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LOCATING VERTICES OF TREES

MARTIN M. ZUCKERMAN

Let X be a nonempty set and let R and S be binary relations on X. Let x, y, x_1, x_2, y_1, y_2 be arbitrary elements of X, then, where $\mathsf{R}(R)$ is the range of R and ω is the set of nonnegative integers, $\langle X, R, S\rangle$ is called a *(dyadic ordered) tree* if the following hold:

(1) If x_1Ry and x_2Ry, then $x_1 = x_2$.
(2) For each $x \in X$, xRy for at most two y.
(3) $X - \mathsf{R}(R)$ is a unit set, $\{x_0\}$.
(4) y_1Sy_2 iff (a) $y_1 \neq y_2$, (b) $y_2 \not S y_1$, and (c) for some $x \in X$, both xRy_1 and xRy_2.
(5) There exists a function l: $X \to \omega$ with the properties: (a) $l(x_0) = 0$ and (b) if xRy, then $l(y) = l(x) + 1$.

This definition, with minor modifications, is essentially the one given in [1].

If $\langle X, R, S\rangle$ is a tree, then the elements of X are called *points* or *vertices*. If xRy holds for a unique $y \in X$, x is called a *simple point*; if xRy holds for two distinct y, x is called a *junction point*. Whenever xRy then y is said to be an *immediate successor of* x. The relation S, in effect, selects one of the two immediate successors of a junction point. Thus if xRy_1, xRy_2 and y_1Sy_2, we say that y_1 is the *left successor* and y_2 the *right successor of* x.

$l(x)$ is called the *level of* x. l_n will denote the set of vertices of level n, $n \in \omega$. Each l_n has at most 2^n vertices; hence for any tree $\langle X, R, S\rangle$, X must be countable. Note that $\langle X, R, S\rangle$ has no junction points iff l is one-one iff $S = \emptyset$.

A *path* of a tree $\langle X, R, S\rangle$ is a finite sequence $[a_0, a_1, \ldots, a_n]$ or a denumerable sequence $[a_0, a_1, \ldots, a_n, \ldots]$ with the properties:

(1) for each a_k appearing in the sequence, $a_k \in X$ and
(2) if a_{k+1} also appears in the sequence, then a_{k+1} is an immediate successor of a_k.

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The path $[a_0, a_1, \ldots, a_n]$ is called a *path from* a_0 *to* a_n. For any distinct $x_i, x_j \in X$, we say x_i *precedes* x_j if there is a path from x_i to x_j.

Theorem 1. (*Induction Principle for Trees*) *Let* $\langle X\ R, S\rangle$ *be a tree. Let* A *be a subset of* X *which satisfies*: (1) $l_0 \subseteq A$ *and*, (2) *whenever* $l_n \subseteq A$, *then* $l_{n+} \subseteq A$, *then* $A = X$.

Theorem 2. *Let* $\mathscr{P}$ *be the set of positive integers. Let* $\langle X, R, S\rangle$ *be a tree. Then there is a unique function* $L : X \rightarrow \mathscr{P}$ *such that* (1) $L(x_0) = 1$ *and such that* (2) *whenever* $y \in \mathrm{R}(R)$ *and* xRy, *then* (a) $L(y) = 2L(x)$ *if either* x *is a simple point or* y *is the left successor of* x, *or* (b) $L(y) = 2L(y) + 1$ *if* y *is the right successor of* x.

Proof. For each $x \in X$, let $A(x)$ be the set of all $a \in x$ such that a is a right successor (of some junction point) and either $a = x$ or a precedes x. Let

$$L(x) = 2^{l(x)} + \sum_{a \in A(x)} 2^{l(x)-l(a)} .$$

We have

$$L(x_0) = 2^{l(x_0)} + \sum_{a \in \emptyset} 2^{l(x)-l(a)} = 1.$$

Let xRy and suppose that either x is a simple point or else y is the left successor of x. Then $A(y) = A(x)$, whereas,

$$l(y) = l(x) + 1. \tag{1}$$

Hence,

$$\begin{aligned} L(y) &= 2^{l(y)} + \sum_{a \in A(y)} 2^{l(y)-l(a)} \\ &= 2^{l(x)+1} + \sum_{a \in A(x)} 2^{l(x)+1-l(a)} \\ &= 2\left(2^{l(x)} + \sum_{a \in A(x)} 2^{l(x)-l(a)}\right) \\ &= 2L(x). \end{aligned}$$

Now suppose that y is the right successor of x. Then $A(y) = A(x) \cup \{y\}$; again, (1) holds.
This time

$$\begin{aligned} L(y) &= 2^{l(y)} + \sum_{a \in A(y)} 2^{l(y)-l(a)} \\ &= 2^{l(x)+1} + \sum_{a \in A(x)} 2^{l(x)+1-l(a)} + 2 \\ &= 2L(x) + 1. \end{aligned}$$

If $L^* : X \rightarrow \omega$ is any function satisfying conditions (1) and (2) of the present theorem, then by the Principle of Induction for trees it follows that $L^* = L$. For let $B = \{x \in X\colon\ L^*(x) = L(x)\}$. Then $l_0 \subseteq L$ because by (1), $L^*(x_0) = L(x_0) = 1$. Suppose $l_n \subseteq B$. If $l_{n+} = \emptyset$, then surely $l_{n+} \subseteq B$. Otherwise, let $y \in l_{n+}$ and let x be the unique vertex satisfying xRy. We have

$x \in L_n \subseteq B$; hence $L^*(x) = L(x)$. If x is a simple point or if y is the left successor of x, then

$$L^*(y) = 2L^*(x) = 2L(x) = L(y).$$

In case y is the right successor of x, then

$$L^*(y) = 2L^*(x) + 1 = 2L(x) + 1 = L(y).$$

Thus $l_{n+} \subseteq B$ and by theorem 1, $B = X$.

