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NON-EXISTENCE DOES NOT EXIST

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The main aims of this paper are to explain criteria for the identity of individuals, to compare various criteria for the existence of properties and for the existence of propositions, and to present certain theses concerning the existence and identity of individuals, of propositions, and of properties. Several other topics are, however, treated incidentally; for example an extended sentential logic designed to take care of certain semantical paradoxes and truth-value gaps by allowing for statement-incapable sentences is sketched.

In order to attack in a formal way the question of the existence of properties and relations and to formalise widely employed criteria for the existence of attributes, i.e. of properties and relations, an extended predicate calculus must first be introduced. As a first move it is valuable to determine how much can be done in the simplest and most accessible of higher order functional calculi, viz. second-order functional calculus. Now this logic has to be so designed that it can express such propositions as "Some properties do not exist" and "All properties, whether possible or impossible, ... (e.g. exist)". At first this suggests that a system like **R***, which allows for quantification over all possible individual items, be extended to second order.[1] Such an extension of **R*** to second order can be obtained by

(i) relaxing a formation rule of **R*** so that predicate and propositional variables as well as individual variables may be bound, i.e. by replacing the formation rule specifying how 'π' (read 'for all possible') can enter into wff by the rule:

> *if A is a wff then* $(\pi u)A$ *is a wff, where u is any variable (individual, predicate, propositional).*

(ii) replacing 'individual variable' whenever it occurs in the axioms and transformation rules of **R*** by 'variable' and using in these axioms and rules extra-systematic (or syntactic) variables which range over both individual, predicate or propositional variables. These extensions are, however, insufficient to yield a Henkin-complete second-order predicate logic.

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In order to strengthen the axioms sufficiently both predicate and propositional schemes have to be extended; for instance the predicate scheme obtained from the enlargement of **R***,

$\vdash (\pi f)A \supset \check{S}^{f}_{g}\, A|$, where g is a predicate variable or a consistent predicate constant,

is widened to allow substitution of *any* wff for f, i.e. the scheme is replaced by:

$\vdash (\pi f)A \supset \check{S}^{f}_{B}\, A|$, where B is any wff.

B need not even be consistent. Thus admitting this new scheme is tantamount to allowing a further extension of attribute quantifiers beyond possible attributes; it is tantamount to allowing that *attribute quantifiers* can, under an interpretation of the logic, *range over any attributes whether possible or impossible*. Consequently there is no reason for selecting notation which indicates possibility-restricted quantifiers; the completely unlimited quantifiers 'A' and 'S' read, respectively, 'for all (whether possible or impossible)' and 'for some' can be used without hazard. For analogous reasons the usual propositional schema

$$\vdash (\forall p)A \supset \check{S}^{p}_{B} A|$$

using a universal but existentially-restricted quantifier '$\forall$' can be replaced by

$$\vdash (\mathrm{A} p)A \supset \check{S}^{p}_{B} A| \; .$$

Individual quantifiers of **R*** could also be extended, only to make the picture clear and to escape the objection that the theory is simply a subsistence theory the underlying quantifier-free logic would have to be amended, for example by distinguishing sentence and predicate negation. To avoid such amendment quantifiers over individual domains are not here extended; consequently individual variables have substitution-ranges restricted to consistent individual expressions.

To attack the problems at hand a modal extension of second order logic is decidedly advantageous. For one reason various criteria for the existence of properties can be expressed in such a system but not simply in second-order logic; for another identity criteria, of philosophical importance, can be separated in the modal extension but not in the second-order logic itself. So consider first a **S5**-modalised second-order predicate logic **+S52R***, a system of some interest on its own account.

Primitive frame of **+S52R*** The primitive symbols are those of pure second-order predicate logic[2], but with quantifiers 'π' and 'A', together with the modal symbol '$\Box$' and with individual, predicate, and propositional constants including the monadic predicate constant '**E**' (read 'exists'). The formation rules combine those of second-order predicate logic with those for modal logic.

Axiom schemata

R.1: $\vdash ((A \supset B) \supset C) \supset ((C \supset A) \supset (D \supset A))$

R.2: $\vdash (\pi v)(A \supset B) \supset (A \supset (\pi v)B)$, where v is an individual variable which does not occur free in A.

R.2B: $\vdash (\mathrm{A}v)(A \supset B) \supset (A \supset (\mathrm{A}v)B)$, where v is a predicate or propositional variable which does not occur free in A.

R.3A: $\vdash (\pi v)A \supset \check{\mathrm{S}}^{v}_{u}\, A \,|$, where v is an individual variable and u is an individual variable or a consistent individual constant.

