

ON A PROBLEM OF TH. SKOLEM

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1. Introduction. As pointed out in [2] the standard definition of an ordered pair, viz. $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$, does not generalize in a natural way to ordered n -tuples. For example, the candidate $\langle x_1, x_2, x_3 \rangle = \{\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\}$ is no good since this gives $\langle x, y, y \rangle = \langle x, x, y \rangle$. The standard generalization to n -tuples is given by $\langle x_1 \rangle = x_1$, $\langle x_1, \dots, x_{n+1} \rangle = \langle \langle x_1, \dots, x_n \rangle, x_{n+1} \rangle$. However, this definition has the unusual property that every n -tuple is also an m -tuple for $2 \leq m \leq n$. Also if x_1, x_2, x_3 are of type k in simple type theory, then $\langle x_1, x_2 \rangle$ is of type $k + 2$, hence $\langle x_1, x_2, x_3 \rangle = \langle \langle x_1, x_2 \rangle, x_3 \rangle$ is not type-theoretically well-defined.

The generalizations proposed in [2] are rather awkward in form. In this paper we offer several solutions to Skolem's problem of finding a "best" definition for ordered n -tuples. The idea is to start with some new definitions of "ordered pair" which in turn do generalize in several natural ways, the "best" choice depending upon what conditions we wish ordered n -tuples to satisfy. Some possible conditions are as follows:

- (C1) $\langle x_1, \dots, x_n \rangle = \langle y_1, \dots, y_n \rangle \implies x_i = y_i$ for $1 \leq i \leq n$;
- (C2) all n -tuples ($n \geq 2$) are actually 2-tuples;
- (C3) $m \neq n \implies \langle x_1, \dots, x_m \rangle \neq \langle y_1, \dots, y_n \rangle$;
- (C4) in simple type theory, if x_1, \dots, x_n are of the same type, then $\langle x_1, \dots, x_n \rangle$ is well-defined.

Of course we want all definitions to satisfy C1. Conditions C2 and C3 are clearly mutually exclusive. C2 is a property possessed by the standard definition of ordered n -tuples, whereas C3 is closer to the intuitive notion of n -tuples. Condition C4 was considered in [2].

Let T_0 be a pure set or set-class theory satisfying the axioms of extensionality and pair set, $T_1 = T_0 +$ null set axiom, and $T_2 = T_1 +$ adjoining set axiom ($x, y \in V \implies x \cup \{y\} \in V$). Small Roman letters denote set variables. Finally, let $x^{[0]} = x$, $x^{[n+1]} = \{x^{[n]}\}$ for $n \geq 0$.

2. First Definition. Consider the basic definition $\langle x, y \rangle = \{\{\emptyset, x\}, \{y\}\}$ which trivially satisfies C1 for case $n = 2$. Several possible generalizations are now defined.

- (a) $\langle x_1 \rangle = \{\emptyset, x_1\}$
 $\langle x_1, \dots, x_{n+1} \rangle = \{\langle x_1, \dots, x_n \rangle, \{x_{n+1}\}\}.$
- (b) $\langle x_1 \rangle = x_1$
 $\langle x_1, \dots, x_{n+1} \rangle = \langle \langle x_1, \dots, x_n \rangle, x_{n+1} \rangle.$
- (c) $\langle x_1 \rangle = \{\emptyset, x_1\}$
 $\langle x_1, \dots, x_{n+1} \rangle = \{\langle x_1, \dots, x_n \rangle, x_{n+1}^{[n]}\}.$

Theorem. *Definitions (a)-(c) are well-defined in T_1 and satisfy C1; in addition (b) satisfies C2 and (c) satisfies C3. Also (c) satisfies C4 if when x_1, \dots, x_n are of type k , then \emptyset represents the null set of type k ; in this case $\langle x_1, \dots, x_n \rangle$ is a set of type $k + n$.*

Proof. All claims are proved by induction. Proof of C1 for (b) is easy; proofs for (a) and (c) use the result

$$(1) \quad (\exists y) (\langle x_1, \dots, x_n \rangle = \{y\}) \implies \langle x_1, \dots, x_n \rangle = \emptyset^{[n]} .$$

C2 obviously holds for (b). To show C3 holds for (c) use induction on $n \geq 1$, proving with the aid of (1) that

$$(\forall m) (m > n \implies \langle x_1, \dots, x_m \rangle \neq \langle y_1, \dots, y_n \rangle) .$$

Remarks. For (a), C2 fails since $\langle \emptyset, \{\emptyset\}, \emptyset \rangle \neq \langle x, y \rangle$ for any x, y and C3 fails since $\langle \emptyset, \emptyset, \emptyset \rangle = \langle \emptyset, \{\emptyset\} \rangle$. However, among all possible definitions of an n -tuple satisfying C1, (a) probably gives the simplest possible unabbreviated expression; e.g.,

$$\langle x_1, x_2, x_3, x_4 \rangle = \{ \{ \{ \emptyset, x_1 \}, \{x_2\} \}, \{x_3\} \}, \{x_4\} \} .$$

Clearly (b) is just the standard generalization to n -tuples of the new definition of ordered pair. Generalization (c) besides having the very desirable properties C3 and C4 also has a simple unabbreviated expression; e.g.,

$$\langle x_1, x_2, x_3, x_4 \rangle = \{ \{ \{ \emptyset, x_1 \}, \{x_2\} \}, \{ \{x_3\} \} \}, \{ \{ \{x_4\} \} \} \} .$$

For what it's worth, for (c) the n -type $\langle \emptyset, \emptyset, \dots, \emptyset \rangle$ is the set $\emptyset^{[n]}$ which is the Zermelo integer n . The standard definition of n -tuple does have the advantage over (a)-(c) of being well-defined even in T_0 .

