri in order to reach a contradiction must resort to some axiom of congruence, which he does by insisting that all right angles must be congruent.[16]

(c) In the Second Part of the First Book Saccheri pretends to give another proof of the absurdity of the hypothesis of the acute angle. Therefore there must be somewhere another paralogism. Here he begins by considering the equidistant line of a straight AB, or the locus of the points K on a perpendicular to AB in the same side and at a fixed constant distance from AB: always under the hypothesis of the acute angle.

In the first three propositions of this part, propositions XXXIV-XXXVI, he proves correctly and substantially the following. The equidistant $C\,F\,K\,N\,H\,D$ (Figure 2) of AB is concave with respect to AB, that is, its chords lie between the curve and the basis AB; the tangent KL at any point K of the equidistant is perpendicular to KM and the curve lies between the tangent and AB. Moreover we have to mention that in proposition III he has proved that any chord CD is greater than its projection AB under the hypothesis of the acute angle and is shorter under the hypothesis of the obtuse angle.

The last three (false) propositions of the First Book are:

> *Proposition XXXVII.* The curve CKD, arising from the hypothesis of the acute angle, must be equal to the opposite base AB.[17]

That is, they must be of equal length.

> *Proposition XXXVIII.* The hypothesis of the acute angle is absolutely false, because it destroys itself.[18]

This is an immediate consequence of the contradiction between the preceding proposition and proposition III just mentioned together with proposition I,20 of the *Elements*, which implies that an arc CKD equidistant to AB is greater than its chord CD.

The last proposition, XXXIX, has the same statement as the fifth postulate of Euclid and its proof follows from the assumed proofs of the absurdity of both hypothesis, that of the obtuse and that of the acute angle.

Obviously the third paralogism of Saccheri must have been committed in the proof of proposition XXXVII and now we are going to analyze it. Immediately after the statement of the proposition, Saccheri premises the following axiom: "If two lines be bisected, then their halves, and again their quarters bisected, and so the process be continued uniformly in infinitum; it will be safe to argue, that those two lines are equal to each other, as often as is ascertained, or demonstrated in that uniform division in infinitum, that at length two of their mutually corresponding parts, must be attained, of which it is certain they are equal to each other."[19] This axiom resembles the Principle of B. Cavalieri and I do not know how much may depend on Newton or Leibniz. Its formulation shows the same lack of precision, which is common with contemporary mathematical analysts, when they deal with "infinitely small quantities". For them the right intuition must supply the deficiencies of the imprecise formulation. I think that the axiom or at least the application of it that Saccheri carries out in his proof is correct, since in modern terms we may formulate it as follows. Let the real variable x be a coordinate on AB and dx an "infinitely small portion" of AB, say at a point M (Figure 2); let $s = s(x)$ be the equation of the equidistant and ds the corresponding "infinitely small portion" of the equidistant, that is at K. The application of the axiom by Saccheri amounts to saying: If $ds = dx$, then the base AB and the equidistant CKD are equal to each other, in corresponding portions. He must prove that $ds = dx$.

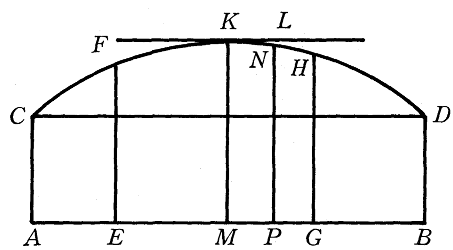

Figure 2

He considers the mixed quadrilateral $ACFE$, with right angles at A and E, $AC = EF$ and CF an arc of the equidistant curve. When AE moves on AB up to the position MG, the arc CF moves on the equidistant, always exactly fitting the curve, up to the position KH, so that it does not matter which point of the curve is considered. One may imagine the segment AC moving always equal to itself and remaining perpendicular to AB up to the position GH.[20]

One may also imagine the segment AB "flowing" from AB to the position CD keeping always a constant distance from AB, that is flowing in such a way that at any time it has the form of an equidistant curve with respect to AB. Now let KL be perpendicular to KM and therefore (Proposition XXXV) tangent at K to the equidistant. "Therefore infinitesimal K, regarding the curve, will be wholly equal to infinitesimal K regarding the tangent. But it is certain that the infinitesimal K regarding the tangent is neither greater nor less than, but exactly equal to the infinitesimal M regarding the base AB; because namely the straight MK may be supposed described by the flow always uniform of point M up to the summit K" (quia nempe recta illa MK intelligi potest descripta ex fluxu semper ex aequo ejusdem puncti M usque ad eam summitatem K.)[21]

We know now that in the hyperbolic plane, that is in the hypothesis of the acute angle, we have

$$ds = \cosh \frac{h}{k} \cdot dx,$$

where $h = AC$ and $-k^{-2}$ is the Gauss curvature of the plane. We may consider the differential of arc KN equal to the differential of tangent KL, and consider also AB as flowing toward CD, but this flow is not "uniform" (semper ex aequo) in the sense that the infinitesimal at M remains constant from M to K, and therefore KL is not equal to MP. The length of the equidistant curve with respect to AB continuously increases during the flow when the equidistant flows away from AB. Still in another way: If we imagine AC moving equal to itself with A moving on AB and remaining perpendicular to AB, then the point C moving on the equidistant CKH moves faster than the point A on the base AMG.

Here Saccheri uses his intuition, and indeed on the unintuitive hyperbolic plane, and fails. He may insist that the flow from AB to CD can be supposed to be described *always uniformly*, but then he is assuming a new postulate, namely, that the Gauss curvature vanishes, and then the paralogism is a *petitio principii.*

Actually the concept of *elementary* geometry is based on the axioms of congruence, which when taken together state the possibility of the motion of rigid bodies in space or of rigid triangles in the plane. We know now (Gauss) that these geometries are characterized by having constant curvature. Since Saccheri's geometry of the plane assumes the axioms of congruence, it follows that Saccheri's plane is of constant curvature.

In such a plane we can define the motion of a segment AB in the two following ways. First the segment AB (Figure 2) "flows" away from AB in such a way that each point E of AB moves on the perpendicular EF to AB and the segmenc AB moves *always uniformly*, that is all points move with constant speed and the same speed for all of them; thus AB during the motion always has the form of an equidistant and reaches CKD; during this motion neither the form nor the length of AB is preserved, unless the curvature of the plane vanishes. Second, since we are in an elementary geometry, we may move AB as a rigid straight line (a rule), that is, during the motion it preserves the form of a straight line and its length, in such a way that A moves on AC and the successive straight positions of AB remain perpendicular to MK; when A reaches C the position of AB will be CQ on the straight line CD; actually each point E moves on a straight line, but the point A moves faster than the interior points of the segment; it is a rigid motion, but not an uniform motion, unless the curvature of the plane is zero.

Saccheri defines the first motion or flow and identifies this flow with the rigid motion. Identifying the two motions he is equivalently assuming that the constant curvature vanishes, or he assumes what was to be proved, that is commits a *petitio principii.*

§3. *Saccheri's method.* Saccheri did not prove what he intended to prove and therefore did not solve the problem of the foundation of the contemporary geometry. But he gave a method that at length would lead to the solution of the problem. In this section we shall analyze in (a) the style of his method. In (b) we shall show historically that he did in fact influence the followers who used Saccheri's method in order to come closer to and finally solve the problem on Euclid's fifth postulate. Finally in (c) we shall show logically how this solution was developed from the initial step by Saccheri up to a final solution by Gauss.

(a) The *type of reasoning* in the pseudo-proof of the fifth postulate by Saccheri is obviously that of *reductio ad absurdum*, since he tries to reach a contradiction. We may be more precise about the main idea of his argumentation.

