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CERTAIN EXTENSIONS OF MODAL SYSTEM S4

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In this paper I present some investigations concerning certain new proper, or probably proper, extensions of Lewis modal system S4. These researches are mostly based on the results in the field of modal logic recently obtained by Schumm, Thomas, Zeman and, indirectly, by Grzegorczyk. There are several open problems connected with the deductions given below, which I was unable to solve. On the other hand, several other of my results related to the topic of this article will not be published here, but will be discussed in a subsequent paper. An acquaintance with modal logic, Łukasiewicz's notation, my method of writing proofs and, especially, with papers [15] and [14] is presupposed.

1 INTRODUCTION

1.1 In [16], [15] and [14] I introduced an enumeration of proper axioms of systems which are the extensions of S4. Subsequently, this enumeration was used by some other authors, but a development of this subject created an inconvenient chaos. For instance, formula $\mathfrak{C}LMpMLp$ which serves as a base for a definition of family $\boldsymbol{K}$ of the non-Lewis modal systems has number *K2* instead of the much more convenient number *K1*. For this reason I decided to change this enumeration, as follows. The letter prefixed to the proper axioms of the given system will remain the same, as in my previous papers, but they will be bold. And, the bold numbers attached to such letters will indicate the different formulas each of which can be adopted as the proper axioms of the discussed system. The proper axiom of the fixed system which for this or that reason I consider, as its principal proper axiom, will always have number **1**. Thus, e.g., the formula *K2* mentioned above will have number **K1**, and McKinsey's formula, *cf*, [5], and [16], p. 77, which previously had number *K1* will be **K2**. A list of this new enumeration is given below:

1) Zeman's system S4.04 (= {S4; **L1**}), *cf.* [18], p. 250:

L1 $\mathfrak{C}LMLpCpLp$

2) Systems S4.1 (= {S4; **N1**}) and S4.1.1 (= {S4; **M1**}), *cf.* [15], p. 306.

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N1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppCMLpp$
M1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCMLpLp$

3) System S4.2 (= {S4;**G1**}):

G1 $\mathfrak{C}MLpLMp$
G2 $\mathfrak{C}MLpLMLp$ [In [16], pp. 73 - 74, formula *L1*]

It is observed by P. T. Geach, *cf.* [1], p. 252, that in the field of S4 these formulas **G1** and **G2** are equivalent.

4) System S4.3 (= {S4;**D1**});

D1 $A\mathfrak{C}Lpq\mathfrak{C}Lqp$ [Previously *D2*]
D2 $A\mathfrak{C}LpLq\mathfrak{C}LqLp$ [Previously *D1*]
D3 $LA\mathfrak{C}Lpq\mathfrak{C}Lqp$
D4 $LA\mathfrak{C}LpLq\mathfrak{C}LqLp$
D5 $\mathfrak{C}KMpMqAMKpMqMKqMp$
[**D5** is Hintikka's axiom of S.4.3, *cf.* [10], p. 176]

The inferential equivalence of **D1** - **D4** in the field of S4 is shown in [15], p. 75.

5) System S4.3.2 (= {S4;**F1**}) of Zeman, *cf.* [18], pp. 296-298:

F1 $A\mathfrak{C}LpqCMLqp$ [In [18], formula (34)]
F2 $A\mathfrak{C}LpLqCLMLqLp$ [In [18], formula (35)]

Equivalence of **F1** and **F2** is proved in [18].

6) System S4.4 (= {S4;**R1**}), *cf.* [15], p. 305:

R1 $\mathfrak{C}MLpCpLp$
R2 $\mathfrak{C}NpCMpLMp$ [Previously, *R1**]

In [15] and [14] instead of *R1* I used its less convenient form $\mathfrak{C}pCMLpLp$. An equivalence of this form, **R1** and **R2** is self evident.

7) System S5 (= {S4;**C1**}) of Lewis:

C1 $\mathfrak{C}MpLMp$ [In [7], p. 497, axiom C11]
C2 $\mathfrak{C}MLpLp$
C3 $\mathfrak{C}LMLpLp$ [Previously *P1*]

Obviously, **C1** is axiom C11 of Lewis system S5, and **C2** is only another form of **C1**. In [1], Dummett and Lemmon have proved metalogically, and in [16], p. 74, it was shown logically, that in the field of S4 **C3** is equivalent to **C1**.

8) Brouwerian axiom:

B1 $\mathfrak{C}pLMp$ [In [7], p, 497, axiom C12]
B2 $\mathfrak{C}MLpp$ [**B2** is only another form of **B1**]

Although in the field of S4 **B1** and **C1** are inferentially equivalent and, therefore, **B1** can be considered as the proper axiom of S5, I prefer to give a special enumeration to it, because of the different properties which **C1** and **B1** have in the field of systems weaker than S4.

9) System VI (= {S4;**V1**}), *cf.* [15], pp. 306-307 and p. 309:

V1 *ALpAℭpqℭpNq*

10) System K1 (= {S4;K1}):

K1 *ℭLMpMLp* [Previously *K2*]
K2 *ℭKLMpLMqMKpq* [Previously *K1*]
K3 *LMCMpLp*
K4 *LMLCpLp*
K5 *LMLCMpp*

Concerning the systems K1 - K4, *cf.* [5], [16], [14], [9] and [3], pp. 265-267. In [16], pp. 77-78, it was proved that in the field of S4 formulas **K1** - **K5** are inferentially equivalent.

10) System K1.1 (= {S4;**J1**}):

J1 *ℭℭℭpLppp*
J2 *ℭℭℭpLpLpLp*

In [14], p. 316 and p. 314, system K1.1 is defined, and the equivalence of **J1** and **J2** is proved.

11) System K1.2 (= {S4;**H1**}):

H1 *ℭpℭMpp*
H2 *ℭLMpCpLp*

In [14], p. 316, system K1.2 is defined. The equivalence of **H1** and **H2** in the field of S4 will be established in Section 2 of this paper.

12) System K4 (= {S4;**P1**} ⇄ {S4.4;**K1**}):

P1 *ℭMLMpCpLp*

In section 2 it will be shown that **P1** is a proper axiom of K4.

13) Modal formula of Grzegorczyk:

T1 *ℭℭℭpqqCℭℭℭNpqqq*

While doing some researches unconnected with modal logic, Grzegorczyk found formula **T1** and recognized that it is a modal formula unprovable in the field of Lewis modal systems. Clearly, **T1** belongs to family *K* of the non-Lewis modal systems. The investigations concerning **T1** will be given in Section **2**.

14) Schumm's system S4.7, *cf.* [12]:

Q1 *AℭMLpLpℭMLMqCqLq*

15) System S4.6

S1 *AℭMLpLpALqAℭqrℭqNr*

16) System S4.5

E1 $A\mathfrak{C}MLpLpALqA\mathfrak{C}qp\mathfrak{C}qNp$

E2 $A\mathfrak{C}MLpLpALpA\mathfrak{C}pq\mathfrak{C}pNq$

The systems S4.5, S4.6 and S4.7 which are between S4.4 and S5 and the formulas **Q1**, **S1**, **E1** and **E2** will be discussed in Section 3 of this paper.

1.2 An enumeration of the systems which are extensions of S4 is established in [16], [15], [14], [18], [19] and [12]. And, although in the future some modifications will be necessary, it remains unaltered in this paper. Only, the numbers for the new systems will be added, and, since in [17] Thomas has proved that systems K4 and K5, *cf.* [14], p. 316, are equivalent, K5 will not be used any more in the old sense. A system which in [15], pp. 306-307, is defined as {S5; V1} will be designated here, as system V2.

1.3 An acquaintance with 4, 8 and 16 valued ordinary logical Boolean matrices for functors C and N is presupposed. These matrices are given explicitly in [15]. In this paper I shall use the following matrices which are presented here only for functors M and L:

	p	1	2	3	4
$\mathfrak{M}1$	Mp	1	2	1	4
	Lp	1	4	3	4

	p	1	2	3	4
$\mathfrak{M}2$	Mp	1	1	1	4
	Lp	1	4	4	4

	p	1	2	3	4	5	6	7	8
$\mathfrak{M}3$	Mp	1	1	1	1	1	1	1	8
	Lp	1	8	8	8	8	8	8	8

	p	1	2	3	4	5	6	7	8
$\mathfrak{M}4$	Mp	1	1	3	4	1	1	3	8
	Lp	1	6	8	8	5	6	8	8

	p	1	2	3	4	5	6	7	8
$\mathfrak{M}5$	Mp	1	2	3	4	1	2	3	8
	Lp	1	6	7	8	5	6	7	8

	p	1	2	3	4	5	6	7	8
$\mathfrak{M}6$	Mp	1	2	1	2	1	2	1	8
	Lp	1	8	7	8	7	8	7	8

	p	1	2	3	4	5	6	7	8
$\mathfrak{M}7$	Mp	1	1	1	1	5	5	5	8
	Lp	1	4	4	4	8	8	8	8

	p	1	2	3	4	5	6	7	8
$\mathfrak{M}8$	Mp	1	1	1	4	1	1	1	8
	Lp	1	8	8	8	5	8	8	8

	p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathfrak{M}9$	Mp	1	1	1	1	5	6	7	8	1	1	1	1	5	6	7	16
	Lp	1	10	11	12	16	16	16	16	9	10	11	12	16	16	16	16

	p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathfrak{M}10$	Mp	1	1	1	4	5	5	5	8	1	1	1	1	13	13	13	16
	Lp	1	4	4	4	13	16	16	16	9	12	12	12	13	16	16	16

	p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathfrak{M}11$	Mp	1	1	1	4	5	5	5	8	1	1	1	4	5	5	5	16
	Lp	1	12	12	12	13	16	16	16	9	12	12	12	13	16	16	16

	p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\mathfrak{M}12$	Mp	1	2	3	4	5	6	7	8	1	10	3	12	5	14	7	16
	Lp	1	10	3	12	5	14	7	16	9	10	11	12	13	14	15	16

In all these matrices 1 is the designated value. And, each of these matrices verifies S4.