3. *Second Definition.* An often suggested definition of $\langle x, y \rangle$ is $\{\{1, x\}, \{2, y\}\}$. This satisfies C1 for $n = 2$. But as is well known C1 fails for $n = 3$ if we use the natural generalization $\langle x, y, z \rangle = \{\{1, x\}, \{2, y\}, \{3, z\}\}$; e.g., $\langle 2, 1, 2 \rangle = \langle 2, 3, 2 \rangle$. The problem is removed if instead one defines $\langle x, y \rangle = \{\{1, \{x\}\}, \{2, \{y\}\}\}$ or alternatively $\{\{0, \{x\}\}, \{1, \{y\}\}\}$. Generalizing the alternate definition to n -tuples we get $\langle x_0, \dots, x_{n-1} \rangle = \{\{i, \{x_i\}\} : i < n\}$. This is well-defined in T_2 . But while we are at it, we can just as well define δ -sequences for any ordinal δ by $\langle x_\alpha \rangle_{\alpha < \delta} = \{\{\alpha, \{x_\alpha\}\} : \alpha < \delta\}$. For $i = 0, 1, 2$ let $T'_i = (\text{class-set theory } T_i) + (\text{class existence theorem for case } n = 1) + (\text{subset axiom } X \subseteq Y \in V \implies X \in V)$. Then, as shown in [1], many properties of the ordinals can be developed in T'_0 , in fact a sufficient amount so that the definitions and proofs of properties of δ -sequences can be carried out within T'_0 . (Of course, if we want to insure the existence of any 2-tuples or

n -tuples for $n \geq 3$, then we must work in T'_1 or T'_2 respectively.) Let T be T'_0 .

Definitions. A set x is a δ -sequence (abbrev.: $\text{Seq}_\delta(x)$) iff

- (i) $(\forall v)_x (\exists \alpha)_\delta (\exists u) v = \{\alpha, \{u\}\}$, and
- (ii) $(\forall \alpha)_\delta (\exists_1 u) \{\alpha, \{u\}\} \in x$;

x is a sequence (abbrev.: $\text{Sq}(x)$) iff x is a δ -sequence for some ordinal δ . The definition in T of the α -th component x_α of a δ -sequence x is given by

$$x_\alpha = \{u : \text{Sq}(x) \cdot (\exists v) (u \in v \cdot \{\alpha, \{v\}\} \in x) \}.$$

Lemma. $\vdash (\forall u, v, \alpha, \beta) (\{\alpha, \{u\}\} = \{\beta, \{v\}\} \implies \alpha = \beta \cdot u = v)$ in T .

Proof. Show $\{x, \{u\}\} = \{y, \{v\}\} \implies x = y$ and $u = v$ for any x, y which aren't singletons of non-empty sets. Then lemma follows since $\alpha = \{v\} \implies v = \emptyset$ by transitivity and ϵ -irreflexivity of ordinals.

The result that our generalized definition of δ -sequences satisfies C1 and C3 follows easily from the lemma and is formally stated as follows:

Theorem. $\vdash (\text{Sq}(x) \cdot \text{Sq}(y) \cdot x = y \implies (\exists \delta) (\text{Seq}_\delta(x) \cdot \text{Seq}_\delta(y) (\forall \alpha)_\delta x_\alpha = y_\alpha))$ in T .

Definition. The Cartesian product of two classes X, Y is defined in T'_1 by

$$X \times Y = \{z : (\exists u, v) (z = \langle u, v \rangle \cdot u \in X \cdot v \in Y)\} ;$$

the generalized Cartesian product $\prod_{\alpha < \delta} x_\alpha$ of a family $\{x_\alpha : \alpha \in \delta\}$ of sets is defined formally by

$$\prod x = \{s : (\exists \delta) (\text{Seq}_\delta(x) \cdot \text{Seq}_\delta(s) \cdot (\forall \alpha)_\delta s_\alpha \in x_\alpha)\}.$$

The generalized Cartesian product is a true generalization of the Cartesian product because of the

Theorem. $\vdash \prod_{\alpha < 2} x_\alpha = \prod \langle x_0, x_1 \rangle = x_0 \times x_1$ in T'_1

REFERENCES

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- [2] Skolem, Th., *Two remarks on set theory*, Math. Scand., 5 (1957), pp. 40-46.

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