R.3B: $\vdash (\mathrm{A}p)A \supset \check{\mathrm{S}}^{p}_{B}\, A \,|$, where p is a propositional variable and B is a wff.

R.3C: $\vdash (\mathrm{A}f)A \supset \check{\mathrm{S}}^{f(v_1 \,..\, v_n)}_{B}\, A \,|$, where f is an n-adic predicate variable and $v_1, v_2, \ldots v_n$ are n distinct individual variables and B is a wff.[3]

R.4: $\vdash \Box(A \supset B) \supset (A \supset \Box B)$, where every variable in A is modalised, i.e. within the scope of '$\Box$'.

R.5A: $\vdash \Box A \supset (\pi v)A$, where v is an individual variable.

R.5B: $\vdash \Box A \supset (\mathrm{A}v)A$, where v is a predicate or propositional variable.

Transformation rules:

R1	$\vdash A,\ \vdash A \supset B$	$\rightarrow \vdash B$	(modus ponens)
R2	$\vdash A$	$\rightarrow \vdash \Box A$	(necessitation)

Derived rules:

$\vdash A \rightarrow\ \vdash (\pi v)A$, where v is an individual variable

$\vdash A \rightarrow\ \vdash (\mathrm{A}v)A$, where v is a predicate or proposition variable (generalisation)

Proof: From **R.5, R1** and **R2.**

Definitions:

D1.	$\not f$	$=_{Df}$	$(\mathrm{A}p)p$
D2.	$\sim A$	$=_{Df}$	$(A \supset \not f)$
D3.	$(\Sigma u)A$	$=_{Df}$	$\sim(\pi u)\sim A$
D4.	$\not t$	$=_{Df}$	$\sim \not f$
D5.	$A \vee B$	$=_{Df}$	$(A \supset B) \supset B$
D6.	$A \,\&\, B$	$=_{Df}$	$\sim(A \supset \sim B)$
D7.	$A \equiv B$	$=_{Df}$	$(A \supset B) \,\&\, (B \supset A)$
D8.	$\Diamond A$	$=_{Df}$	$\sim \Box \sim A$
D9.	$A \mathrel{-\!\!\!3} B$	$=_{Df}$	$\Box (A \supset B)$
D10.	$A \equiv B$	$=_{Df}$	$(A \mathrel{-\!\!\!3} B) \,\&\, (B \mathrel{-\!\!\!3} A)$
D11.	$\Diamond(f)$	$=_{Df}$	$\Diamond (\Sigma v_1) \ldots (\Sigma v_n) f(v_1 \ldots v_n)$
D13.	$(\pi p)A(p)$	$=_{Df}$	$(\mathrm{A}p)(\Diamond p \supset A(p))$
D14.	$(\mathrm{S}v)A$	$=_{Df}$	$\sim(\mathrm{A}v)\sim A$, where v is a predicate or propositional variable.

D15. $(\exists v)A(v) =_{Df} (\Sigma v)(\mathsf{E}(v) \,\&\, A(v))$ }
D16. $(\forall v)A(v) =_{Df} (\pi v)(\mathsf{E}(v) \supset A(v))$ } where v is an individual variable.
D18. $(\pi f)A(f) =_{Df} (\mathrm{A}f)(\Diamond(f) \supset A(f))$

General schemes for '$\exists$' and '$\forall$' cannot be introduced at this stage since '$\mathsf{E}(f)$' and '$\mathsf{E}(p)$' (where 'f' and 'p' range respectively over predicate variables and constants and propositional variables and constants) have not been defined.

Theorems and metatheorems:

T1. *Every theorem of propositional calculus is a theorem of* **+S52R***.

Proof indication: **R.1** is Łukasiewicz's shortest axiom for the pure implicational calculus[4], and it, together with **R.2B** and **R.3B**, modus ponens and generalisation and definitions D1-D7, gives the full propositional calculus.[5]

T2. $\vdash \Box A \supset A$

Proof: By **R.5B**, **R.3B** & T1

T3. $\vdash \Box(A \supset B) \supset. \Box A \supset \Box B$

Proof:
(i) $\vdash (\Box A \supset A) \supset ((A \supset B) \supset (\Box A \supset B))$; by T1, T2
(ii) $\vdash ((A \supset B) \supset (\Box A \supset B))$; from (i)
(iii) $\vdash \Box(A \supset B) \supset (\Box A \supset B)$; from T2, (ii), by T1
(iv) $\vdash \Box(A \supset B) \supset \Box(\Box A \supset B)$; from (iii), T2, **R.4**
(v) $\vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$; by **R.4**
(vi) $\vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$; from (iv) and (v)

T4. $\vdash \Box A \supset \Box\Box A$

Proof: $\vdash \Box(\Box A \supset \Box A) \supset (\Box A \supset \Box\Box A)$; from **R.4**
Result by **R2, R1,** T1.