Matrices $\mathfrak{M}1$ and $\mathfrak{M}2$ (in [15] they are $\mathfrak{M}2$ and $\mathfrak{M}3$) are familiar Groups II and III of Lewis-Langford, *cf.* [7], p. 493. $\mathfrak{M}3$ and $\mathfrak{M}4$ (in [15] $\mathfrak{M}5$ and $\mathfrak{M}6$) are mine. $\mathfrak{M}5$ (in [15] $\mathfrak{M}7$) is due to Parry, *cf.* [8], and also [9], §§1-3. $\mathfrak{M}6$ was constructed by Zeman, *cf.* [18], p. 297. Matrices $\mathfrak{M}7$ and $\mathfrak{M}8$ (in [12] $\mathfrak{M}'$ and $\mathfrak{M}''$) were found by Schumm. $\mathfrak{M}9$ is defined by Prior in [9], but is published explicitly, as $\mathfrak{M}8$, in [15], p. 310. $\mathfrak{M}10$ is mine. Matrix $\mathfrak{M}11$ is a translation of a set-theoretical construction used by McKinsey and Tarski in [6], p. 7, into a matrix form. I found an important matrix $\mathfrak{M}12$ checking some matrices among several thousand 16 valued modal matrices which for my researches Professor T. W. Scharle of West Virginia University (Morgantown) kindly computed using Computer IBM System 360/75 of that University.

2 FAMILY *K*

In this Section I shall give several proofs related to the structure of family *K*. On the other hand, several open problems connected with formula **T1** of Grzegorczyk will be presented. A familiarity with the definition of family *K* of the non-Lewis modal systems, *cf.* [14], is presupposed.

2.1 System K4. In [14] K4 is defined as a non-Lewis modal theory generated by an addition of **K1** to S4.4. It will be shown that the following formula

P1 $\mathfrak{C}MLMpCpLp$

is a proper axiom of K4.

2.1.1 Assume system K4. Hence we have S4, **R1** and **K1**. Then:

Z1	$\mathfrak{C}MLMpMMLp$	[**K1**; S2°]
Z2	$\mathfrak{C}MLMpMLp$	[*Z1*; S4°]
P1	$\mathfrak{C}MLMpCpLp$	[**R1**; *Z2*; S1°]

Whence, **P1** is a thesis of K4.

2.1.2 Now, let us assume S4 and **P1**. Then:

Z1	$\mathfrak{C}pMp$	[S1]
Z2	$\mathfrak{C}MLpMLMp$	[*Z1*; S2°]
R1	$\mathfrak{C}MLpCpLp$	[*Z2*; **P1**; S1°]
H2	$\mathfrak{C}LMpCpLp$	[*Z1*, p/LMp; **P1**; S1°]
Z3	$\mathfrak{C}LMp\mathfrak{C}pLp$	[**H2**; S2°; S4]
Z4	$\mathfrak{C}LMpCLMpLMLp$	[*Z3*; S3°]
K1	$\mathfrak{C}LMpMLp$	[*Z4*; S1]

Thus, S4 together with **P1** implies **R1** and **K1**. Therefore, it has been proved that {K4} ⇄ {S4.4; **K1**} ⇄ {S4; **R1**; **K1**} ⇄ {S4; **P1**}. Hence **P1** is a proper axiom of K4. By the way **H2** is proved, *cf.* system K1.2.

2.2 {K4} ⇄ {S3; **P1**}. Obviously, {K4} → {S3; **P1**}. Now, assume S3 and **P1**. Then:

Z1 $\mathfrak{C}pCMLMpLp$ [P1; S1°]
Z2 $\mathfrak{C}CMpLq\mathfrak{C}pq$ [S1; *cf.* [13], p. 156]
Z3 $\mathfrak{C}p\mathfrak{C}LMpp$ [*Z1*; *Z2*, *p/LMp*, *q/p*; S1°]
Z4 $\mathfrak{C}\mathfrak{C}MpLqLLCpq$ [*Z2*; S2°]
Z5 *LLCpCLMMpMp* [*Z4*, *q/CLMMpMp*; *Z3*, *p/Mp*]

Since in [8], p. 148, Parry has proved that an addition of any formula of the form *LLα* to S3 yields S4, the proof is completed.

2.3 System VI. In [17] Thomas has proved that system K5 which in [14] was defined as {V1; **K1**} is inferentially equivalent to K4, i.e., he has shown that formula

V1 *ALpACpqCpNq*

is a consequence of K4. From this important result of Thomas it follows that system V1 which in [15], p. 306, was defined as {S4; V1} is a subsystem of K4. Below, I shall make some remarks which will be used in Section 3.

2.3.1 The deductions given below are a formalized version of Thomas' proof that {K4} → {V1}. In Section **3, 3.4.1**, a modification of this version, but much more complicated and longer will be used for some purposes. Let us assume system K4. Hence, we have S4, **R1**, **K1** and, *cf.* [15], p. 307, **G1**. Then:

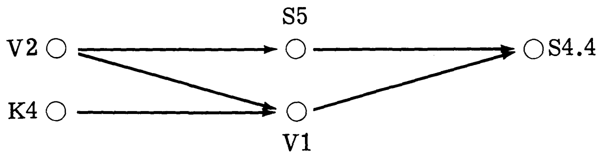
Z1 $\mathfrak{C}KpMLpLp$ [**R1**; S1°]
Z2 $\mathfrak{C}MLpLMp$ [**G1**; **K1**; S1°]
Z3 $\mathfrak{C}MCpqCLpMq$ [S2°; *cf.* [16], p. 71]
Z4 $\mathfrak{C}MLCpqCMLpMLq$ [S3°; *Z3*; *Z2*; S4°]
Z5 $\mathfrak{C}MLqMLCpq$ [S2°]
Z6 $\mathfrak{C}MLNpMLCpq$ [S2°]
Z7 $\mathfrak{C}NMLpMLCpq$ [*Z6*; S1°; *Z3*]
Z8 $\mathfrak{C}CMLpMLqMLCpq$ [*Z7*; *Z5*; S1°]
Z9 $\mathfrak{C}CMLpMLqMLCpq$ [*Z4*; *Z8*; S1°]
Z10 *AKprAKCpqCrsKCpNqCrNs* [S1°; *cf.* [17]]
Z11 *ALpAKCpqMLCpqKCpNqCMLpNMLq* [*Z10*, *r/MLp*, *s/MLq*; *Z1*; *Z9*; S1°]
Z12 $ALpA\mathfrak{C}pqKCpNqCMLpMLq$ [*Z11*; *Z1*, *p/Cpq*; S1°; *Z2*, *p/Np*]
Z13 $ALpA\mathfrak{C}pqKCpNqMLCpNq$ [*Z12*; *Z9*, *q/Nq*; S1°]
V1 $ALpA\mathfrak{C}pq\mathfrak{C}pNq$ [*Z13*; *Z1*, *p/CpNq*; S1°]

Thus, {K4} → {**V1**}.

2.3.2 In [15], p. 309, it was proved that in the field of S2 **V1** yields **R1** and, therefore, S4.4 is a subsystem of V1. Schumm's matrix $\mathfrak{M}8$ which verifies S4.4 falsifies **V1** for *p*/2 and *q*/3: $AL2A\mathfrak{C}23\mathfrak{C}2N3 = A8AL3\mathfrak{C}26 = CN8CN8L5 = C1C15 = C15 = 5$. Hence, system V1 is a proper extension of S4.4. Matrix $\mathfrak{M}3$ verifies S5, but falsifies **V1** for *p*/2 and *q*/3: $AL2A\mathfrak{C}23\mathfrak{C}2N3 = CN8CNL3L5 = C1CN85 = C1C15 = C15 = 5$. Hence, system **V1** is not contained in S5. Matrix $\mathfrak{M}1$ verifies S5 and **V1** which shows that V1 is a subsystem of {V2} = {S5; **V1**}. On the other hand, matrix $\mathfrak{M}10$ verifies **V1**, but it falsifies the proper axioms of S5 and K4. Namely, **C1** is rejected for *p*/8: ℭ*M*8*LM*8 = ℭ8*L*8 = ℭ816 = *L*9 = 9, and **P1** for *p*/2: ℭ*MLM*2*C*2*L*2 = ℭ*ML*1*C*24 = *LCM*13 = *LC*13 = *L*3 = 4. Therefore, system V1 includes neither S5 nor K4.