Let Σ be the set of axioms of Saccheri's geometry and P the proposition stating the fifth axiom of Euclid. Following G. Vailati [25] and G. B. Halsted [9] we may say that Saccheri wants to prove P by proving

$$\Sigma, \sim \mathsf{P} \vdash \mathsf{P}.$$

Since Σ is a set of axioms, given the law of Clavius,

$$(\sim \mathsf{P} \supset \mathsf{P}) \supset \mathsf{P}$$

a tautology, P follows.

That this is the main idea of Saccheri's argumentation follows from the otherwise strange form of the statements of propositions XIII and XXXIX; also from the first proof of proposition XIV; and above all from the last Scholion, that terminates the Book. We quote from this Scholion:

> "For chiefly this seems to be as it were the character of every primal verity, that precisely by a certain recondite argumentation based upon its very contradictory, assumed as true, it can be at length brought back to its own self. And I can avow that thus it has turned out happily for me right on from early youth in reference to the consideration of certain primal verities, as is known from my *Logica demonstrativa*".[22]

In this important book for the history of logic Saccheri claims as his most important contribution to be the first to give a systematic treatment of that type of reasoning. Indeed his *Euclides* constitutes a monumental example of this figure of reasoning. This matter is made clear enough by G. Vailati [25] and G. B. Halsted [9] in his Introduction, where he shows also how Saccheri defines and handles very sharply the distinction between definition *quid nominis* or *nominales* and definition *quid rei* or *reales*. The confusion between them is specially to be feared in the context of complicated definitions (*definitiones complexae*), as happens when parallel straight lines are defined as equidistant straight lines.

What is new in Saccheri's method is that he is the first to adjoin the proposition $\sim \mathsf{P}$ to the first 28 propositions of the first book of the *Elements* in order to prove P; and what deserves also great merit is the remarkable large number of original and deep theorems that he actually proves so that he discovers, without being conscious of it, two new continents or geometries while looking to find a new way to the old one.

(b) The *influence of Saccheri* on the creators of hyperbolic geometry is substantial in the sense that he made a decisive contribution or rather that he is a first link in the development and branching off of the geometries.

The importance of the *Euclides ab omni naevo vindicatus* is that it deals with an old and primary question for the very foundations of geometry and mathematics, and that it contains the very method and a remarkably accurate and extensive beginning of the ensuing development up to the creation of hyperbolic geometry simultaneously by C. F. Gauss, J. Bolyai and N. I. Lobačevskij. Moveover the development has gone further up to the creation of the formal or existential axiomatic method and consequently to a new interpretation of mathematics itself.

The most important book on the history of non-Euclidean geometry up to Gauss is the one by P. Staeckel and F. Engel (1895) [21]. Many authors of texts on non-Euclidean geometry give an historical account of the origins of the non-Euclidean geometry; we quote as specially valuable R. Bonola (1906) [3], D. M. Y. Sommerville (1914) [20], H. S. Carslaw (1961) [4] and the already mentioned book of H. S. M. Coxeter (1942).[23] On the very subject of the influence of Saccheri in the building of the non-Euclidean Geometry we have a well documented paper by C. Segre [18].

After an attentive reading of this literature, and specially of the book of Staekel and Engel and of the paper of Segre, it seems to me, that there is no doubt of the direct, or at least indirect but not less illuminating, influence of Saccheri on Lambert, Gauss, Bolyai and Lobačevskij. During the eighteenth and nineteenth centuries there was a general interest in the theory of parallels and no doubt the work of Saccheri contributed to it. The royal libraries of Berlin and Dresden and four university libraries including that of Goetingen (since 1770) possessed a copy. The *Euclides* was mentioned by the two well known and highly regarded *History of Mathematics* one by J. C. Heilbronner (Leipzig, 1742) and the second by J. E. Montucla (Paris, 1758). It was mentioned by the *Acta Eruditorum* (1736). The dissertation of G. S. Kluegel on the *History of the Theory of Parallels* (Goetingen, 1763) mentions it as "sonderbare Buch" (singular book) and gives a long account of the main results, and this dissertation is mentioned and praised by J. H. Lambert (1766 ?). The director of the dissertation of Kluegel was A. G. Kaestner (1719-1800), who was professor at the University of Goetingen for many years until his death.

It is clear that J. H. Lambert depends on Saccheri, at least through Kluegel's dissertation.[24]

With respect to C. F. Gauss (1777-1855), who began his investigations on the fifth postulate in 1792,[25] it is difficult to establish how much he may depend on the *Euclides*. To this end there are many references and contributions in the mentioned publications of Staeckel and Segre. I may add the following remarks.

The second item in the section on "Grundlagen der Geometrie" of volume VIII of the *Gesammtausgabe von Gauss' Werken* [8] is a letter (1804) of Gauss to W. Bolyai. In it he answers a letter of Bolyai asking Gauss for a sincere and frank opinion about a presumed new proof of the fifth postulate. The argument deals with a regular polygon, that reminds one of the final pseudo-proof of Lambert in §88 of the *Theorie der Parallellinien*.

The third item is a *Note* containing five propositions on the equidistant curve and related straight lines.[26]

Now, the content of these two items fits well into the continuation of the works of Saccheri and Lambert. There is the quadrilateral (of Saccheri) $ACEFG$, the use of the equality of angles C and F, the use of the obtuse angle and the perpendicularity of AC with the tangent CE to the parallel (equidistant) CF, as in proposition XXXV of *Euclides*. That is, it seems that the proof and conclusions are similar to those in *Euclides* and *Theorie der Parallellinien*.

But probably at 1808, perhaps as soon as 1799, and certainly at 1820 Gauss had already gone much further in the development of the non-Euclidean[27] geometry than Saccheri and Lambert, so that it seems difficult to draw any certain conclusion. Only in the *Notes* dated approximately at 1831 (pages 202-209) we find a theory on the parallels (now in the sense of asymptotic straight lines), that although deep, is elementary and also fits well with the methods of Saccheri and Lambert.[28] Also the letter (1829) of Bessel to Gauss ([8], p. 201) suggests that the work of Lambert was well known to Gauss.

By contrast, it seems that the work on the proofs of the fifth postulate by the great French mathematicians was independent of Saccheri and Lambert. They do not use the hypothesis $\sim \mathsf{P}$ and try to give a proof in the old way before Saccheri. Actually they lead to a dead end. I quote from Staeckel:

> "Almost all of the great French mathematicians of this time have turned their interest toward the Foundations of Geometry".[29]

The most well known of them is A. M. Legendre (1752-1833) through the several editions of his *Élêments de Géomêtrie* (first in 1794, the twelfth in 1823) and his *Mêmoire de l'Acadêmie des Sciences* (1833) on the same matter. About him Staeckel says:

> "Neither the results of his investigations nor the methods, that led him to these results, can be marked as an essential progress compared with the accomplishments of Wallis, Saccheri and Lambert".[30]

(c) One is now inclined to see as an inevitable *consequence of the method* introduced by Saccheri that at length mathematicians would establish a new geometry, in fact that of the acute angle, which would enter into competition with the old Euclidean geometry as to which one is the true geometry.

If we want to historically evaluate the progress contributed by the creation of the non-Euclidean geometries, then we cannot forget that the present theory of formal axiomatic systems and the more abstract concept of geometry as a pure mathematical science, independent in its method from the outside world, are precisely in good part a consequence of that creation. As different from these formal or existential axiomatic systems, we find in Euclid's *Elements*, and more clearly according to the philosophy of Aristotle and St. Thomas, an axiomatic system, that now we qualify as material or genetic.