T5. $\vdash \sim\Box A \supset \Box\sim\Box A$

Proof: $\vdash \Box(\sim\Box A \supset \sim\Box A) \supset (\sim\Box A \supset \Box\sim\Box A)$; from **R.4**
Result by **R2, R1,** T1.

T6. *Every theorem of* **S5** *is a theorem of* **+S52R***

Proof: From T1, **R2**, T2, T3, T5.

T7. *Every theorem of* **S5R*** *is a theorem of* **+S52R*** [6]

Proof: **+S52R*** includes **R***, and **S5R*** has as postulates postulates of **R*** and of **S5.**

T8. **+S52R*** *is consistent*

Proof: Consider the map M which maps 'π' to 'A', '$\Box$' to '(Av)', and otherwise is an identity map. Under M **+S52R*** maps onto a system equivalent to that obtained from Church's $\mathbf{F}^2$ by replacing '$\forall$' by 'A'. For the restriction of **R.3A** to consistent constants can be lifted. $\mathbf{F}^2$ is consistent (in various senses)[2]. Hence **+S52R*** is consistent.

T9. $\vdash (\pi v)(A \supset B) \supset . A \supset (\pi v)B$, *where* v *is a variable of any sort which does not occur free in* A.

Not all results which hold for unrestricted quantifiers 'A' and 'S' hold for consistent quantifiers 'π' and 'Σ' or existential quantifiers '$\forall$' and '$\exists$'. In particular, universal instantiation and particular generalisation are qualified.

T10. $\vdash (\pi p)A(p) \supset \check{S}^{p}_{B}\, A(p)\,|$, *provided* B *is consistent,* i.e., $\Diamond B$

T11. $\vdash (\pi f)A(f) \supset \check{S}^{f(v_1 \ldots v_n)}_{B}\, A(f)\,|$, *provided* B *is consistent*

T12. $\vdash \check{S}^{p}_{B} A\,| \supset (\mathrm{S}p)A$, *with provisos as in* **R.3B**

T13. $\vdash \check{S}^{f(v_1, v_2 \ldots v_n)}_{B} A\,| \supset (\mathrm{S}f)A$, *with provisos as in* **R.3C**

T14. $\vdash \check{S}^{p}_{B} A(p)\,| \supset (\Sigma p)A(p)$, *provided* B *is consistent*

T15. $\vdash \check{S}^{f(v_1 \ldots v_n)}_{B} A(f)\,| \supset (\Sigma f)A(f)$, *provided* B *is consistent*

T16. *Scheme* (A) (of footnote 3) *is a theorem-schema of* **+S52R***

Proof: Substituting $(\pi x_1, x_2 \ldots x_n)(A \equiv A)$ for A in T13.

T17. $\vdash (\Sigma f)(\pi v_1, v_2 \ldots v_n)(f(v_1, v_2 \ldots v_n) \equiv A)$, *where* A *is a consistent wff containing distinct individual variables* $v_1, v_2 \ldots v_n$ *and* f *is an* n*-adic predicate which does not occur free in* A.

The restriction to consistent wff in preceding theorems plays an important role in those solutions of logical paradoxes which class paradoxical items as impossibilia, once orders are dropped.

T18. $\vdash (\mathrm{S}f)(\pi x)f(x)$, i.e. some property is a property of every possible item.

$\vdash (\Sigma f)(\pi x)f(x)$

Proof: Putting $A(x) \vee \sim A(x)$, where A is consistent, for A in T16 and T17.

T19. $\vdash (\pi x)(\Sigma f)f(x)$, i.e. every consistent item has some consistent property.

Despite its initial attractiveness **+S52R*** has some serious defects, exhibited in particular in the following theorems:

T20. $(\Sigma x)A \supset \Diamond A$. Proof from **R.5A**

T21. $(\Sigma x)A \supset (\pi x)\Diamond A$

Proof by T20, generalisation, and **R.5A**.