Thus, system V1 is a proper subsystem of K4 and V2 and a proper extension of S4.4. And, moreover, it is entirely independent from S5. From this it follows clearly that, although K4 contains V1, the latter system does not belong to family ***K***. In fact, it belongs to another family of the non-Lewis modal systems which are extensions of S4. However, this family, which I call family $\mathcal{V}$, will not be discussed in this paper.

The following diagram explains the position of V1 in regard to V2, S5, K4 and S4.4:



However, it will be proved in Section 3 that this diagram should be substituted by a much more complicated one.

2.4 System K3.2. In [18], pp. 296-298, Zeman has shown that there is an extension of S4, which he called S4.3.2, such that it is a proper extension of S4.3, and it is properly contained in S4.4. Moreover, it is neither contained in nor does it contain S4.3.1. This new system which was unknown at the time when [15] was published is generated by the addition of the following formula

F1 *A*ℭ*LpqCMLqp*

as a new axiom, to S4. Zeman has also remarked that an addition of the proper axiom of S4.1.1:

M1 ℭℭℭ*pLpLpCMLpLp*

to S4.3.2 yields S4.4. This means, since S4.3.2 contains S4.2, cf. [15], p. 305, that an addition of the proper axiom of S4.1:

N1 ℭℭℭ*pLppCMLpp*

to this system gives the same result. On the other hand, Zeman's matrix 𝔐6 which verifies S4.3.2 and rejects systems S4.4 and S4.3.1, and which falsifies **J1** for *p*/5: ℭℭℭ5*L*555 = ℭℭ*LC*5755 = ℭ*LCL*355 = *LCL*15 = *LC*15 = *L*5 = 7 verifies **K1**. Hence, an addition of **K1** to S4.3.2 generates a new system which belongs to family ***K*** and which I call system K3.2. Matrix 𝔐4 which verifies system K3.1 and, therefore, also K3, falsifies formula **F1** for *p*/5 and *q*/2: *A*ℭL52*CML*25 = *A*ℭ52*CM*65 = *ALC*52*C*15 = *CNL*25 = *CN*65 = *C*35 = 5. Hence, system K3.2 is a proper extension of K3 and, since 𝔐6 rejects **R1** for p/3: ℭ*ML*3*C*3*L*3 = ℭ*M*7*C*37 = *LC*15 = *L*5 = 7, it is a proper subsystem of K4. On the other hand, K3.2 neither contains nor is contained in K3.1. Thus, system K3.2 is a fullfledged member of family ***K***.

2.5 Systems K1.2 and S4.04. In [19], pp. 249-251, Zeman constructed another proper system, called S4.04, between S4 and S4.4 by adding to S4 the following formula:

L1 *ℭLMLpCpLp*

as a new axiom. It is obvious that **L1** is a simple consequence of **R1** and S2. Therefore, S4.4 contains S4.04. As Zeman points out, S4.04 is a proper subsystem of S4.4, because matrix 𝔐5 which, *cf.* [9], falsifies formula **G1**, and, therefore, also **R1** verifies **L1**. Matrix 𝔐9 which falsifies **L1** for *p*/2: *ℭLML*2*C*2*L*2 = *ℭLM*10*C*210 = *ℭL*19 = *LC*19 = *L*9 = 9 shows that S4.04 is a proper extension of S4.

It is self evident that an addition of **L1**, as a new axiom to any extension of S4 which contains the proper axiom of S4.2, viz.:

G1 *ℭMLpLMp*

yields S4.4. Hence, in the field of family *K* an addition of **L1**, as a new axiom could be interesting only in regard to the systems K1, K1.1 and K1.2. I shall prove that an addition of **L1** to K1 or K1.1 gives K1.2, and that system K1.2 contains **L1**. This last result is very important for some deductions which will be given in the next part **2.6**. By the way, it will be proved here, as mentioned in Section 1, that in the field of S4 the formulas **H1** and **H2** are inferentially equivalent.

2.5.1 Obviously, S4.04 does not contain the proper axiom of K1, i.e. axiom **K1**. And it is confirmed by 𝔐11 which verifies S4.04, but falsifies **K1** for *p*/2: *ℭLM*2*ML*2 = *ℭL*1*M*12 = *LC*14 = *L*4 = 12. On the other hand, matrix 𝔐4 which verifies systems K1 and K1.1 falsifies **L1** for *p*/2: *ℭLML*2*C*2*L*2 = *ℭLM*6*C*26 = *CL*15 = *LC*15 = *L*5 = 5. Thus, in the field of S4, **L1** is not a consequence of K1 and even of K1.1, Hence, S4.04 neither contains K1 or K1.1, nor is contained in K1 or K1.1.

2.5.2 Assume K1.2. Hence, we have S4 and **H1**. Then:

Z1	*ℭpCLMpLp*	[**H1**; S1°]
H2	*ℭLMpCpLp*	[*Z1*; S1°]

2.5.3 Now, let us assume S4 and **H2**. Then:

Z1	*ℭLMpℭpLp*	[**H2**; S2°; S4]
Z2	*ℭLMpℭLMpLMLp*	[*Z1*; S3°]
K1	*ℭLMpMLp*	[*Z2*; S1]
Z3	*ℭCLpMqLMCpq*	[S2; *cf.* [16], pp. 71-72]
Z4	*LMCMpLLp*	[*Z3*, *p*/*Mp*, *q*/*Lp*; **K1**; S4]
Z5	*ℭLMCMpLqLMLCpq*	[S1, *cf.* [13], p. 156; S2°]
K4	*LMLCpLp*	[*Z5*, *q*/*Lp*; *Z4*]
Z6	*ℭLMLpLMp*	[S2]
L1	*ℭLMLpCpLp*	[*Z6*; **H2**; S1°]

Thus, **L1** follows from S4 and **H2**. Since, *cf.* **2.5.2**, **H2** is a consequence of K1.2, it proves that K1.2 contains **L1**.

Z7	*ℭCpLpLCpLp*	[**L1**, *p*/*CpLp*; **K4**; S4°]

Z8 $\mathfrak{C}NpCpq$ [S1°]
Z9 $\mathfrak{C}Np\mathfrak{C}pLp$ [*Z8*, q/Lp; *Z7*; S1°]
H1 $\mathfrak{C}p\mathfrak{C}Mpp$ [*Z9*, p/Np; S1°]

Hence, it is shown that {K1.2} $\rightleftarrows$ {S4; **H1**} $\rightleftarrows$ {S4; **H2**}, and that {K1.2} $\rightarrow$ {S4.04}.

2.5.3. Now, assume K1 and **L1**. Then, we have axiom **K1** and, therefore:

Z1 $\mathfrak{C}LMpLMLp$ [**K1**; S4°]
H2 $\mathfrak{C}LMpCpLp$ [*Z1*; **L1**; S1°]

Since K1.1 does not imply L1 it follows from **2.5.2** and **2.5.3** that {K1.2} $\rightleftarrows$ {S4; **H1**} $\rightleftarrows$ {S4; **H2**} $\rightleftarrows$ {K1; **L1**} $\rightleftarrows$ {K1.1; **L1**}.

2.6 Formula of Grzegorczyk. In [2], using reasonings analogous to Cohen's method of forcing, Grzegorczyk tried to construct models for such propositional calculi which would correspond to methodological patterns of scientific investigation. As far as I know, calculi obtained in this way are not yet systematically investigated. One of the models constructed by Grzegorczyk verifies a peculiar propositional calculus in which not all classical propositional theorems are valid, but all modal theses of S4 are. Moreover, this theory contains the following formula

$$\{[(Z \rightarrow \Box Y) \rightarrow \Box Y] \wedge [(\sim Z \rightarrow \Box Y) \rightarrow \Box Y]\} \rightarrow \Box Y$$

which, if we accept the symbols "$\rightarrow$", "$\Box$" "$\sim$" and "$\wedge$" as the symbols of strict implication, necessity, negation and conjunction respectively, in Łukasiewicz's notation would have the following form:

$$\mathfrak{C}K\mathfrak{C}\mathfrak{C}pLqLq\mathfrak{C}\mathfrak{C}NpLqLqLq$$

And, it is self evident that in the field of S4 the latter form is inferentially equivalent to:

T1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqqC\mathfrak{C}\mathfrak{C}Npqqq$

As Grzegorczyk points out in his paper, **T1** is a formula which does not belong to S4. Here I shall not analyze the Grzegorczyk propositional calculus, but only **T1** and its connections with family $\mathcal{K}$.