I want now to analyze the evolution of the concept of geometry from Saccheri to Gauss. During this time the *Critique of Pure Reason* (1781) by I. Kant was published. I shall consider only Saccheri, Lambert, Taurinus and Gauss, who was the first to fully realize that his non-Euclidean geometry had a chance, as well as Euclid's geometry, of being the true one.

(1) It seems that Girolamo Saccheri (1767-1733) never doubted the truth of the fifth postulate and therefore the truth of Euclidean geometry. Thus in the Preface to the Reader, after a proposition stating the fifth postulate, he comments:

> "No one doubts the truth of this proposition; but solely they accuse Euclid in respect to it, because he has used for it the name axiom, as if obviously from the right understanding of its terms alone came conviction. Whence not a few (withal retaining Euclid's definition of parallels) have attempted its demonstration".[31]

Why such a conviction? Although in the next section I shall give some motivations for the parallogisms committed by Saccheri, here I give the following possible reasons of his adherence to the fifth postulate.

One was his own proofs of the postulate, although it seems clear that he realized that the proofs of the propositions that contain the second or third parallogisms were not rigorous.

Another reason was experience. Actually, in Scholion II after proposition XXI, he proposes three ideal experiments. The first assumes a Saccheri quadrilateral *ABDC*, with right angles at *A* and *B* and *AC* = *BD*. He writes:

> But in so far as may be here permissible to cite physical experimentation, I forthwith bring forward three demonstrations physico-geometric to sanction the Euclidean postulate.
>
> Therewith I do not speak of physical experimentation extending into the infinite, and therefore impossible for us; such as of course would be requisite to cognizing that all points of the straight line *DC* are equidistant from the straight *AB*, which is supposed to be in the same plane with this *DC*.
>
> For a single individual case will be sufficient for me; as suppose, if, the straight *DC* being joined, and any one point of it *N* being assumed, the perpendicular *NM* let fall to the underlying *AB* is ascertained to be equal to *BD* or *AC*.

Then, it follows that "we shall have demonstrated the Euclidean Postulate."

> I pass over to the second. Let there be a semicircle, of which the center is *D*, and diameter *AC*. If then (Figure 17) any point *B* is assumed in its circumference, to which *AB*, *CB* joined are ascertained to contain a right angle, this single case will be sufficient (as I have demonstrated in P. XVIII.) for establishing the hypothesis of right angle, and consequently (from the aforesaid P. XIII.) for demonstrating that famous postulate.

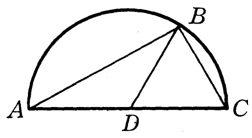

Figure 17

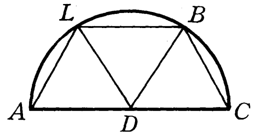

Figure 22

There remains the third demonstration physico-geometric, which I think the most efficacious and most simple of all, inasmuch as it rests upon an accessible, most easy, and most convenient experiment.

For if in a circle, whose center is D, are fitted (Figure 22) three straight lines CB, BL, LA, each equal to the radius DC, and it is ascertained that the join AC goes through the center D, this will be sufficient for demonstrating the assertion.[32]

The most important reason, amounting almost to impossibility of doubting, is to be found, I think, in Saccheri's background of philosophical ideas. I may assume that Saccheri's concept of geometry was the same that we find in St. Thomas, where geometry is that part of mathematics which deals with continuous magnitudes. I will not give here an exposition of the theses of St. Thomas on mathematics, but I quote some references in a note[33] in order to justify what follows.

The crucial point is that at least up to and including Gauss, the concept of geometry, either in the Scholastic or in the *Critique of Pure Reason*, was too simple in two aspects: first, it did not distinguish, as we now do, between geometry as a part of pure mathematics, which is a rather clear-cut subject created or constructed with precisely defined terms, and geometry as a part of Physics, which is a very complex subject dealing with the discovery of some type of properties of the previously given outside world, submerged in space-time and filled with matter, and using therefore empirically defined concepts. Second, because geometry was still in its beginnings and because of the mentioned identification of the two concepts of geometry, only the obvious notion of congruence derived from the motion of rigid bodies in physical space is considered, and therefore only elementary geometry comes into question; moreover, it follows that only one geometry is conceivable, which simultaneously will be both a part of exact mathematics (either by abstraction in Aristotle, or by a prior intuition in Kant) and a description of the given outside world.

Now, the thirteen Books of Euclid's *Elements* constitute such an impressive block of coherent geometry, that it must have been taken to be the true geometry. Only after having built a second block comparable with the *Elements*, and having failed during a century to find any contradiction, the idea will arise for the first time (Riemann (1854) ?) of separating the two concepts of geometry and of admitting the ramification of the mathematical geometries as equally valid.

I think that this is the first example in the history of philosophy of two large, very deep and certainly coherent systems of propositions, the first assuming a postulate P and the second assuming $\sim$ P, but otherwise both of them being deduced from the same set of axioms.

(2) Johann Heinrich Lambert (1728-1777) goes much further than Saccheri in the development of the geometries. In §39 of the *Theorie der Parallellinien* (1766) [13] he establishes the same three hypotheses as Saccheri (using a "Lambert quadrilateral", with three right angles and two opposite sides equal), and after a sequence of theorems he reduces the hypothesis of the obtuse angle *ad absurdum* in §64. In §82 he first shows that under the hypothesis of the acute angle the area of a triangle ABC will be proportional to the defect, that is we may write

$$\text{Area } ABC = m(\pi - A - B - C);$$

and shows also that in the hypothesis of the obtuse angle it will be

$$\text{Area } ABC = m(A+B+C-\pi).$$

Then he continues:

> Here it seems to me remarkable, that the second [obtuse angle] hypothesis is fulfilled when instead of a plane triangle one takes a spherical one, because in this both the sum of the angles is greater than 180 degrees and also the excess is proportional to the area of the triangle.
>
> Still it is more remarkable, that what I say here of the spherical triangle, may be proved without regard to the difficulty of the parallels, and does not assume any other axiom, than that each plane through the center of a sphere divides the sphere in two equal parts.
>
> I should almost conclude from it that the third hypothesis takes place in an imaginary sphere. At least there must be ever something, why in the plane it [the third hypothesis] cannot be overthrown far so easy, as can be done in the case of the second hypothesis.[34]

We see in this text an important contribution of Lambert on the determination of areas. He also realizes that the geometry of the obtuse angle is in this respect fulfilled on the sphere. Moreover he rightly conjectures that the geometry of the acute angle would be fulfilled on an "imaginary sphere". If in the second formula above we put $m = R^2$, R being the radius of the sphere, then the imaginary sphere would have radius iR and the second formula goes over to the first. Finally, he shows also how convinced he is of the truth of the fifth postulate and of the possibility of also reducing the hypothesis of the acute angle *ad absurdum*.

In §10 he grants that, of course the fifth postulate is true. Why?

> Because the truth of it is shown to such a degree as illuminating and necessary from all the consequences that for all purposes are deduced from it, that these consequences taken together may be regarded in many ways as a complete *induction*.[35]

What is not said, but is implicitly assumed, is that, if the fifth postulate is true, then its negation cannot be true, and therefore a geometry assuming this negation as an axiom must necessarily lead to a contradiction.

We quote still another interesting text of §11.