T22. $(\Sigma x)(\Diamond A \supset \Box A) \supset (\pi x)(\Diamond A \supset \Box A)$ (principle of predication)

Proof: $(\Sigma x)(\Diamond A \supset \Box A) \supset. \Diamond(\Diamond A \supset \Box A)$, by T20
$\supset. \Diamond A \supset \Box A$, by **S5** principles.
Result by generalisation and **R.2A**.

For the principle of predication, a principle tied to essentialism in modal logic, is a completely unsatisfactory principle (for details see IE), and T21 is false if essentialism is: for some consistent item is red but not every consistent item, e.g. a non-red one, is possibly red.

A logic **S52R***, which escapes these consequences, is obtained from +**S52R*** by dropping postulates **R.5A** and **R.5B** and adding instead as postulates T1, T2, T5 and the rules of generalisation. **S52R*** is consistent since it is a subsystem of +**S52R***.

Theorem. T20 *and* T21 *are not theorems of* **S52R***. *Hence* **S52R*** *is a proper subsystem of* +**S52R***.

Proof outline: A countermodel to T21, and thereby to T20, can be constructed by taking a two individual model, with distinct elements a and b, for Church's system $\mathbf{F}^2$ and equating $\Box A$ with A. Then every theorem of **S52R*** is true in the model, but instances of T21 such as

$$a = a \vee a = b \supset. \sim\sim(a = a \ \& \ a = b)$$

are false.

Identity of individuals with respect of **S52R*** When extensional identity and strict identity of individuals are distinguished the Leibniz identity criterion, usually adopted in higher predicate logics, proves inadequate.[7] And short of extending **S52R***, for instance in the fashion explained below, there is no decent option to introducing the extensional identity sign, written '≡', as a further primitive. Call the resulting system **S52R***. The axiom-schemes for '≡' are

1. $\vdash x \equiv x$
2. $\vdash (x \equiv y) \supset (A \supset B)$,

where x and y are individual variables or constants and B is obtained from A by replacing one particular occurrence of x by y, this occurrence of x being neither within the scope of (πx) or (πy) nor modalised (*proviso* (α)).

Strict identity is then defined.

D17A. $x \equiv y =_{Df} \Box(x \equiv y)$,

and it can be shown that strict identity satisfies

1. $\vdash x \equiv x$
2. $\vdash (x \equiv y) \supset (A \supset B)$,

with proviso (β), where proviso (β) differs from proviso (α) in omitting the clause 'nor modalised'[8]. It further follows, in ≡ **S52R***, but not in more comprehensive intensional logics, that the Leibniz criterion holds for strict identity: that is

$$\vdash (x \overset{\equiv}{\equiv} y) \equiv (\mathrm{A}f)(f(x) \equiv f(y)).$$

Proof: (1) $\vdash (x \overset{=}{\equiv} y) \supset (\mathrm{A}f)(f(x) \equiv f(y))$
For $\vdash (x \overset{=}{\equiv} y) \supset . \Box(f(x) \equiv f(y))$ by **D17A** and $\overset{=}{\equiv}$ 2
$\supset . (\mathrm{A}f)(f(x) \equiv f(y))$
(2) $\vdash (\mathrm{A}f)(f(x) \equiv f(y)) \supset (x \overset{=}{\equiv} y)$
$\vdash (\mathrm{A}f)(f(x) \equiv f(y)) \supset (x \overset{=}{\equiv} x \supset . x \overset{=}{\equiv} y)$

and (2) follows by T1 and $\overset{=}{\equiv}$ 1.

To introduce '≡' as an extra primitive is displeasing, since it sacrifices a major gain in economy of second-order systems over first-order ones. Worse, no purely extensional (i.e. non-strict extensional) identity can be asserted in ≡**S52R***: for if such an identity were asserted a strict identity could be derived using necessitation, so contradicting the assumption that the identity is not strict. Indeed no purely contingent truth can be asserted in **S5R*** or in **S52R***.

To overcome these defects two moves are made. First analytic assertion, symbolised '⊢', is distinguished from (synthetic or analytic) assertion, symbolised '⊩'. Second the formation rules are relaxed so as to allow wff of the form $f(p_1, p_2 \ldots p_n)$, where f is an n-adic predicate (or functor) and $p_1, p_2 \ldots p_n$ are propositional variables. Given the further symbolism this relaxation permits, *extensionality* of properties can be defined. A system +**R***, in which these moves are carried through, is now presented.

The system +**R*** The primitive frame of +**R*** differs from that of **S52R*** with respect to analytic assertion as follows:

(i) The formation rule of **S52R*** for predicates is replaced by the rule:

If f is an n-adic predicate (or functorial) variable or constant and if each $u_1, u_2 \ldots u_n$ is either an individual variable or a consistent individual constant or a propositional variable or constant then $f(u_1, u_2 \ldots u_n)$ is a wff.