2.6.1 Lewis matrix $\mathfrak{M}2$ falsifies **T1** for $p/2$ and $q/4$: $\mathfrak{C}\mathfrak{C}\mathfrak{C}244C\mathfrak{C}\mathfrak{C}N2444$ = $\mathfrak{C}\mathfrak{C}LC244C\mathfrak{C}LC3444$ = $\mathfrak{C}\mathfrak{C}L34C\mathfrak{C}L244$ = $\mathfrak{C}LC44CLC444$ = $\mathfrak{C}L1CL14$ = $\mathfrak{C}1C14$ = $LC14$ = $L4$ = 4. On the other hand, **T1** is verified by $\mathfrak{M}1$. Hence **T1** is not contained in the systems V2, V1 and S5, but its addition, as a new axiom, to S4 does not reduce the latter system to the classical propositional calculus. In fact, it can be proved at once that in the field of S4 **T1** implies **J1** which is a proper axiom of K1.1 and, therefore, *cf.* [15], pp. 314-314, also **K1**. Namely:

Z1 $\mathfrak{C}\mathfrak{C}NpLpLp$ [S2°]
J2 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpLp$ [**T1**, q/Lp; *Z1*; S1°]
J1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppp$ [**J2**; S4; *cf.* [14], p. 314]
K1 $\mathfrak{C}LMpMLp$ [**J1**; S4; *cf.* [14], pp. 314-315]

Since **K1** follows from S4 and **T1**, system {S4; **T1**} clearly belongs to family ***K***.

2.6.2 Let us assume K1.2, whence we have at our disposal S4, **H1**, **K1** and **L1**. Then:

Z1 *ℭNMLpCNNLMpq* [**K1**; S1°]

Z2 *ℭNMLpCNMLNpq* [*Z1*; S1°]

Z3 *ℭpCqq* [S1°]

Z4 *ℭNMLpCCMLNpqq* [*Z3*, *p/NMLp*; *Z2*; S1°]

Z5 *ℭpCqp* [S1°]

Z6 *ℭCMLpqCCMLNpqq* [*Z5*, *p/q*, *q/CMLNpq*; *Z4*; S1°]

Z7 *ℭℭpqCMLpMLq* [S3°]

Z8 *ℭℭLpqCMLpMLq* [*Z7*, *p/Lp*; S4°]

Z9 *ℭℭLpqCCMLNpMLqMLq* [*Z8*; *Z6*, *q/MLq*; S1°]

Z10 *ℭℭLpqCℭLNpqMLq* [*Z8*, *p/Np*; *Z9*; S1°]

Z11 *ℭMLqMLCpq* [S2°]

Z12 *ℭMLNpMLCpq* [S2°]

Z13 *ℭNLMpMLCpq* [*Z12*; S1°]

Z14 *ℭCLMpMLqMLCpq* [*Z11*; *Z13*; S1°]

Z15 *ℭCCpqrCCpvCCvqr* [S1°]

Z16 *ℭCMLpMLqMLCpq* [*Z15*, *p/LMp*, *q/MLq*, *r/MLCpq*, *v/MLp*; *Z14*; **K1**; S1°]

Z17 *ℭℭLpqCℭLNpqCrMLq* [*Z10*; S1°]

Z18 *ℭℭLpqCℭLNpqMLCrq* [*Z17*, *r/MLr*; *Z16*, *p/r*; S1°]

Z19 *ℭℭLpqCℭLNpqLMLCrq* [*Z18*; S3°; S4°]

Z20 *ℭpCNpq* [S1°]

Z21 *ℭCpCqrCCsCvpCCtqCsCvCNrCtw* [S1°]

Z22 *ℭℭLpqCℭLNpqCNℭNpqCpq* [*Z21*, *p/LMLCNpq*, *q/CNpq*, *r/ℭNpq*, *s/ℭLpq*, *v/ℭLNpq*, *t/p*, *w/q*; **L1**, *p/CNpq*; *Z19*, *r/Np*; *Z20*; S1°]

Z23 *ℭCpCqrCCsCvpCCsCvCtqCsCvCtCNrw* [S1°]

Z24 *ℭℭLpqCℭLNpqCNℭNpqCNℭpqq* [*Z23*, *p/LMLCpq*, *q/Cpq*, *r/ℭpq*, *s/ℭLpq*, *v/ℭLNpq*, *t/NℭNpq*, *w/q*; **L1**, *p/Cpq*; *Z19*, *r/p*; *Z22*; S1°]

Z25 *ℭpCrCsCqq* [S1°]

Z26 *ℭℭLpqCℭLNpqCNℭNpqCCℭpqqq* [*Z25*, *p/ℭLpq*, *r/ℭLNpq*, *s/NℭNpq*; *Z29*; S1°]

Z27 *ℭpCrCqCsq* [S1°]

Z28 *ℭℭLpqCℭLNpqCCℭNpqqCCℭpqqq* [*Z27*, *p/ℭLpq*, *r/ℭLNpq*, *s/Cℭpqq*; *Z26*; S1°]

Z29 *ℭLNpℭpq* [S2°]

Z30 *ℭℭℭpqrℭLNpr* [*Z29*; S2°]

Z31 *ℭℭℭNpqrℭLpr* [*Z30*; *p/Np*; S2°]

T1 *ℭℭℭpqqCℭℭNpqqq* [*Z28*; *Z31*, *r/q*; *Z30*, *r/q*; S4°]

Thus, it is proved that **T1** is a consequence of K1.2. It should be noticed that the given proof is based upon the availability of **L1** in the field of K1.2.

Matrix 𝔐4 verifies **T1**, and, *cf.* [14], p. 316, falsifies **H1**. Hence, {S4; **T1**} is a proper subsystem of K1.2. Matrix 𝔐5 which verifies {S4; **T1**} falsifies, as it is well known, *cf.* [9] and [14], p. 316, formula **G1**. Hence, {S4; **T1**} does not contain S4.2. On the other hand, as Schumm has proved in [12], 𝔐7 verifies K2 and K3, but rejects K2.1 and K3.1. This matrix also falsifies **T1** for $p/2$ and $q/4$: ℭℭℭ244CℭℭN2444 = ℭℭLC244CℭLC7444 = ℭℭL34CℭL244 = ℭLC44CLC444 = ℭL1CL14 = ℭ1C14 = LC14 = L4 = 4. This proves that neither K1, nor K2, nor even K3 imply **T1**. Moreover, matrix 𝔐9 which verifies K2.1, but rejects K3, *cf.* [9], and [14], pp. 316-317, also verifies **T1**.

These matrix calculations suggest that between K1.1 and K1.2 there is a proper system {S4; **T1**} which I call K1.1.1, and that between K2.1 and K3.1 there is another proper system, called K2.2, namely {S4.2; **T1**}. Unfortunately, although it is very probable, I do not have yet the proofs that K1.1 does not imply **T1**, and that **T1** is not a consequence of K2.1

2.6.3 In [11] Schumm has proved that system D* established by Makinson in [4] by the way of defining its characteristic matrix is inferentially equivalent to my system K3.1. Therefore, Makinson's matrix is also a characteristic matrix of the latter system, and {D*} ⇄ {K3.1} ⇄ {S4.3; **J1**}. Below, in **2.7** I shall prove that Makinson's matrix 𝔇* verifies **T1**, and, therefore, there is a metalogical proof that **T1** is a consequence of K3.1, i.e., that {K3.1} ⇄ {S4.3; **T1**}. But, as yet I was unable to find a logical proof of this fact, i.e., to deduce **T1** from the axioms of K3.1. Maybe, a very tedious proof could be obtained by an application of an idea which is included in Schumm's Lemma 1, *cf.* [11], p. 263.

2.7 Makinson defines system D* as the non-Lewis modal system whose characteristic matrix:

$$\mathfrak{D}^* = \langle V, d, -, \cap, \mathrm{P}^* \rangle$$

satisfies the following four conditions:

i) V is a set of all ω sequences $\{x_n\}_{n<\omega}$ of 0's and 1's.
ii) d is the designated element: $\{1_n\}_{n<\omega}$.
iii) - and ∩ are the operations in V defined in pointwise fashion from the familiar Boolean operations - and ∩ in {0, 1}.
iv) P* is the operation in V such that if $\{x_0, x_1, \ldots\} \in V$, then $\mathrm{P}^*\{x_0, x_1, \ldots\} = \{y_0, y_1, \ldots\}$ where, for each i, $y_i = 1$ iff $x_j = 1$ for some $j \leqslant i$.

It should be noticed that Makinson's matrix 𝔇* is a modification of a characteristic matrix 𝔇 introduced by Prior in order to define his Diodorian system D. In both matrices the first three conditions are the same, but in Prior's matrix the last condition is formulated as follows:

iv) P is the operation in V such that if $\{x_0, x_1, \ldots\} \in V$, then $\mathrm{P}\{x_0, x_1, \ldots\} = \{y_0, y_1, \ldots\}$ where, for some i, $y_i = 1$ iff $x_j = 1$ for some $j \leqslant i$. *Cf.* [4], pp. 406-407, and [11], p. 263.

A Diodorian system of Prior is identical with my system S4.3.1, *cf.* [10], p. 176, [3], pp. 262-264, and [14], p. 316.

It follows from the definition of $\mathfrak{M}^*$ that if for arbitrary well formed propositional formula p we shall assign a sequence $\phi(p) \epsilon V$ such that $\phi(p) = \{x_n\}_{n<\omega}$ in which for some j, $0 \leqslant j < \omega$, $x = 1$, then for the formula Np there is one and only one sequence corresponding to $\phi(p)$, $\psi(Np) \epsilon V$ such that $\psi(Np) = \{y_n\}_{n<\omega}$ in which term $y_j = 0$. And, vice versa, if for some i, $0 \leqslant i < \omega$, in $\phi(p)$ term $x_i = 0$, then in $\psi(Np)$ term $y_i = 1$.