> *The question* is, *can it* [the fifth axiom] *be correctly deduced from the Euclidean postulates together with the other axioms? Or, if these were not sufficient, can other postulates or axioms or both be given which would have the same evidence as the Euclidean ones and from which the eleventh* [fifth] *axiom could be proved?*
>
> For the first part of this question one can abstract from all that I have previously called *representation of the matter*. And since Euclid's postulates and remaining axioms are already expressed in words, it can and must be required that one in the proof never leans on the matter itself, but carries forward the proof in an absolutely symbolic way. In this respect Euclid's postulates are as so many algebraic equations, that one already has as previously given, and that must solve for x, y, z, . . . , without looking back to the matter itself.[36]

Here Lambert clearly formulates the question of the possible dependence or independence of the fifth postulate, and in connection with other texts[37] also establishes clearly, for the first time, the distinction between the concepts of explicit definitions ("Namen", names) and of implicit definitions ("Gleichungen", equations), which will be fully explained by M. Pasch (1882).

In the last section, §88, Lambert surrenders to the desire of proving the fifth postulate as Saccheri had done. In the proof there is also a paralogism. It seems clear that Lambert himself realized that the proof was not rigorous enough.[38]

(3) Franz Adolph Taurinus (1794-1874) in his *Theorie der Parallellinien* (1825) [22] and *Geometriae prima elementa* (1826) [23] developed much further than Saccheri and Lambert, whose method follows,[39] the geometries of the obtuse and of the acute angle. He calls logarithmo-spherical geometry the one under the hypothesis of the acute angle and develops the corresponding trigonometry. He is convinced of the, so to say, independence of the three hypotheses and that his new geometry does not lead to any contradiction. Nevertheless, he cannot give up his conviction that the Euclidean geometry is the only true one. So unbelievably difficult is it to free oneself from the historically and philosophically given contemporary situation!

I shall only quote some texts that bear on the concept of geometry.

> A geometry, in which more than two right angles are contained in a triangle, leads to a clear contradiction with the axiom of the straight line; since in any system of this kind the straight lines would intersect each other in two points, without being coincident.[40]

Thus the geometry of the obtuse angle is rejected, because Taurinus cannot abstract from the representation of the matter, in terms of Lambert. Why cannot the two points be the same? Klein (1871) will identify the two antipodal points on a sphere and obtain in this way a model for elliptic geometry.

Now on the geometry of the acute angle:

> We have to object to the acceptance of such a system as a system of straight lines the following:
>
> 1. It contradicts every intuition. It is true, that such a system could offer in the small the same appearances as the Euclidean one: but, if the representation of the space may be considered as the pure form of the outer sense, then the Euclidean system is certainly the true one, and it cannot be assumed that a limited experience can produce a sensory delusion.
>
> 3. Should the third system be the true one, then there would be absolutely no Euclidean geometry, but indeed its possibility [of the Euclidean geometry] cannot be denied.[41]

In the *Postscript* of the *Theorie der Parallellinien* he writes:

> It is easy to show that a geometrical system, in which less than two rights is contained in the triangle, is not determined in itself, but requires one special determining-magnitude or constant. Hence it follows immediately, that there is for us absolutely no other geometry than the Euclidean one, because such a constant can be taken with absolute arbitrarity.[42]

Obviously this constant cannot be stated a priori, but would have to be determined from experience. But according to Kant, the space is a necessary a priori representation and geometry determines its properties also a priori through intuition. Therefore such a constant cannot be admitted and consequently the new geometry must be rejected.

Before writing the following texts of the *Supplement* to the *Theorie* and its *Proscript,* Taurinus received the letter of Gauss, that we partially quote in the next subsection (4). He does not speak any more of apriority, but still he cannot accept the new geometry. In the Preface of the *Geometriae prima elementa* (1826) he writes commenting on this letter: "...certainly I could not guess completely his [Gauss'] view on the matter".[43] Perhaps he writes this because the letter was private, but according to the following paragraphs probably it was also true.

> That geometry, in which it is assumed that the sum of the angles of a triangle is smaller than two rights, contains in itself—according to the concept—no contradiction. . . .
>
> The contradiction must be sought in it, that there are not one, but infinitely many systems of this kind, each of which would have equal claim to validity; . . .[44]

(4) Carl Friedrich Gauss (1777-1855) already in 1799, after seven years of having begun his investigations on the theory of parallels,[45] writes to W. Bolyai that in spite of having found some arguments in order to prove the fifth postulate, his results rather lead "to make doubtful the truth of the geometry".[46] He writes analogously in 1808 to H. Ch. Schumacher,[47] more clearly in a letter to Ch. L. Gerling (1816).

> It is easy to prove, that, if Euclid's geometry is not the true one, then there are absolutely no similar figures: the angles in an equilateral triangle are then also different according to the magnitude of the side, whereby I do not find abso-

lutely anything absurd. . . . It seems somewhat paradoxical, that a constant line as it were a priori can be possible; but I find in it nothing contradictory. It would be even desirable that Euclid's geometry were not true, because then we would have a priori an universal measure. . . .[48]

Here Gauss substantially follows Kant. Assuming that space is essentially unique and that geometrical intuition is the a priori form of the outer sense, he does not here hesitate considering the constant as given a priori or "as it were a priori" (...dass eine constante Linie gleichsam a priori möglich sein könne) or "we would have a priori an universal measure" (wir ein allgemeines Mass a priori hätten). He follows Kant, but trusts mathematics more than philosophy.

Kant had set geometry at the same level of pure apriority as arithmetic. In a letter to Olbers (1817), Gauss[49] doubts this thesis. But the mind of Gauss is better revealed in the following important letter to Taurinus (1824):

. . . I do not believe, that anybody has spent more time than I precisely in this second part [geometry of the acute angle], although I have never published anything about it. The assumption, that the sum of the three angles is smaller than 180°, leads to a peculiar geometry, completely different from ours (Euclidean), which in itself is absolutely consequent, and that I have built for myself totally satisfactorily, so that I can solve any exercise in it, except the determination of a constant, that cannot be ascertained a priori. . . . All my efforts to find a contradiction, an inconsequence in this non-Euclidean geometry have been fruitless, and the unique thing that resists our understanding is that, if it were true, then there must exist in space a linear magnitude, determined in itself (although unknown to us). But it seems to me, that, in spite of metaphysicists' word-wisdom that says nothing we know properly too little or absolutely nothing about the true essence of space, that we may take for Absolutely Impossible something that presents itself unnaturally. If the non-Euclidean geometry is the true one and if that constant is in a certain relation to such magnitudes that lie in the region of our measurements on earth or in the heavens, then it could be found out a posteriori.[50]

I give now two paragraphs of two letters to F. W. Bessel (1829) and 1830):

. . . and my conviction, that we cannot found the geometry completely a priori, has, if possible, become still stronger. Meanwhile I decide that for a long time I shall not work out my very extensive investigations about it, and perhaps this will never happen in my lifetime, because I fear the outcry of the boeotians, if I would express completely my view.[51]

According to my most inner conviction, the doctrine on space has a completely different position in our knowledge a priori, than the pure doctrine on magnitudes; our knowledge of the former departs thoroughly from the complete conviction of its necessity (hence also of its absolute truth), that is proper to the latter; we must grant in humility that, while number is a pure product of our spirit, space has also out of our spirit a reality, whose laws we cannot completely prescribe a priori. . .[52]

We remark that by "boeotians" the kantians are designated as stupid. I may still mention two letters to Schumacher (1836 and 1846)[53] and the contribution of W. Sartorius von W.[54] that substantially do not contain anything new.

C. F. Gauss was the first to be concious of having a consequent fairly complete non-Euclidean geometry; the same originated by Saccheri under the hypothesis of the acute angle about one hundred years earlier. Now we may look at Gauss' creation both from the viewpoint of the modern concept of geometry and mathematics as dealing with formal systems, and from the viewpoint of geometry as a part of physics.