It follows that '$\mathsf{E}(p)$' is a wff. No expression of the form $f(\ldots g(\ldots p\ldots)\ldots)$ with one propositional function nested in another is however a wff.

(ii) Scheme **R.3C** is extended to

+**R.3C**: $\vdash (\mathrm{A}f)A \supset \check{\mathsf{S}}_B^{f(v_1 \ldots v_n)} A|$, where f is an n-adic predicate (or functorial) variable and $v_1, v_2 \ldots v_n$ are n distinct individual or propositional variables.

(iii) Scheme **R.3B** is curtailed to

+**R.3B**: $\vdash (\mathrm{A}p)A \supset \check{\mathsf{S}}_B^{p} A|$, where B is a wff and p is a propositional variable which is not a place holder in a predicate wff, i.e. p is not within the scope of a predicate symbol; or else B is a propositional variable or constant and p is a propositional variable.

Otherwise the postulates of +**R*** for analytic assertion are those of **S52R***. But the following postulates for assertion also belong to the frame of +**R***.

Axioms:

+**R.6**: $\Vdash (\Sigma x)\mathsf{E}(x)$
+**R.7**: $\Vdash (\Sigma x) \sim \mathsf{E}(x)$

Transformation rules:

R+R.3 $\vdash A \rightarrow \Vdash A$
R+R.4 $\Vdash A, \Vdash A \supset B \rightarrow \Vdash B$[9]

Definitions D1-D16 are transferred to +**R***

Theorems and Metatheorems

T1 *Every theorem of* **S52R*** *is a theorem, both asserted and analytically asserted, of* +**R***.

On the other hand not every analytically asserted theorem of +**R*** is a theorem of **SR2R***.

T2 +**R*** is consistent

Proof sketch: The argument turns on showing that every theorem of +**R*** has a tautology as afp, i.e. as associated wff of a propositional calculus which contains $\not{t}$ and $\not{f}$ as primitive symbols. [The semantical motivation of the argument is this: To establish consistency a model is selected over which +**R*** reduces to propositional calculus. The model has two individuals, the existent $\not{t}$ and the non-existent $\not{f}$ and these are identified with the two propositions, the true and the false, of the model. Further all functions of propositions (and individuals) are, over the model, reckoned to be truth functional, and thus predicates can be eliminated.] The afp of a wff A of +**R*** is obtained by the following steps. Delete all vacuous individual quantifiers and all occurrences of '□' and bind all free individual variables. Replace each expression of the form $(\pi x)B$ by $(\check{\mathbf{S}}^{x}_{\not{t}}B| \& \check{\mathbf{S}}^{x}_{\not{f}}B|)$. Replace $\mathsf{E}(\not{t})$ by $\not{t}$ and $\mathsf{E}(\not{f})$ by $\not{f}$. The wff A' obtained from A by these steps is a wff of a protothetic which contains $\not{t}$ and $\not{f}$ as primitive symbols. By treating all propositional functors of A' as truth-functional A' can be replaced by a conjunction A'' of wff of an extended propositional calculus which contains $\not{t}$ and $\not{f}$ as primitive symbols[10]. Finally the afp A''' of A is obtained by replacing all wf parts of A'' of the form $(\mathrm{A}p)C$ by $(\check{\mathbf{S}}^{p}_{\not{t}}C| \& \check{\mathbf{S}}^{p}_{\not{f}}C|)$. (Assertion signs can simply be deleted).

T2.1 *Every theorem of* +**R*** *has a tautology as afp.*

To show this in full it would have to be shown that every axiom has a tautology as afp and that the rules of inference preserve this property. From T2.1 it follows that a propositional variable standing alone, since not a tautology, is not a theorem of +**R***. Hence

T2.2 +**R*** *is Post-consistent, absolutely consistent, and consistent, with respect to the transformation of A into $\sim A$.*

Consistency of +**R*** may appear to be purchased at a price: for it is not possible to add as further postulate-sentences such sentences as

(1p): *It is possible that the only Cretan statement is that all Cretan statements are false*, *without sacrificing consistency*[11].