Moreover, since in $\mathfrak{M}^*$ for an arbitrary element $\alpha \epsilon V$ we have:

(1) $\alpha \cap \{0_n\}_{n<\omega} = \{0_n\}_{n<\omega} \cap \alpha = \{0_n\}_{n<\omega}$,

an assignment of $\{0_n\}_{n<\omega}$ for p which occurs in the formulas Kqp or Kpq gives for both these formulas the corresponding assignment $\{0_n\}_{n<\omega}$ regardless of the assignment given for q.

2.7.1 Written in the primitive functors of modal logic formula **T1** has the following form:

T1* $NMKNMKNMKpNqNqKNMKNMKNpNqNqNq$

which, obviously, in the field of S4 is inferentially equivalent to:

T $NMKNMKNMKpqqKNMKNMKNpqqq$ [**T***, q/Nq; S1°]

Whence, instead of **T1** or **T1*** it is sufficient to investigate formula **T**.

2.7.2 Let us assume that formula **T** is falsified by matrix $\mathfrak{M}^*$. Then, there must exist sequences α and β belonging to V such that for $\alpha = \alpha(p)$ and $\beta = \beta(q)$ the corresponding sequence $\rho(\mathbf{T}) \neq \{1_n\}_{n<\omega}$.

From the properties of $\mathfrak{M}^*$ observed above it clearly follows that it cannot be $\beta = \{0_n\}_{n<\omega}$, because in such a case $\rho(\mathbf{T})$ would be $\{1_n\}_{n<\omega}$. Hence, $\beta = \beta(p)$ should be a sequence belonging to V in which, for certain j, $0 \leqslant j < \omega$, term $x_j = 1$. Let us assume that in β x_j is the first term which is equal to 1. Since in the assignment for p: $\alpha = \alpha(p) = \{y_n\}_{n<\omega}$ term y_j is either 1 or 0, we have two and only two possible cases:

Case I The assignments α and β for p and q determine for formula Kpq a sequence in V $\gamma = \gamma(Kpq) = \alpha \cap \beta = \{z_n\}_{n<\omega}$ such $z_j = 1$ and, for $0 \leqslant i < j$, $z_i = 0$. And, therefore, due to the properties of $\mathfrak{M}^*$, mentioned above, for formula $KNpq$ there is, determined by α and β, a sequence $\delta = \delta(KNpq) = -\alpha \cap \beta = \{v_n\}_{n<\omega}$ such that its term $v_j = 0$, and, for $0 \leqslant i < j$, $v_i = 0$.

or

Case II The assignments α and β for p and q determine for formula Kpq a sequence in V $\gamma' = \gamma'(Kpq) = \alpha \cap \beta = \{z_n\}_{n<\omega}$ such that that $z_j = 0$ and for $0 \leqslant i < j$, $z_i = 0$. And, therefore, due to the properties of $\mathfrak{M}^*$ mentioned above, for formula $KNpq$ there is, determined α and β, a sequence $\delta' = \delta'(KNpq) = -\alpha \cap \beta = \{v_n\}_{n<\omega}$ such that its term $v_j = 0$, and, for $0 \leqslant i < j$, $v_i = 0$.

2.7.3 Since, obviously, matrix $\mathfrak{M}^*$ verifies S4, $\mathfrak{M}^*$ verifies the rules of procedure and thesis $\mathfrak{C}pNNp$ of S4. From this remark and an inspection of the structure of **T** (and even better of **T1**) it follows at once that cases I and II are entirely analogous. Namely, if in the first case the assignments α and β for p and q induce assignments, say, μ and ν, for the formulas $NMKpq$

and *NMKNpq* respectively and if $\mu(NMKpq)$ and $\nu(NMKNpq)$ possess certain properties, say, *a* and *b* respectively, then in the second case the assignments μ' and ν' for the formulas *NMKpq* and *NMKNpq* are such that $\nu'(NMKNpq)$ has property *a* and $\mu'(NMKpq)$ has property *b*. And, it is self evident that an inverse of the cases in this reasoning gives the same result. Hence, it is quite sufficient for our end to investigate only one of these case.

2.7.4 Let us analyze Case I. Then, according to the definition of this case, for formula *KNpq* there is, determined by α and β, a sequence:

(2) $\delta = \delta\ (KNpq) = \{v_n\}_{n<\omega}$

such that its term $v_j = 0$, and, for $0 \leqslant i < j$, $v_i = 0$. Hence, for formula *NMKNpq* there is a sequence:

(3) $\xi = \xi\ (NMKNpq) = -\mathrm{P}(-\alpha \cap \beta) = -\mathrm{P}(\delta) = \{r_n\}_{n<\omega}$

such that, for $0 \leqslant i \leqslant j$, its term $r_i = 1$. Therefore, since in β the first term which is equal to 1 is y_j, for formula *KNMKNpqq* we have a sequence:

(4) $\eta = \eta(KNMKNpqq) = \xi \cap \beta = \{s_n\}_{n<\omega}$

such that its term $s_j = 1$ and, for $0 \leqslant i < j$, $s_i = 0$, whence, for formula *NMKNMKNpqq* there is a sequence:

(5) $\kappa = \kappa(NMKNMKNpqq) = -\mathrm{P}(\eta) = \{t_n\}_{n<\omega}$

such that, for $0 \leqslant i < j$, its terms $t_i = 1$, and, for $j \leqslant k < \omega$, $t_k = 0$. Therefore since in β the first term which is equal to 1 is y_j, and in κ, for $j \leqslant k < \omega$, $t_k = 0$, a sequence λ which is an assignment for formula *KNMKNMKNpqqq* is such that:

(6) $\lambda = \lambda(KNMKNMKNpqqq) = \kappa \cap \beta = \{0_n\}_{n<\omega}$

which gives at once $\rho(\mathbf{T}) = \{1_n\}_{n<\omega}$

Therefore, there are no assignments for *p* and *q* in **T** such that for them formula **T** would be falsified by matrix $\mathfrak{P}^*$. This completes the proof that **T1** is a consequence of K3.1. Since in the field of S4 **T1** implies **J1**, we have: {K3.1} ⇄ {S4.3; **J1**} ⇄ {S4.3; **T1**}. But this result is proved metalogically.

2.8 Two new proper axioms of system K1. It will be shown that the following two formulas:

K6 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqpMLp$

and

K7 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppMLp$

whose structure is very akin to the structures of **J1** and **J2**, are such that each of them can serve as a proper axiom of K1.

2.8.1 Assume K1. Hence, we have S4 and

K1 $\mathfrak{C}LMpMLp$

and, therefore, McKinsey's theorem

K2 $\mathfrak{C}KLMpLMqMKpq$ [S4; **K1**; *cf.* [16], p. 76, point γ]

Then:

Z1 $\mathfrak{C}MLNpM\mathfrak{C}pq$ [S2°]
Z2 $\mathfrak{C}LMNpM\mathfrak{C}pq$ [**K1**, p/Np; *Z1*; S1°]
Z3 $\mathfrak{C}LMNpLM\mathfrak{C}pq$ [*Z3*; S4°]
Z4 $\mathfrak{C}pp$ [S1°]
Z5 $\mathfrak{C}CpqCCprCCKqrsCps$ [S1°]
Z6 $\mathfrak{C}LMNpMK\mathfrak{C}pqNp$ [*Z5*, $p/LMNp$, $q/LM\mathfrak{C}pq$, $r/LMNp$, $s/MK\mathfrak{C}pqNq$; *Z3*; *Z4*, $p/LMNp$; **K2**, $p/\mathfrak{C}pq$, q/Np; S1°]
K6 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqpMLp$ [*Z6*; S1°]

2.8.2 Obviously, **K6** implies:

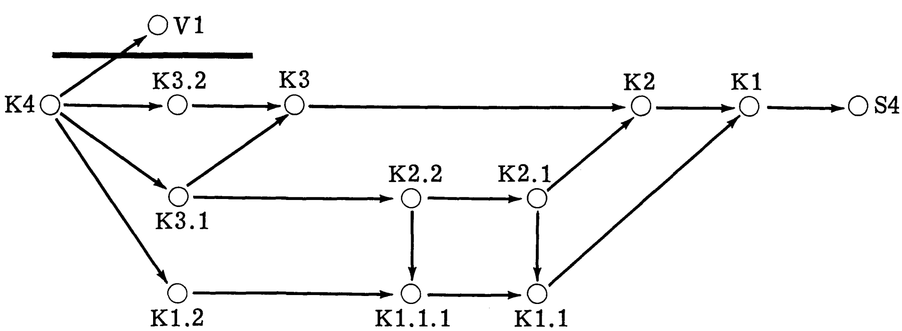
K7 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppMLp$ [**K6**, p/Lp]

Now, let us assume S4 and **K7**. Then:

Z1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqr\mathfrak{C}LNpr$ [S2°]
Z2 $\mathfrak{C}LN\mathfrak{C}pLpMLp$ [*Z1*, $p/\mathfrak{C}pLp$, q/p, r/MLp; **K7**]
Z3 $\mathfrak{C}LMNpM\mathfrak{C}pLp$ [*Z2*; S1]
Z4 $\mathfrak{C}LMNpMCMpMLp$ [*Z3*; S3°]
Z5 $\mathfrak{C}LMNpCLMpMMLp$ [*Z4*; S3; *cf.* [16], pp. 71-72]
K1 $\mathfrak{C}LMpMLp$ [*Z5*; S4°]

Thus, {K1} $\rightleftarrows$ {S4; **K1**} $\rightleftarrows$ {S4; **K6**} $\rightleftarrows$ {S4; **K7**}

2.9 The known members of family ***K*** of the non-Lewis modal systems can be arranged in the following diagram:



in which the bold horizontal line indicates that, although system V1 is a proper subsystem of K4, and only of K4 among the known members of family ***K***, it really does not belong to this family. Comparing this diagram with that which was published in [16], p. 317, we see that:

1. In the present diagram there is no K5, since Thomas reduced this system to K4, cf. [17] and 2.3. Instead, V1 is added, as a proper subsystem of K4.
2. On the other hand, three new systems which are not occurring in the former diagram are added, namely, K3.2, K2.2 and K1.1.

There is no problem concerning K3.2, since in **2.4** it has been proved that K3.2 is a proper extension of K3 and a proper subsystem of K4. But, in connection with K2.2 and K1.1.1 there are two open problems, viz.:

a) Whether K2.2 is a proper subsystem of K3.1?
b) Whether K1.1.1 is a proper extension of K1.1?

To these problems, obviously, a third one has to be added, viz.:

c) To obtain a logical proof that K3.1 implies **T1**.

3 SYSTEMS BETWEEN S4 AND S5

In [16], p. 311, I put, as an open problem, a question whether there exists or does not exist a system being a proper extension of S4.4 and in the same time being a proper subsystem of S5. In [12] Schumm has solved this problem positively proving that the addition of the following formula:

A1 *AℭMpLMpℭLMqMLq*

as a new axiom, to S4.4 generates a new system, called by him S4.7, which satisfies the properties which I required: namely, S4.7 properly contains S4.4 and is a proper subsystem of S5. In this Section I shall show that there is a proper axiom of S4.7, and, moreover, it will be proved that besides S4.7 there are other systems which are probably weaker than S4.7 and at the same time are intermediate systems between S4.4 and S5.

3.1 It is easy to prove that in the field of S4.4 Schumm's axiom *A1* is inferentially equivalent to:

Q1 *AℭMLpLpℭMLMqCqLq*

and, moreover, that the addition of **Q1** to S4 gives S4.7 so that **Q1** is a proper axiom of the latter system.

3.1.1 Let us assume S4.7. Hence, we have *A1* and S4.4, and, therefore, also S4 and

R1 *ℭMLpCpLp*

i.e., the proper axiom of S4.4. Then:

Z1	*ℭℭpqCMpMq*	[S1°]
Z2	*ℭℭLMpMLqCMLMpMLq*	[*Z1*, *p*/*LMp*, *q*/*MLq*; S4°]
Z3	*ℭCpqCCrCspCrCsq*	[S1°]
Z4	*ℭℭLMqMLqCMLMqCqLq*	[*Z3*, *p*/*MLq*, *q*/*CqLq*, *r*/*ℭLMqMLq*, *s*/*MLMq*; **R1**, *p*/*q*; *Z2*, *p*/*q*; S1°]
Z5	*ℭℭLMqMLqℭMLMqCqLq*	[*Z4*; S4°]
Z6	*ℭℭMNpLMNpℭMLpLp*	[S2°]
Q1	*AℭMLpLpℭMLMqCqLq*	[*A1*, *p*/*Np*; *Z6*; *Z5*; S1]

Thus, S4.7 implies **Q1**

3.1.2 Assume now S4 and **Q1**. Then:

Z1 $\mathfrak{C}pMp$ [S1]
Z2 $\mathfrak{C}MLpMLMp$ [*Z1*; S2°]
Z3 $\mathfrak{C}\mathfrak{C}MLMpq\mathfrak{C}MLpq$ [*Z2*; S2°]
Z4 $\mathfrak{C}\mathfrak{C}MLpLp\mathfrak{C}MLpCpLp$ [S2°]
R1 $\mathfrak{C}MLpCpLp$ [**Q1**, q/p; *Z4*; *Z3*, $q/CpLp$; S1°]

Thus, {S4; **Q1**} implies **R1**

Z5 $\mathfrak{C}\mathfrak{C}Mpq\mathfrak{C}pq$ [*Z1*; S2°]
Z6 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}LpLq$ [S3°]
Z7 $\mathfrak{C}\mathfrak{C}LMqCqLq\mathfrak{C}LMqCqLq$ [*Z6*, p/LMq, $q/CqLq$; S4]
Z8 $\mathfrak{C}\mathfrak{C}LMqCqLq\mathfrak{C}LMqCLMqLMLq$ [*Z7*; S4]
Z9 $\mathfrak{C}\mathfrak{C}LMqCqLq\mathfrak{C}LMqMLq$ [*Z8*; S2]
Z10 $\mathfrak{C}\mathfrak{C}MLMqCqLq\mathfrak{C}LMqMLq$ [*Z5*, p/LMq, $q/CqLq$; *Z9*; S1°]
Z11 $\mathfrak{C}\mathfrak{C}MLNpLNp\mathfrak{C}MpLMp$ [S2°]
A1 $A\mathfrak{C}MpLMp\mathfrak{C}LMqMLq$ [**Q1**, p/Np; *Z11*; *Z10*; S1°]

Hence, {S4; **Q1**} yields Schumm's axiom *A1*. Therefore, it follows from **3.1.1** and **3.1.2** that {S4.7} $\rightleftarrows$ {S4.4; *A1*} $\rightleftarrows$ {S4; **Q1**}. Hence, **Q1** is a proper axiom of S4.7

3.2 As Schumm has remarked in [12], his matrix $\mathfrak{M}8$ verifies S4.4, but falsifies his axiom *A1*. Hence, S4.7 is a proper extension of S4.4. This matrix, obviously, falsifies **Q1** for $p/5$ and $q/6$: $A\mathfrak{C}ML5L5\mathfrak{C}MLM6C6L6 = A\mathfrak{C}M55\mathfrak{C}ML1C68 = ALC15\mathfrak{C}M13 = CNL5LC13 = CN5L3 = C48 = 5$. It is self evident that S4.7 is a subsystem of S5 and at the same time a subsystem of K4, since **C2** and **P1** are proper axioms of S5 and K4 respectively. Matrix $\mathfrak{M}10$ which verifies S4.7 falsifies **C2** for $p/9$: $\mathfrak{C}ML9L9 = \mathfrak{C}M99 = LC19 = L9 = 9$, and, $\mathfrak{M}10$ falsifies **P1** for $p/2$: $\mathfrak{C}MLM2C2L2 = \mathfrak{C}ML1C24 = LCM13 = LC13 = L3 = 4$. This proves that S4.7 is a proper subsystem of S5 and at the same time a proper subsystem of K4.

3.3 As an immediate consequence of **Q1** we have:

R4 $A\mathfrak{C}MLpLp\mathfrak{C}MLMpCpLp$ [**Q1**, q/p]

It is clear that in the field of S4 applying deductions entirely analogous to those which were given in **3.1.2** we can obtain **R1** from **R4**, but, the addition of **R4** as a new axiom to S4 does not generate a new system, since **R4** and, even, a little stronger formula are provable in S4.4.

3.3.1 Let us assume S4.4. Hence we have S4 and **R1**. Then:

G1 $\mathfrak{C}MLpLMp$ [S4.4; *cf.* [16], p. 307]
G2 $\mathfrak{C}MLpLMLp$ [**G1**, p/Lp; S4]
Z1 $\mathfrak{C}LMLpLCpLp$ [**R1**; S2°]
Z2 $\mathfrak{C}MLpLCpLp$ [*Z1*; **G2**; S1°]
Z3 $\mathfrak{C}MLpCrLCpLp$ [*Z2*; S1°]
Z4 $\mathfrak{C}CMpLq\mathfrak{C}pq$ [S1; *cf.* [13], p. 156]
Z5 $\mathfrak{C}MLp\mathfrak{C}MLMpCpLp$ [*Z3*, $r/MMLMp$; *Z4*, $p/MLMp$, $q/CpLp$; S1°]
Z6 $\mathfrak{C}NCpqp$ [S1°]
Z7 $\mathfrak{C}NCMLpq\mathfrak{C}MLMpCpLp$ [*Z6*, p/MLp; *Z5*; S1°]

Z8 *ℭNCMMLpqℭMLMpCpLp* [*Z7*; S4]
Z9 *ℭNℭpqNCMpLq* [*Z4*; S1°]
Z10 *ℭNℭMLpqℭMLMpCpLp* [*Z9*, *p/MLp*; *Z8*, *q/Lq*; S1°]
R3 *AℭMLpqℭMLMpCpLp* [*Z10*; S1]
R4 *AℭMLpLpℭMLMpCpLp* [**R3**, *q/Lp*]

Thus, it is proved that {S4.4} $\rightleftarrows$ {S4; **R1**} $\rightleftarrows$ {S4; **R3**} $\rightleftarrows$ {S4; **R4**}. Therefore, formulas **R3** and **R4** can serve as proper axioms of S4.4, and their addition to S4 does not generate a new system.