From the first viewpoint Gauss does not reach to formulate the notions of consistency and independency (formulated by Lambert) of axioms, much less the possibility of equal consistency of the two geometries. E. Beltrami (1868) for two dimensions and F. Klein (1871) for three will be the first ones to prove the equal consistency of the elementary geometries.

Although Gauss is able to free himself from Kant's theses on the apriority of the Euclidean geometry[55] more than Taurinus, he does not from the thesis on the uniqueness of geometry and does not reach to formulate the question on validity rather than that of truth. Repeatedly he assumes implicitly that only one geometry can be true, that which is tight with physics, and it is assumed that this can be only one, and precisely an elementary one; much less can he think of a ramification of geometries. We have seen that Lambert foresaw the possibility of cutting the contact with the physical world ("abstracting from the representation of the matter" and "carrying forward the proof in an absolutely symbolic way"); M. Pasch (1882), stating the projective geometry, introduced the implicit definitions by axioms, thus completely abstracting from the physical world; finally D. Hilbert introduced the existential or formal systems.

From the second viewpoint he still adheres too much on the apriority of the geometry: not only one constant, but the whole metric structure of the space depends on experience. Much less can he foresee that the question on the truth of physical space, including geometric measurements through actual straight (geodetic) lines either rules (solids) or light rays, may depend on time or speed and on the mechanical properties of the entities that fill or constitute the space and support the measuring instruments; that is on elastic properties for rules and on optical properties for light rays. B. Riemann (1854) was the first to formulate this possibility and A. Einstein the first to take full advantage of it.

§4. *Motivation of the paralogisms.* (a) The first motive was that, a priori of his proofs, Saccheri was absolutely convinced of the truth of the fifth postulate, because of experience and because of the historical and philosophical situation, as we have explained in §3, (c,1).

In a certain context one could ask why he did not think in terms of pure logic, rather than in terms of geometry, which was absolutely connected with the physically continuous magnitude. But to take such an approach was much more impossible, precisely because of the historical and philosophical situation, as the development given in §3 seems to show with evidence.

Already in the beginning, in the statements of the Propositions V, VI, VII, Saccheri speaks of physical truth; the three hypotheses of the obtuse, right, acute angles, are physical hypotheses; and the three experiments that we have mentioned in §3 are given as physical experiments belonging to our experience.

It seems that still many other paralogisms should be committed and a greater block of coherent propositions should be deduced to make it actually possible, that the fifth postulate were doubted without the risk of being taken as a fool or at least of being overcome by the outcry of those contemporary philosophers.

(b) In the Introduction to the English translation, G. B. Halsted [9], and previously in his own work G. Vailati [25], point out that one of the reasons why Saccheri attached the proof of the fifth postulate was to test his figure of reasoning through the law of Clavius. He was first professor of Logic and wrote his *Logica demonstrativa* and, after that, became professor of mathematics in the University of Pavia (1697 or later) until his death (1733), so that he could spend much time on this task, as he certainly did. What we have said in §3, (a) and the text that we have quoted there, would confirm the above suggestion. The way in which he speaks of his method in the quoted text strongly suggests also that not only was he convinced of the fifth postulate, but that a proof of it should necessarily be possible through his method. And the desire to show it, motivated also his paralogisms.

(c) Saccheri leaning on Euclid proved, and was convinced of it, that: "If the angle [of his quadrilateral] is obtuse, then it follows that it must be right." Therefore he was equally convinced that he had rejected the hypothesis of the obtuse angle, because he never doubted the correctness of Euclid's *Elements*.

In the proofs of the propositions from the first say to XXI and specially in XII, XIII, XIV, where the proof is carried out, there is no hesitation, no vagueness, and in fact all the proofs are correct as we have said in §2,(a); and he never suspected the occult paralogism in the proof of I,16 of the *Elements*. But even Lambert, Taurinus and Gauss himself[56] never doubted that conclusion and never suspected the paralogism of Euclid.

Moreover I may quote the following text of Saccheri from the last Scholion of the First Book:

> *Scholion.* It is well to consider here a notable difference between the foregoing refutations of the two hypotheses. For in regard to the hypothesis of obtuse angle the thing is clearer than midday light; since from it assumed as true is demonstrated the absolute universal truth of the controverted Euclidean postulate, from which afterward is demonstrated the absolute falsity of this hypothesis; as is established from P. XIII. and P. XIV.
>
> But on the contrary I do not attain to proving the falsity of the other hypothesis, that of acute angle, without previously proving that the line, all of whose points are equidistant from an assumed straight line lying in the same plane with it, is equal to this straight, which itself finally I do not appear to demonstrate from the viscera of the hypothesis, as must be done for a perfect refutation.[57]

Saccheri must have considered the achievement of this proof both an important contribution and a further confirmation of the power of his method.

Of course this paralogism is very understandable: given the open, oriented straight segment $\overrightarrow{AB}$, one may always continue it as straight up to C, being $\overrightarrow{BC} = \overrightarrow{AB}$. It turns out that the assumption that the segments AB and BC are always disjoint, excludes the geometry of the obtuse angle. Since this assumption is comparable to the assumption of the fifth postulate and the latter was stated explicitly by Euclid, it seems that the omission of stating the former should imply a paralogism in the proof by Euclid of Proposition I,16.

Now, Saccheri thought he had been successful in the hypothesis of the obtuse angle. Why should there be a substantial difference in the hypothesis of the acute angle? It is illuminating to consider the following argumentation that Saccheri gives at least twice (propositions XVII and XXVII). Summarizing: he proves that given the angle BAC as small as pleased, there is a perpendicular to AB such that is not met by AC; this looks as an awkward conclusion and therefore he remarks that if this conclusion were false, "then there will be no place for the hypothesis of the acute angle" (XXVII). The propositions which contain the paralogisms start from awkward and correctly proven premises and attempt to show their absurdity.

No doubt that the argumentation *a pari* comparing with the case of the hypothesis of the obtuse angle must have been an important motivation for the paralogisms.

(d) Reading the long proofs of propositions XXXIII and XXXVII that contain the paralogisms, it is obvious that Saccheri thought that they had a convincing value, specially the first one.

But it seems also clear that he was not completely satisfied with these two proofs. This is explicitly granted in the second paragraph of the text quoted in (c) above, at least for the second proof.

One may ask, why two proofs? The answer seems to be, because none of the two is mathematically rigorous. It is almost only in these two propositions, that Saccheri uses words alien to the mathematical method, thus departing from his usually precise, clear, categoric writing. Immediately after finishing the first proof, in the final Scholion of the First Part he says:

> *Scholion*. And here I might safely stop. But I do not wish to leave any stone unturned, that I may show the hostile hypothesis of acute angle, torn out by the very roots, contradictory to itself.[58]

And in the scholion following the second proof, that is, following the last text quoted in §2, (note 21), he says:

> *Scholion 1*. But perchance to some one the enunciated exact equality between the infinitesimals M, and K will seem by no means evident. Wherefore to remove this scruple I again proceed thus.[59]

But, then, if he had some doubt about the rigor of these proofs, why did he put them in the Book? Perhaps otherwise there would have been no book, and that would have been much worse. Apparently he surrendered, like Lambert and many others, to the desire of giving a higher interpretation or value to his work. For him the proof of the fifth postulate was the only possible outcome. The book was published the very year of his death. And certainly none of the two paralogisms is trivial.