In the context (possible world) in which 'All Cretan statements are false' is the only Cretan declarative sentence, the sentence is statement-incapable, i.e. does not yield a statement, because of vicious content self-dependence[12]. Thus the sentence (1p) cannot be taken as expressing a proposition at all and so cannot be added to the postulate list of +**R***. [If (1p) were taken, under an alternative solution proposal for the paradoxes (which can however be subsumed under the proposal just indicated), to express a proposition then it would express, applying **S5**, a logically false proposition. So again (1p) could not consistently be added to the postulate-list]. These issues do raise important questions as to the interpretation of +**R***. To simplify an attack on these issues consider just the sentential or propositional segment of +**R***. The variables (and constants) of this segment are amenable to various rival though related intended-interpretations, i.e. as sentential variables, as truth-value variables, as propositional variables, as variables with sentences as substitution-ranges and truth-values as designation-ranges. Interpretations have also been proposed[13] which synthesize these interpretations: e.g. a variable has sentences as substitution-ranges, truth values as extensional designation-ranges or value-extensions, and propositions as intensional designation-ranges or value-intensions. If not all significant declarative sentences express propositions, such a synthesis breaks down. Here a different combination of interpretations, which allows for statement-incapable sentences, is sketched.

Interpretation 1: Variables have significant declarative sentences as substitution-ranges. Each of these sentences either is statement-incapable, in which case it is assigned value $+i = \{+1, -1\}$, or is statement-capable, in which case it is assigned value $+1$ or -1 according as the statement it yields (in its context) is true or false. The connectives which connect the variables are *sentential* connectives. Values for such connectives as '&', 'v', '⊃' (here 'if ... then materially ---': not 'implies') are determined by treating a variable which has value $+i$ as if it had bracketed with it the ordered pair of values $\langle +1, -1 \rangle$. Thus the matrix for '~' can be written in full:

~	
1	-1
+1, -1 } +i	-1, + } +1
-1	1

The values +1 and -1 whether bracketed or not are combined according to the usual truth-table valuations. Resulting bracketed values are assessed according to the schemes:

$$\left.\begin{matrix}1\\1\end{matrix}\right] = 1, \quad \left.\begin{matrix}1\\-1\end{matrix}\right] = +i, \quad \left.\begin{matrix}-1\\+1\end{matrix}\right] = +i, \quad \left.\begin{matrix}-1\\-1\end{matrix}\right] = -1\ ;$$

these schemes are also used in reverse. The following matrices result:

v	1	+i	-1
1	1	1	1
+i	1	$\left.\begin{matrix}1\\+1\end{matrix}\right]$	+1
-1	1	+i	-1

&	1	+i	-1
1	1	i	-1
i	i	$\left.\begin{matrix}+i\\-1\end{matrix}\right]$	-1
-1	-1	-1	-1

⊃	1	i	-1
1	1	+i	-1
i	1	$\left.\begin{matrix}i\\1\end{matrix}\right]$	i
-1	1	1	1

where the paired values are alternative possibilities for bracketed values of either p or q. For instance when p and q both have value +i this scheme for 'v' appears -

$$\begin{array}{ccc} p & \vee & q \\ \left[\begin{matrix}1\\1\end{matrix}\right] & \left(\begin{matrix}1\\1\end{matrix}\right) & \left.\begin{matrix}1\\-1\end{matrix}\right] \\ \left[\begin{matrix}-1\\-1\end{matrix}\right] & \left(\begin{matrix}1\\-1\end{matrix}\right) & \left.\begin{matrix}1\\-1\end{matrix}\right] \end{array}$$

with values 1 and +i for both p bracketings and q bracketings. Apart from paired values the matrices are those of Łukasiewicz's Ł3.[5]

Besides their intended role, the matrices can be adapted to provide a logical model which allows for truth-value gaps, for example for Aristotelian neuter sentences as usually explained. To illustrate: even if the sentence 'There will be a sea battle tomorrow' is neuter, the sentence 'Either there will be a sea battle tomorrow or there will not be a sea battle tomorrow' has value 1 and yields a true statement. For consider the valuation:

$$\begin{array}{ccc} p & \vee & \sim p \\ 1 & 1 & -1 \\ \left.\begin{matrix}+1\\-1\end{matrix}\right] i \left.\begin{matrix}1\\1\end{matrix}\right] & 1 & \left.\begin{matrix}-1\\+1\end{matrix}\right] i \\ -1 & 1 & 1 \end{array}$$

The matrices are not fully truth-valued. But although the matrices only provide partial truth-tables (in the many-valued sense), there is an effective procedure for deciding whether any given wff of sentential logic is a *+-tautology*, i.e. has as a value designated value +1 only. It follows