3.4 The following formula:

S1 *AℭMLpLpALqAℭqrℭqNr*

which, evidently, is a consequence of S5 and of V1 follows from S4.7.

3.4.1 Let us assume S4.7. Hence, we have at our disposal S4 and **Q1** and, therefore, also, **R1**. Then:

Z1 *ℭKpMLpLp* [**R1**; S1°]
Z2 *ℭℭMLMpCpLpℭLMpMLp* [S4; *cf.* in **3.1.2** proof of *Z10*]
Z3 *ℭℭMLMpCpLpℭNMLpNLMp* [*Z2*; S2°]
Z4 *ℭMLNpMLCpq* [S2°]
Z5 *ℭNLMpMLCpq* [*Z4*; S1°]
Z6 *ℭMLqMLCpq* [S2°]
Z7 *ℭCpℭqrCCrsCpCqs* [S1]
Z8 *ℭℭMLMpCpLpCNMLpMLCpr*
[*Z7*, *p/ℭMLMpCpLp*, *q/NMLp*, *r/NLMp*, *s/MLCpr*; *Z3*; *Z5*; *q/r*; S1°]
Z9 *ℭCqrCCsCNprCsCCpqr* [S1°]
Z10 *ℭℭMLMpCpLpCCMLpMLqMLCpq*
[*Z9*, *p/MLp*, *q/MLq*, *r/MLCpq*, *s/ℭMLMpCpLp*; *Z6*; *Z8*, *r/q*; S1°]
Z11 *AKprAKCpqCrsKCpNqCrNs* [S1°]
Z12 *ℭNMLpLMNp* [S1°]
Z13 *ALpAKCpqCMLpMLrKCpNqCMLpLMNr*
[*Z11*, *r/MLp*, *s/MLr*; *Z1*; *Z12*, *p/r*; S1°]
Z14 *ℭCpCqrCAsAKvqtCpAsAKvrt* [S1°]
Z15 *ℭℭMLMpCpLpALpAKCpqMLCprKCpNqCMLpLMNr*
[*Z14*, *p/ℭMLMpCpLp*, *q/CMLpMLr*, *r/MLCpr*, *s/Lp*, *v/Cpq*, *t/KCpNqCMLpLMNr*; *Z10*, *q/r*; *Z13*; S1°]
Z16 *ℭℭMLMpCpLpCLMpMLp* [*Z2*; S1]
Z17 *ℭCpCqrCCsAtAuKvCwqCKpsAtAuKvCwr* [S1°]
Z18 *ℭKℭMLMNrCNrLNrℭMLMpCpLpALpAKCpqMLCprKCpNqCMLpMLNr*
[*Z17*, *p/ℭMLMNrCNrLNr*, *q/LMNr*, *r/MLNr*, *s/ℭMLMpCpLp*, *t/Lp*, *u/KCpqMLCpr*, *v/CpNq*, *w/MLp*; *Z16*, *p/Nr*; *Z15*; S1°]
Z19 *ℭCpCqrCCKspAtAvKwqCKspAtAvKwr* [S1°]
Z20 *ℭKℭMLMNrCNrLNrℭMLMpCpLpALpAKCprMLCprKCpNrMLCpNr*
[*Z19*, *p/ℭMLMpCpLp*, *q/CMLpMLNr*, *r/MLCpNr*, *s/ℭMLMNrCNrLNr*, *t/Lp*, *v/KCpqMLCpr*, *w/CpNq*; *Z10*, *q/Nr*; *Z18*; S1°]
Z21 *ℭCpqCCrsCCtAvAprCtAvAqs* [S1°]

Z22 ℭKℭMLMNrCNrLNrℭMLMpCpLpALpAℭprℭpNr
[*Z21*, p/KCprMLCpr, q/ℭpr, r/KCpNrMLCpNr, s/ℭpNr,
t/KℭMLMNrCNrLNrℭMLMpCpLp, v/Lp; *Z1*, p/Cpr;
Z1, p/CpNr; *Z20*; S1°]

Z23 ℭApqCAprApKqr [S1°]

Z24 AℭMLpLpKℭMLMNrCNrLNrℭMLMqCqLq [*Z23*, p/ℭMLpLp,
q/ℭMLMNrCNrLNr, r/ℭMLMqCqLq; **Q1**, q/Nr; **Q1**; S1°]

S1 AℭMLpLpALqAℭqrℭqNr [*Z24*; *Z22*, p/q; S1°]

Thus, S4.7 yields formula **S1**.

3.5 As an immediate consequences of **S1** we have the following two formulas:

E1 AℭMLpLpALqAℭqpℭqNp [**S1**, r/p]

and

E2 AℭMLpLpALpAℭpqℭpNq [**S1**, q/p, r/q]

I shall show here that each of the systems {S4; **E1**} and {S4; **E2**} contains S4.4 and, moreover, that in the field of S4 formulas **E1** and **E2** are inferentially equivalent.

3.5.1 Let us assume S4 and **E2**. Then:

Z1 ℭALpALqLrℭNLrCNLqLp [S2°]

Z2 ℭNLCpNqMKpq [S2°]

Z3 ℭNLCpqMKpNq [S2°]

Z4 ℭALpAℭpqℭpNqCMKpqCMKpNqLp [*Z1*, q/Cpq, r/CpNq; *Z2*; *Z3*; S1°]

Z5 AℭMLpLpℭMKpqCMKpNqLp [**E2**; *Z4*; S1°]

Z6 ℭMLpMKpLp [S2]

Z7 ℭpCqMKpq [S1]

Z8 ℭCNppp [S1°]

Z9 ℭCpqCCrCtsCCCtmuCCqCsuCpCru [S1°]

Z10 ℭCMKpLpCMKpNLpLpCMLpCpLp [*Z9*, p/MLp, q/MKpLp, r/p, t/NLp,
m/Lp, u/Lp, s/MKpNLp; *Z6*; *Z7*, q/NLp; *Z8*, p/Lp; S1°]

Z11 ℭℭMKpLpCMKpNLpLpℭMLpCpLp [*Z10*; S2°]

Z12 ℭℭpqℭpCrq [S2°]

R1 ℭMLpCpLp [*Z5*, q/Lp; *Z12*, p/MLp, q/Lp, r/p; *Z11*; S1°]

Thus, **R1** follows from {S4; **E2**} and, therefore, {S4; **E2**} → {S4.4}.

3.5.2 Let us assume S3 and **E1**. Then:

Z1 ℭLpℭqp [S2°]

Z2 ℭCpqCKpsArAqt [S1°]

Z3 ℭKLpsArAℭqpt [*Z2*, p/Lp, q/ℭqp; *Z1*; S1°]

Z4 ℭCpArAsqCCqtCpArAst [S1°]

Z5 ℭKpqArAsq [S1°]

Z6 ℭKtLNpArAsℭqNp
[*Z4*, p/KtLNp, q/LNp, t/ℭqNp; *Z5*, p/t, q/LNp; *Z1*, p/Np; S1°]

Z7 ℭℭpqCℭNpqLq [S3°]

Z8 *ℭCpCqrCKpqArs* [S1°]
Z9 *ℭKℭpqℭNpqALqs* [*Z8*, *p*/*ℭpq*, *q*/*ℭNpq*, *r*/*Lq*; *Z7*; S1°]
Z10 *ℭℭNpNqℭqp* [S2°]
Z11 *ℭCpqCKspArAqt* [S1°]
Z12 *ℭKsℭNpNqArAℭqpt* [*Z11*, *p*/*ℭNpNq*, *q*/*ℭqp*; *Z10*; S1°]
Z13 *ℭℭpNqℭqNp* [S2°]
Z14 *ℭℭpqCKpsArAtq* [S1°]
Z15 *ℭKℭpNqsArAtℭqNp* [*Z14*, *p*/*ℭpNq*, *q*/*ℭqNp*; *Z13*; S1°]
Z16 *ℭCKsptCCKsqtCCKsrtCKsApAqrt* [S1°]
Z17 *ℭKℭpqALNpAℭNpqℭNpNqALqAℭqpℭqNp*
[*Z16*, *s*/*ℭpq*, *p*/*LNp*, *t*/*ALqAℭqpℭqNp*, *q*/*ℭNpq*, *r*/*ℭNpNq*;
Z6, *t*/*ℭpq*, *r*/*Lq*, *s*/*ℭqp*; *Z9*, *s*/*Aℭqpℭ qNp*;
Z12, *s*/*ℭpq*, *r*/*Lq*, *t*/*ℭqNp*; S1°]
Z18 *ℭCKpstCCKqstCCKrstCKApAqrst* [S1°]
Z19 *ℭKALpAℭpqℭpNqALNpAℭNpqℭNpNqALqAℭqpℭqNp*
[*Z18*, *p*/*Lp*, *s*/*ALNpAℭNpqℭNpNq*, *t*/*ALqAℭqpℭqNp*, *q*/*ℭpq*, *r*/*ℭpNq*;
Z3, *s*/*ALNpAℭNpqℭNpNq*, *r*/*Lq*, *t*/*ℭqNp*; *Z17*;
Z15, *s*/*ALNpAℭNpqℭNpNq*, *r*/*Lq*, *t*/*ℭqp*; S1°]
Z20 *ℭCKpqrCAspCAsqAsr* [S1°]
E2 *AℭMLpLpALpAℭpqℭpNq* [*Z20*, *p*/*ALqAℭqpℭqNp*, *q*/*ALNqAℭNqpℭNqNp*,
r/*ALpAℭpqℭpNq*, *s*/*ℭMLpLp*; *Z19*, *p*/*q*, *q*/*p*; **E1**; **E1**, *q*/*Nq*; S1°]

Thus, {S3; **E1**} implies **E2**. Therefore, it follows from **3.5.1** that {S4; **E1**} contains S4.4.