NOTES

1. Gerolamo Saccheri was born in 1667 at San Remo, then Republic of Genoa. In 1685 he joined the Society of Jesus. Staying at the Collegio di Brera in Milan he made acquaintance with the mathematicians: brothers Giovanni Ceva and Father Tommaso Ceva (also a Jesuit). He taught philosophy and apologetics in Turin and in 1697 moved to Pavia, where be became Professor of Mathematics in the University and died on October 25, 1733. He was an outstanding chess player and Gamborana says of him: "He did not care for his person, food, dress, comfort, but only the truth, the welfare of others and the defense and propagation of the holy catholic faith affected him at heart." He published the *Logica demostrativa,* 1701, the *Euclides ab omni naevo vindicatus* (Euclid freed of every fleck), 1733, and a few other works. For references see Staeckel, pages 31-41 and 318, 319, 324; and Halsted, pages VII-XXX.

2. There is no difficulty in establishing a sufficient set of axioms for each of the three geometries:

   For the Euclidean plane one may take I-III of the Appendix III and IV-V of the First Chapter in Hilbert. Or in Coxeter's book the axioms 8.311, 8.313-8.317 of intermediacy, 9.11-9.15 of congruence, 9.51 of Euclid and 2.13 of continuity.

   For the hyperbolic geometry the same set of axioms but substituting 9.51 in Coxeter's book for Euclid's axiom.

   For elliptic plane geometry the axioms of incidence 2.111-2.114, (2.32 without the word "coplanar") and 2.31 that denies the existence of parallels; the axioms 2.121-2.126 of order; the axioms of congruence 9.11-9.15, but taking into account the present axioms of order, that is, where it says that B lies between A and C, i.e. $[ABC]$, the segment AB must be specified in accordance with what is said in the axioms of separation, [5], page 23; and the same 2.13 of continuity.

3. See for instance A. Dou [6]. The present paper is an extension and also a correction of some statements of that paper, published in Spanish.

4. The axioms of Saccheri's geometry may be stated assuming disjunctively either the axioms of Euclidean-hyperbolic geometry (without any axiom of parallels) for a non-obtuse angle or the axioms of the elliptic geometry (without 2.31) for the obtuse angle. I presume that these axioms may be stated simpler as being those of the elliptic geometry (without 2.31) together with the Pasch axiom as formulated by Hilbert, Chapter 1, but where an elliptic triangle must be understood, that is a figure with an interior region separated from the exterior, so that three points determine exactly four different triangles.

5. Saccheri in the statement of proposition XII seems to assume that the two lines meet the transversal with angles, whose sum is *less* than two rights. But, of course, in the hypothesis of the obtuse angle, if two lines meet a transversal making angles that sum two rights, the case is immediately reduced to the previous one, and indeed in both directions so that the two lines must meet in both directions.

6. I refer only to the publications mentioned in §3,(b) and specially to the bibliographies by Staeckel [21] and by Sommerville [20].

7. The English translation that we give in the text is that by Halsted [9]. The original text in latin is also taken from the same book.

   "Proposition XIII.—Si recta *XA* (quantaelibet designatae longitudinis) incidens in duas rectas *AD, XL*, efficiat cum eisdem ad easdem partes (fig. 11.) angulos internos *XAD, AXL* minores duobus rectis: dico, illas duas (etiamsi neuter illorum angulorum sit rectus) tandem in aliquo puncto ad partes illorum angulorum invicem coituras, et quidem ad finitam, seu terminatam distantiam, dum consistat alterutra hypothesis aut anguli recti, aut anguli obtusi." Remark that "ad easdem partes" (same side) must be understood locally.

   "Proposition XIV.—Hypothesis anguli obtusi est absolute falsa, quia se ipsam destruit."

8. Staeckel [21], footnote in page 52.

9. Segre [18], *Opere*, page 445.

10. In fact Saccheri uses I,26 in Proposition XI, but inessentially, since he does under the hypothesis of the right angle, when I,26 is obviously valid. And in Euclid's *Elements* the proof of proposition I,18 depends also on I,16 and the proofs of propositions I,19-21 and I,24-25 depend on I,18; and Saccheri applies several of these propositions in the proof of propositions VIII, X-XII. But I,18 can be proved without depending on I,16 and therefore all of these propositions are valid in Saccheri's geometry.

11. "...; nunquam idcirco adhibens ex ipsis prioribus Libri primi Euclidaei Propositionibus, non modo vigesimam septimam, aut vigesimam octavam, sed nec ipsas quidem decimam sextam, aut decimam septimam, nisi ubi clare agatur de triangulo omni [xi] ex parte circumscripto".

12. "Propositio XXV.—Si duae rectae (fig. 30) *AX, BX* in eodem plano existentes (una quidem sub angulo acuto in puncto *A*, et altera in puncto *B* perpendiculariter insistens ipsi *AB*) ita ad se invicem semper magis accedant versus partes punctorum *X*, ut nihilominus earundem distantia semper major sit assignata quadam longitundine, destruitur hypothesis anguli acuti."

13. Mansion [14]. Quoted also by Halsted [9], pages VIII-IX.

14. "Propositio XXXIII.—Hypothesis anguli acuti est absolute falsa; quia repugnans naturae lineae rectae."

15. Pages 172-3.

16. This paralogism is already suggested when Saccheri speaks of infinite distance as of an ordinary distance (corollary II of proposition XXIX) and in the words "in two distinct points" of the statement of proposition XXX. The point *X* at infinity may be considered correctly introduced (real definition) as the intersection of two parallel straights lines; and analogously the limit of the common perpendicular as the straight line that is perpendicular at *X* to *BX*. But the axiom of congruence of triangles is not satisfied for these points and lines. All this is obvious considering the Beltrami model.

17. "Propositio XXXVII.—Curva *CKD*, ex hypothesi anguli acuti enascens, aequalis esse deberet contrapositae basi *AB*."

18. "Propositio XXXVIII.—Hypothesis anguli acuti est falsa, quia se ipsam destruit."

19. "Si duae lineae bifariam dividantur, tum earum medietates, ac rursum quadrantes bifariam, atque ita in infinitum uniformiter procedatur; certo argumento erit, duas istas lineas esse inter se aequales, quoties in ista uniformi in infinitum divisione comperiatur, seu demonstretur, deveniri tandem debere ad duas illarum sibi invicem respondentes partes, quas constet esse inter se aequales."

20. One is tempted to conclude that the equidistant *CKD* must be a straight line, because "quaecunque ex aequo punctis in ea sitis iacet" (it lies homogeneously with respect to the points in itself), according to Euclid's definition. Saccheri shows that such a proof cannot be admitted (Scholion II of proposition XIX). He also rejects the comparison between the "flowing" of the straight line from *AB* to *CKD* with the motion of a circle into another concentric circle.

21. "Igitur infinitesima *K*, spectans ad curvam, aequalis omnino erit infinitesimae *K* spectanti ad tangentem. Constat autem infinitesimam *K* spectantem ad tangentem, nec majorem, nec minorem, sed omnino aequalem esse infinitesimae *M* spectanti ad basim *AB*; quia nempe recta illa *MK* intelligi potest descripta ex fluxu semper ex aequo ejusdem puncti *M* usque ad eam summitatem *K*."

    We have translated *nempe* by "namely" instead of "obviously" (Halsted). The same Saccheri admits in the next lines that the argument may be doubted; moreover this is the most common meaning of *nempe* in the Scholastic (See latin *Lexicon* by R. J. Deferrari).

22. "Nam hic maxime videtur esse cujusque primae veritatis veluti character, ut non nisi exquisita aliqua redargutione, ex suo ipsius contradictorio, assumpto ut vero, illa ipsa sibi tandem restitui possit. Atque ita a prima usque aetate mihi feliciter contigisse circa examen primarum quarundam veritatum profiteri possum, prout constat ex mea Logica demonstrativa."