Corollary. $2^{l(x)} \leq L(x) < 2^{l(x)+1}$ *for all* $x \in X$.

By means of L we can locate elements of the tree; thus we call L the *location function* of the tree. L is especially useful in trees whose vertices are (occurrences of) (1) subformulas of a given formula or (2) probability events. In particular, in the case of an analytic tableau for a formula P (see [1]), various subformulas of P are repeated again and again. It might be convenient to index the subformulas of the tableau by their locations. Thus if Q is a subformula of P which occurs in the tableau, we replace each occurrence of Q by Q_n—so that $L(Q_n) = n$. Moreover, L can be used to specify the relations R and S (in the definition of "tree") in the following sense.

Theorem 3. *Let* X *be a countable set. Let* $L : X \to \mathcal{P}$ *be any one-one function satisfying* (a) $1 \in \mathrm{R}(L)$ *and* (b) *for any* $n \geq 1$, *whenever* $2n + 1 \in \mathrm{R}(L)$, *then* $2n \in \mathrm{R}(L)$ *and whenever* $2n \in \mathrm{R}(L)$, *then* $n \in \mathrm{R}(L)$. *Then there is a unique tree* $\langle X, R, S\rangle$ *for which*

(i) xRy *iff either* $L(y) = 2L(x)$ *or* $L(y) = 2L(x) + 1$, *and*
(ii) $y_1 S y_2$ *iff* $L(y_1)$ *is even and* $L(y_2) = L(y_1) + 1$.

Proof. Suppose R and S are binary relations on X defined by (i) and (ii) of theorem 3. In order to show that $\langle X, R, S\rangle$ is a tree we must show that clauses (1)-(5) of the definition of "tree" hold.

(1) Suppose $x_1 Ry$ and $x_2 Ry$. Then for $i = 1, 2$, $L(y) = 2L(x_i)$ or $L(y) = 2L(x_i) + 1$. Parity considerations assure that either $L(y) = 2L(x_1) = 2L(x_2)$ or else $L(y) = 2L(x_1) + 1 = 2L(x_2) + 1$. Thus $x_1 = x_2$ because L is one-one.

(2) follows from (i).

(3) According to (a), $1 \in \mathrm{R}(L)$; since L is one-one, there is a unique element x_0 in X such that $L(x_0) = 1$. $1 \notin \mathrm{R}(R)$, by (i). Suppose $x \in X - \{x_0\}$. Then $L(x) > 1$; hence for some $n \geq 1$, $L(x) = 2n$ or $L(x) = 2n + 1$. Thus $\{n, L(x)\} \subseteq \mathrm{R}(L)$ by (b), and by (i), $L^{-1}(n)Rx$. Consequently, $X - \mathrm{R}(R) = \{x_0\}$.

(4) Suppose $y_1 S y_2$. Then $L(y_2) = L(y_1) + 1$. Since L is a function, $y_1 \neq y_2$. Were $y_2 S y_1$ as well as $y_1 S y_2$ to hold, then $L(y_1) = L(y_2) + 1 = L(y_1) + 2$. Contradiction! Finally, let $L(y_1) = 2n$. Then $n \in \mathrm{R}(L)$ and $L^{-1}(n)$ is unique. $L(y_1) = 2L(L^{-1}(n))$ and $L(y_2) = 2L(L^{-1}(n)) + 1$; hence $L^{-1}(n)Ry_1$ and $L^{-1}(n)Ry_2$.

Now suppose that (a), (b), and (c) of (4) hold, and let x be as in (c). Then for $i = 1, 2$, xRy_i; hence by (i), $L(y_i) = 2L(x)$ or $L(y_i) = 2L(x) + 1$. Since L is one-one and $y_1 \neq y_2$, we have either $L(y_1) = 2L(x)$ and $L(y_2) =$

$2L(x) + 1$, or else $L(y_2) = 2L(x)$ and $L(y_1) = 2L(x) + 1$. In the latter case, by (ii), we would have y_2Sy_1, contradicting (b) of (4). Thus the former case holds and y_1Sy_2.

(5) Define $l : X \to \omega$ by $l(x) = m$ if $2^m \leq L(x) < 2^{m+1}$. Then $l(x_0) = 0$ because $L(x_0) = 2$. Let xRy. First suppose $L(y) = 2L(x)$. Then $2^m \leq L(x) < 2^{m+1}$ iff $2^{m+1} \leq L(y) < 2^{m+2}$. Now suppose $L(y) = 2L(x) + 1$. Then $2^m \leq L(x) < 2^{m+1}$ iff $2^{m+1} \leq L(y) - 1 < 2^{m+2}$ iff

$$2^{m+1} < L(y) < 2^{m+2} + 1 \ . \tag{2}$$

Since $L(y)$ is odd, (2) holds iff

$$2^{m+1} \leq L(y) < 2^{m+2} \ .$$

Thus in either case for $L(y)$, we have $L(y) = l(x) + 1$.

The uniqueness of the tree $\langle X, R, S\rangle$ follows from the fact that the relations R and S are completely determined in terms of the given function L.

REFERENCE

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City College of the City University of New York
New York, New York