T3. *Every sentential theorem is a +-tautology.*

This result also holds when the logic contains $\not t$ or $\not f$ where these are constants with fixed values +1 and -1 respectively. '$\not f$' can still be satisfactorily defined as '(Ap)p', because some sentence has value -1, and the quantifier 'A' has the following value stipulation:

(Ap)B(p) has value 1, -1, +i according as B(p) has value +1 for all substitution-values for p, has value-1 for some substitution-value for p, or has value +i for some substitution-value for p and +1 for all other substitution-values for p if any. These values practically coincide with values which would be obtained by treating universal expressions as conjunctions. Further the values are precisely those needed to lop off sophisticated semantical paradoxes. Very briefly,

(2p): 'All Cretan statements are false', in contexts in which all other Cretan statements, if any, are true, is statementally equivalent to (3p): 'This very Cretan statement is false', which, for reasons of vicious content self-reference, has value +i. Hence (2p) has value +i. For (2p) can be symbolised '$(\mathrm{A}p)(\mathbf{c}(p) \supset \sim p)$'; and all values of p in this are 1 except for (3p) when the value is $+i$.

Under this interpretation 1 the sentential calculus is not functionally complete once fully truth-valued connectives which have uniform values of +1 or -1 for components with value +i are admitted. Thus such semantical connectives as '$\not{\mathbf{C}}$', read 'is statement-incapable' and '$\not{\mathbf{T}}$' read (in contrast with '$\mathbf{T}$' i.e. 'that . . . is true') 'that . . . is unlimitedly true', with matrices

$\not{\mathbf{C}}$	
1	+1
+i	-1
-1	+1

$\mathbf{T}$	
1	1
+i	-1
-1	-1

Contrast: $\not{\mathbf{T}}$	
1	1
+1, -1] +i	+1, -1] +i
-1	-1

can be introduced. Axioms for a functional complete set of connectives are not adduced here. Relevant points can, however, be made using the matrices given and the definitions:

D1': $\mathbf{F}p =_{Df} \mathbf{T} \sim p$

D2': $\mathrm{Prop}(p) =_{Df} \mathbf{T}p \vee \mathbf{F}p$.

'Prop(p)' reads 'that p expresses a proposition'. First, the law of excluded middle in the form $\mathbf{T}p \vee \mathbf{F}p$ is not a theorem. Thus Prop(p) though always true or false is not a theorem: not every sentence yields a statement. Second, the theorem

$$\vdash \mathrm{Prop}(p) \equiv \not{\mathbf{C}}p \equiv \mathbf{T}p \vee \mathbf{F}p$$

opens the way to

Interpretation 2: The variables of sentential calculus have as substitution-values statement-capable sentence, and thus they yield, or have indirectly, as values propositions. Interpretation 2 can be derived from interpretation 1 by introducing propositional variables, distinguished by Clarendon type, and defined through

D3': $(\mathrm{A}\underset{\sim}{p})B(\underset{\sim}{p}) =_{Df} (\mathrm{A}p)(\mathrm{Prop}(p) \supset B(p))$

With propositional variables a generality interpretation can also be employed. Thus if $A(p)$ is an axiom of sentential calculus then $(\mathrm{Prop}(p) \supset A(p))$ is a theorem. Hence $A(\underset{\sim}{p})$ is a theorem. Also

$(\mathrm{A}\underset{\sim}{p})(A \supset B(\underset{\sim}{p})) \supset. (\mathrm{A}p)(A \supset. \mathrm{Prop}(p) \supset B(p))$
$\supset. A \supset (\mathrm{A}\underset{\sim}{p})B(\underset{\sim}{p})$, provided $\underset{\sim}{p}$ does not occur free in A.

Continuing for further axioms and rules, it follows

T4: *The extended sentential calculus holds for propositional variables.*

Propositional connectives to replace sentential connectives may also be defined as in

D4': $(\mathrm{A}\underset{\sim}{p},\underset{\sim}{q})\ .\underset{\sim}{p} \underset{\sim}{\supset} \underset{\sim}{q} =_{Df} (\mathrm{A}p,q)\ .\mathrm{Prop}(p) \supset. \mathrm{Prop}(q) \supset. p \supset q$, where '$\underset{\sim}{\supset}$' reads 'that ... materially implies that ---'.

Finally, in a wff where all connectives and variables are propositional, connexions with sentences may be cut completely (and propositional expressions used where expressions are wanted). A logic with only wff of this sort is a purely propositional logic.