23. Other accounts are not so accurate. One, precisely by a historian of mathematics, is unbelievably ignorant of the sources, and naive. (See E. T. Bell, *The Magic of Numbers*, pages 345-356).

24. See Lambert [13], §3 or Staeckel [21], page 155; Segre [18], article 4 and note (17).

25. See Gauss [8], p. 238.

26. I remark that the word "equidistant" as a curve, that I have used here and in §2 for sake of clarity, is never used by Saccheri. Here the equidistant is called by Gauss "Parallellinie", as a nominal definition, very much as many others had done before and did contemporarily, and Saccheri had done on page 100, (page 236 in Halsted [9]).

27. Gauss uses first the name "Astralgeometrie" following the term "astralische Groessenlehren" of F. K. Schweikart (1780-1857), pp. 180, 182 and 183 of [8]. Then the name "anti-Euklidische Geometrie", page 175. Finally the name "Nicht-Euklidische Geometrie" (1824), page 187. N. Lobačevskij says "imaginary geometry" and J. Bolyai says "absolute geometry" ("scientiam spatii absolute veram").

28. Gauss uses here the word "Perpendikel", instead of "Senkrecht". It reminds me that Lambert also uses it and that Saccheri, in propositions IV-VII, the ones given by Kluegel (whose Dissertation I have not seen), uses the word "perpendiculum" instead of "perpendicular" that he uses ordinarily.

29. Staeckel [21], p. 211.

30. Staeckel [21], p. 213.

31. "Porro nemo est, qui dubitet de veritate expositi Pronunciati; sed in eo unice Euclidem accusant, quod nomine Axiomatis usus fuerit, quasi nempe ex solis terminis rite perspectis sibi ipsi faceret fidem. Inde autem non pauci (retenta caeteroquin Euclidaea parallelarum definitione) illius demonstrationem aggrèssi sunt. . . ."

32. "Sed quatenus ad experientiam physicam provocare hic liceat; tres statim afferam demonstrationes Physico-Geometricas ad comprobandum Pronunciatum Euclidaeum. Ubi non loquor de experientia physica tendente in infinitum, ac propterea nobis impossibili; qualis nempe requireretur ad cognoscendum, quod puncta omnia junctae rectae *DC* aequidistent a recta *AB*, quae supponitur in eodem cum ipsa *DC* plano consistens. Nam mihi satis erit unicus individuus casus; ut puta, si juncta recta *DC*, assumptoque uno aliquo ejus puncto *N*, perpendicularis *NM* demissa ad subjectam *AB* comperiatur esse aequalis ipsi *BD*, sive *AC*."

Transeo ad secundam. Esto semicirculus, cujus centrum *D*, et diameter *AC*. Si ergo (fig. 17.) in ejus circumferentia assumatur punctum aliquod *B*, ad quod junctae *AB*, *CB* comperiantur continere angulum rectum, satis erit hic unicus casus (prout demonstravi in 18. hujus) ad stabiliendam hypothesim anguli recti, ac propterea (ex praedicta 13. hujus) ad demonstrandum notum illud Pronunciatum. [36]

"Superest tertia demonstratio Physico-Geometrica, quam puto omnium efficacissimam, ac simplicissimam, utpote quae subest communi, facillimae, paratissimaeque experientiae. Si enim in circulo, cujus centrum *D*, tres coaptentur (fig. 22.) rectae lineae *CB*, *BL*, *LA*, aequales singulae radio *DC*, comperiaturque juncta *AC* transire per centrum *D*, satis id erit ad demonstrandum intentum."

33. For our purpose the references given in St. Thomas [24] are enough; specially interesting in these texts is *In Boeth. de Trinitate* q.5, a.1 ad 2. For a more complete information see J. Alvarez Laso, C.M.F.: *La Filosofía de las Matemâticas en Santo Tomâs*, Mêxico, Jus, 1952.

34. I quote the texts of Lambert [13] from the edition of Staeckel [21]. The English translation is mine.

"Hierbey scheint mir merkwürdig zu seyn, dass die zwote Hypothese statt hat, wenn man statt ebener Triangel *sphärische* nimmt, weil bey diesen sowohl die Summe der Winkel grösser als 180 Gr. als auch der Ueberschufs dem Flächenraume des Triangels proportional ist.

Noch merkwürdiger scheint es, dass, was ich hier von den sphärischen Triangeln sage, sich ohne Rücksicht auf die Schwierigkeit der Parallellinien erweisen lasse, und keinen andern Grundsatz voraussetzt, als dass jede durch den Mittelpunkt der Kugel gehende ebene Fläche die Kegel in zween gleiche Theile theile.

"Ich sollte daraus fast den Schluss machen, die dritte hypothese komme bey einer imaginären Kugelfläche vor. Wenigstens muss immer Etwas seyn, warum sie sich bey ebenen Flächen lange nicht so leicht umstossen lasst, als es sich bey der zwoten thun liess."

35. "Es wird aber die Wahrheit desselben auch aus allen Folgen, die in allen Absichten daraus gezogen werden, dergestalt erwiesen, einleuchtend und nothwendig, dass man diese Folgen, zusammengenommen, als eine auf vielfache Arten vollständige *Induction* ansehen kann."

36. "...*die Frage* ist, *ob derselbe aus den Euklidischen Postulatis mit Zuziehung seiner übrigen Grundsätze in richtiger Folge hergeleitet werden könne*? Oder, *wenn diese nicht hinreichend wären, ob sodann noch andre Postulata oder Grundsätze, oder Beydes könnten vorgebracht werden, die mit den Euklidischen gleiche Evidenz hätten, und aus welchen sein 11-ter Grundsatz erwiesen werden könnte*?

"Bey dem ersten Theile dieser Frage kann man nun von Allem, was ich im Vorhergehenden *Vorstellung der Sache* genennt habe, abstrahieren. Und da *Euklid's Postulata* und übrigen Grundsätze einmal mit Worten ausgedrückt sind: so kann und soll gefordert werden, dass man sich in dem Beweise nirgends auf die | Sache selbst berufe, sondern den Beweis durchaus symbolisch vortrage—wenn er möglich ist. In dieser Absicht sind *Euklid's Postulata* gleichsam wie eben so viele algebraische Gleichungen, die man bereits vor sich hat, und aus welchen x, y, z, & c herausgebracht werden soll, ohne dass man auf die Sache selbst zurücke sehe."

37. See in particular the letter (1765) of Lambert to G. J. von Holland quoted by Staeckel [21], pages 141-142.

38. As Staeckel says, precisely because Lambert realized this lack of rigor, he did not publish in his lifetime this beautiful and deep *Theorie der Parallellinien.*

39. See Staeckel [21], page 248, and Segre [18], article 8.

40. I quote the texts of Taurinus [23] and [22] from the partial edition of Staeckel [21]. The English translation is mine.

"Eine Geometrie, in welcher mehr als zwei Rechte im Dreieck enthalten sind, führt auf einen offenbaren Widerspruch mit dem Axiom der geraden Linie; denn in jedem System der Art wurden die geraden Linien sich in zwei Puncten schneiden, ohne zusammenzufallen."

41. "Wir haben gegen die Annahme eines solchen Systems als geradlinig noch folgendes einzuwenden:

"1. Es widerspricht aller Anschauung. Es ist wahr, ein solches System wurde im Kleinen die nemlichen Erscheinungen darbieten können, wie das Euklidische: allein, wenn die Vorstellung des Raumes als die blosse Form der äussern Sinne betrachtet werden darf, so ist unstreitig das Euklidische System das wahre und es lässt sich nicht annehmen, dass eine beschränkte Erfahrung eine sinnliche Tauschung erzeugen könne.