3.5.3 Let us assume S4 and **E2**. Hence, we have **R1**, *cf*. **3.5.1**. Then:

Z1 *ℭpCqp* [S1°]
Z2 *ℭMLpMLCqp* [*Z1*; S2°]
Z3 *ℭpp* [S1°]
Z4 *ℭCpqCCsrCCqCrvCpCsv* [S1°]
Z5 *ℭMLpCpℭqp* [*Z4*, *p*/*MLp*, *q*/*MLCqp*, *s*/*p*, *r*/*Cqp*, *v*/*ℭqp*;
Z2; *Z1*; **R1**, *p*/*Cqp*; S1°]
Z6 *ℭMLpCCqpℭqp* [*Z4*, *p*/*MLp*, *q*/*MLCqp*, *s*/*Cqp*, *r*/*Cqp*, *v*/*ℭqp*;
Z2; *Z3*; *p*/*Cqp*; **R1**, *p*/*Cqp*; S1°]
Z7 *ℭℭpqCMLpMLq* [S3°]
Z8 *ℭCpCqrCCrCstCKpqCNtCsv* [S1°]
Z9 *ℭKℭpqMLpCNLqCqv*
[*Z8*, *p*/*ℭpq*, *q*/*MLp*, *r*/*MLq*, *s*/*q*, *t*/*Lq*; *Z7*; **R1**, *p*/*q*; S1°]
Z10 *ℭCpCqCrvCCvCNstCpCqCNtCrs* [S1°]
Z11 *ℭKℭpqMLpCNLqCNℭqNpCqp* [*Z10*, *p*/*KℭpqMLp*, *q*/*NLq*, *r*/*q*, *v*/*MLNp*,
s/*p*, *t*/*ℭqNp*; *Z9*, *v*/*MLNp*; *Z5*, *p*/*Np*; S1°]
Z12 *ℭCKpqCrCNstCCqCtvCKpqCrCNvs* [S1°]
Z13 *ℭKℭpqMLpCNLqCNℭqpℭqNp*
[*Z12*, *p*/*ℭpq*, *q*/*MLp*, *r*/*NLq*, *s*/*ℭqNp*, *t*/*Cqp*, *v*/*ℭqp*; *Z11*; *Z6*; S1°]
Z14 *ℭKpqp* [S1°]
Z15 *ℭℭpNqℭqNp* [S2°]
Z16 *ℭCpqCCqrCpCsCtr* [S1°]
Z17 *ℭKℭpNqrCsCtℭqNp*
[*Z16*, *p*/*KℭpNqr*, *q*/*ℭpNq*, *r*/*ℭqNp*; *Z14*, *p*/*ℭpNq*, *q*/*r*; *Z15*; S1°]

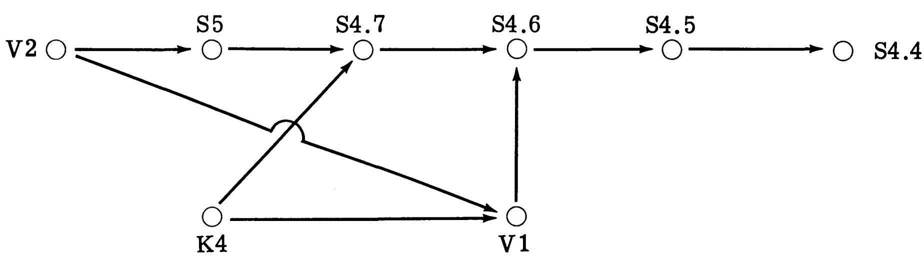
Z18 ℭ*Lp*ℭ*qp* [*Z1*; S2°]
Z19 ℭ*CpqCCqrCpCsCNrt* [S1°]
Z20 ℭ*KLprCsCN*ℭ*qpt* [*Z19*, *p*/*KLpr*, *q*/*Lp*, *r*/ℭ*qp*; *Z14*, *p*/*Lp*, *q*/*r*; *Z18*; S1°]
Z21 ℭ*CKpsvCCKqsvCCKrsvCKApAqrsv* [S1°]
Z22 ℭ*KALpA*ℭ*pq*ℭ*pNqMLpALqA*ℭ*qp*ℭ*qNp*
[*Z21*, *p*/*Lp*, *s*/*MLp*, *v*/*ALqA*ℭ*qp*ℭ*qNp*, *q*/ℭ*pq*, *r*/ℭ*pNq*; *Z20*, *r*/*MLp*, *s*/*NLq*, *t*/ℭ*qNp*; *Z13*; *Z17*, *r*/*MLp*, *s*/*NLq*, *t*/*N*ℭ*qp*; S1°]
Z23 ℭ*Lpp* [S1]
Z24 *ACMLpLpALpA*ℭ*pq*ℭ*pNq* [**E2**; *Z23*, *p*/*CMLpLp*; S1°]
Z25 *ACpqp* [S1°]
Z26 ℭ*ArpCArqArKpq* [S1°]
Z27 *ACMLpLpKALpA*ℭ*pq*ℭ*pNqMLp* [*Z26*, *r*/*CMLpLp*, *p*/*ALpA*ℭ*pq*ℭ*pNq*; *q*/*MLp*; *Z24*; *Z25*, *p*/*MLp*, *q*/*Lp*; S1°]
Z28 *ACMMLpLLpKALpA*ℭ*pq*ℭ*pNqMLp* [*Z27*; S4]
Z29 ℭ*CMpLq*ℭ*pq* [S1; *cf.* [13], p. 156]
Z30 *A*ℭ*MLpLpKALpA*ℭ*pq*ℭ*pNqMLp* [*Z28*; *Z29*, *p*/*MLp*, *q*/*Lp*; S1°]
E1 *A*ℭ*MLpLpALqA*ℭ*qp*ℭ*qNp* [*Z30*; *Z22*; S1°]

Thus, {S4; **E2**} implies **E1**. Therefore, it is proved that system {S4; **E1**} (and {S4; **E2**}) contains S4.4 and that in the field of S4 **E1** and **E2** are inferentially equivalent.

3.6 I call system {S4; **E1**} (and {S4; **E2**}) which is an extension of S4.4 system S4.5. And, system {S4; **S1**} which is an extension of S4.5 system S4.6. Matrix 𝔐8 which verifies S4.4, falsifies the proper axiom of S4.5 **E1** (and, naturally, also **E2**) for *p*/5 and *q*/2: *A*ℭ*ML*5*L*5*AL*2*A*ℭ25ℭ2*N*5 = *ALCM*55*CN*8*ALC*25*LC*24 = *ALC*15*C*1*AL*5*L*3 = *CNL*5*C*1*CN*58 = *CN*5*C*1*C*48 = *C*4*C*15 = *C*45 = 5. This proves that S4.5 is a proper extension of S4.4.

I have no proof that S4.5 is a proper subsystem of S4.6, but it is very probable. Since it is self evident that S4.6 is not only a subsystem of S4.7, but also a subsystem of V1, it should be proved that both these systems, S4.7 and V1, are proper extension of S4.6. Matrix 𝔐3 verifies S4.6, but falsifies a proper axiom of V1. Namely, **V1** is valsified for *p*/2 and *q*/3, *cf.* [15], p. 306. Thus, S4.6 is a proper subsystem of V1. It is an open problem whether S4.7 is a proper extension of S4.6.

3.7 The following diagram:



visualizes the relations among the systems which were discussed in this Section. There are many, open problems connected with this diagram. I mention here only two, namely:

a) Whether S4.5 is a proper subsystem of S4.6, and whether S4.7 is a proper extension of S4.6?

b) To prove that there exist or does not exist an intermediate system between S4.4 and S5 which at the same time is not a subsystem of K4.

Added in proof: 1. Concerning Grzegorczyk's formula which was discussed in Section 2, cf. [20] and [21]. 2. In a letter of 4/14/70, Professor J. Jay Zeman informed the author that the systems S4.5, S4.6 and S4.7 are inferentially equivalent.

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