"3. Wäre das dritte system das wahre, so gäbe es überhaupt keine Euklidische Geometrie, da noch ihre Möglichkeit nicht geläugnet werden kann."

42. "Es lässt sich sehr leicht zeigen, dass ein geometrisches System, in welchem weniger als zwei Rechte im Dreieck enthalten sind, an sich nicht bestimmt ist, sondern eine besondere Bestimmungsgrösse oder Constante erfordert. Hieraus ergiebt sich sogleich, dass es a priori gar keine andere Geometrie, als die | Euklidische für uns giebt, weil eine solche Constante ganz willkührlich angenommen werden kann."

43. "Gauss hat einiges über den Gegenstand hinzugefuegt, woraus ich freilich seine Ansicht ueber die Sache nicht vollstaendig habe erraten koennen."

    This text is given by Staeckel [21] in page 248.

44. "Jede Geometrie, in welcher die Winkelsumme im Dreieck kleiner, als zwei Rechte, angenommen wird, enthaelt in sich selbst—dem Begriff nach—keinen Widerspruch...

    "Der Widerspruch muss darin gesucht werden, dass es nicht ein, sondern eine unendliche Menge von Systemen der Art giebt, von welchen jedes auf Gueltigkeit gleichen Anspruch haben wuerde;"

45. Gauss himself says explicitly in a letter to Schumacher (1846), ([8], page 238) that he began these investigations in 1792 [when he was 15 years old].

46. Gauss [8], page 159.

47. Gauss [8], page 165.

48. Gauss [8], page 169.

    "Es ist leicht zu beweisen, dass wenn Eulikds Geometrie nicht die wahre ist, es gar keine ähnliche Figuren gibt: die Winkel in einem gleichseitigen Dreieck sind dann auch nach der Groesse der Seite verschieden, wobei ich gar nichts absurdes finde. Es ist dann der Winkel Function der Seite und die Seite Function des Winkels, natürlicher Weise eine solche Function, in der zugleich eine constante Linie vorkommt. Es scheint etwas paradox, dass eine constante Linie gleichsam a priori moeglich sein koenne; ich finde aber darin nichts widersprechendes. Es waere sogar wuenschenswerth, dass die Geometrie Euklids nicht wahr waere, weil wir dann ein allgemeines Mass a priori haetten,..."

49. Gauss [8], page 177.

50. Gauss [8], page 187.

    "...ich glaube nicht, dass jemand sich eben mit diesem 2^n. Theil mehr beschaeftigt haben koenne als ich, obgleich ich niemals darueber etwas bekannt gemacht habe. Die Annahme, dass die Summe der 3 Winkel kleiner sei als 180°, fuehrt auf eine eigene, von der unsrigen (Euklidischen) ganz verschiedene Geometrie, die in sich selbst durchaus consequent ist, und die ich fuer mich selbst ganz befriedigend ausgebildet habe, so dass ich jede Aufgabe in derselben aufloesen kann mit Ausnahme der Bestimmung einer Constante, die sich a priori nicht ausmitteln laesst. ...

    "Alle meine Bemuehungen, einen Widerspruch, eine Inconsequenz in dieser Nicht-Euklidischen Geometrie zu finden, sind fruchtlos gewesen, und das Einzige, was unserm Verstande darin widersteht, ist, dass es, waere sie wahr, im Raum eine an sich bestimmte (obwohl uns unbekannte) Lineargroesse geben muesste. Aber mir deucht, wir wissen, trotz der nichtssagenden Wort-Weisheit der Metaphysiker eigentlich zu wenig oder gar nichts ueber das wahre Wesen des Raums, als dass wir etwas uns unnatuerlich vorkommendes mit Absolut Unmoeglich verwechseln duerfen. Waere die Nicht-Euklidische Geometrie die wahre, und jene Constante in einigem Verhaeltnisse zu solchen Grosssen, die im Bereich unserer Messungen auf der Erde oder am Himmel liegen, so liesse sie sich a posteriori ausmitteln."

51. Gauss [8], page 200:

"...und meine Ueberzeugung, dass wir die Geometrie nicht vollstaendig a priori begruenden koennen, ist, wo moeglich, noch fester geworden. Inzwischen werde ich wohl noch lange nicht dazu kommen, meine sehr ausgedehnten Untersuchungen darueber zur oeffentlichen Bekanntmachung auszuarbeiten, und vielleicht wird diess auch bei meinen Lebzeiten nie geschehen, da ich das Geschrei der Boeotier scheue, wenn ich meine Ansicht ganz aussprechen wollte."

52. Gauss [8], page 201:

"Nach meiner innigsten Ueberzeugung hat die Raumlehre in unserm Wissen a priori eine ganz andere Stellung, wie die reine Groessenlehre; es geht unserer Kenntniss von jener durchaus diejenige vollstaendige Ueberzeugung von ihrer Nothwendigkeit (also auch von ihrer absoluten Wahrheit) ab, die der letztern eigen ist; wir muessen in Demuth zugeben, dass, wenn die Zahl bloss unsers Geistes Product ist, der Raum auch ausser unserm Geiste eine Realitaet hat, der wir a priori ihre Gesetze nicht vollstaendig vorschreiben koennen. ..."

53. Gauss [8], pages 230 and 247. In this last letter Gauss writes that, through the difference of the concepts of right and left, "he finds a decisive refutation of Kant's presumption, that the space is *purely* the form of outer intuition."

54. Gauss [8], page 267-268.

55. Kant [11], *Transcendental Aesthetic*, pp. 65-91; see also *Transcendental Doctrine of Method*, Chapter I, Section I, where he takes as an example precisely proposition I,32 of the *Elements* stating that the sum of the angles of a triangle is two rights.

56. With respect to Lambert and Taurinus we have shown their conviction in §3. Gauss in the quoted letter to Taurinus (1824) says: "There is no doubt, that that impossiblity [of a sum greater than 180° for the three angles of a triangle] can be proved most rigorously."

Moreover in a *Note* (1828) ([8], page 190) he carries out such a pseudo-proof.

57. "*Scholion.* Sed juvat expendere hoc loco notabile discrimen inter praemissas duarum hypothesium redargutiones. Nam circa hypothesin anguli obtusi res est meridiana luce clarior; quandoquidem ex ea assumpta ut vera demonstratur absoluta universalis veritas controversi Pronunciati Euclidaei, ex quo postea demonstratur absoluta falsitas ipsius talis hypothesis; prout constat ex XIII. et XIV. hujus. Contra vero non devenio ad probandam falsitatem alterius hypothesis, quae est anguli acuti, nisi prius demonstrando; quod linea, cujus omnia puncta aequidistent a quadam supposita recta linea in eodem cum ipsa plano existente, aequalis sit ipsi tali rectae; quod ipsum tamen non videor demonstrare ex visceribus ipsiusmet hypothesis, prout opus foret ad perfectam redargutionem."

58. "*Scholion.* Atque his subsistere tutus possem. Sed nullum non movere lapidem volo, ut inimicam anguli acuti hypothesim, a primis usque radicibus revulsam, sibi ipsi repugnantem ostendam."

59. "*Scholion I.* Sed forte minus evidens cuipiam videbitur enunciata exactissima aequalitas inter illas infinitesimas *M*, et *K*. Quare ad avertendum hunc scrupulum sic rursum procedo."

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