Although +**R*** is consistent it is not complete under the intended interpretation.[14] For the interpretation law $\Box(\pi x)\Box\mathbf{E}(X)$ is valid under the intended interpretation but it is not a theorem of +**R***, as the representation used to establish consistency shows: $(\not t\ \&\not f)$ is the afp. Similarly Meinong's law $\Box(\pi x){\sim}\Box\mathbf{E}(x)$ is not a theorem. The interpretation law follows immediately once 'A' and 'S' quantifiers supplant 'π' and 'Σ'. This replacement would provide several gains. To illustrate $(\mathrm{S}x)\Box\sim\mathbf{E}(x)$ could be deduced, e.g. from $\Box\sim\mathbf{E}((\iota x)(f(x)\&\sim f(x)))$ by particular generalisation, though a parallel conclusion does not follow for 'Σ'.

Identity of individuals To define identity auxiliary definitions of 'extensional in the i^{th} place', 'extensional', 'extensional or modal in the i^{th} place', and 'extensional or modal', abbreviated respectively 'ext_i' 'ext ', 'em_i' and 'em', are introduced.

D17 $\mathrm{ext}_i(f^n) \equiv_{Df} (\mathrm{A}p)(\mathrm{A}q)(p \equiv q \supset. (\pi u_1)..(\pi u_{i-1})(\pi u_{i+1})...(\pi u_n)$

$$(f(u_1..u_{i-1}\,p,..u_n) \equiv f(u_1..qu_{i+1}..u_n))$$

D18 $\mathrm{em}_i(f^n) \equiv_{Df} (\mathrm{A}p)(\mathrm{A}q)\ p \overset{=}{\equiv} q \supset. (\pi u_1)..(\pi u_n)(f(u_1..p..u_n) \equiv f(u_1..q..u_n))$
D19 $\mathrm{ext}(f^n) \equiv_{Df} \mathrm{ext}_1(f^n)\ \&\ \mathrm{ext}_2(f^n)\ \&...\&\ \mathrm{ext}\,(f^n)$.
D20 $\mathrm{em}(f^n) \equiv_{Df} \mathrm{em}_1(f^n)\ \&....\&\ \mathrm{em}_n(f^n)$.

It follows for monadic predicates:

$\vdash \mathrm{ext}(f) \overset{=}{\equiv} (\mathrm{A}p)(\mathrm{A}q)(p \equiv q \supset. f(p) \equiv f(q))$
$\vdash \mathrm{em}(f) \overset{=}{\equiv} (\mathrm{A}p)(\mathrm{A}q)(p \overset{=}{\equiv} q \supset. f(p) \equiv f(q))$;

and it follows

$\vdash \mathrm{ext}(\vee)\ \&\ \mathrm{ext}(\supset)\ \&\ \mathrm{ext}(\sim)\ \&\ \mathrm{ext}(\&)$
$\vdash \mathrm{em}(\Box)\ \&\ \sim\mathrm{ext}(\Box)\ \&\ \mathrm{em}(\Diamond)\ \&\ \sim\mathrm{ext}(\Diamond)$
$\vdash \mathrm{ext}(f) \supset \mathrm{em}(f)$

If 'f' is a predicate true only over some individuals, like 'is (unlimitedly) tall', then as both $f(p)$ and $f(q)$ are false 'f' is extensional.
Extensional and strict identity are defined:

D21. $x \equiv y \equiv_{Df} (\mathrm{A}f)(\mathrm{ext}(f) \supset . f(u) \equiv f(v))$
D22. $x \stackrel{\equiv}{\equiv} y \equiv_{Df} (\mathrm{A}f)(\mathrm{em}(f) \supset . f(u) \equiv \bar{f}(v))$

Actually more general definitions can be formulated:

D21'. $u \equiv v \equiv_{Df} (\mathrm{A}f)(\mathrm{ext}(f) \supset . f(u) \equiv f(v))$
D22'. $u \stackrel{\equiv}{\equiv} v \equiv_{Df} (\mathrm{A}f)(\mathrm{em}(f) \supset . f(u) \equiv f(v))$

It follows:

$\vdash x \equiv x;$
$\vdash x \stackrel{\equiv}{\equiv} x$
$\vdash x \equiv y \supset y \equiv x;$
$\vdash x \stackrel{\equiv}{\equiv} y \supset y \stackrel{\equiv}{\equiv} x$
$\vdash (x \equiv y) \;\&\; (y \equiv z) \rightarrow (x \equiv z).$

Here the premisses can be strengthened to strict identity since

$\vdash (x \stackrel{\equiv}{\equiv} y) \prec (x \equiv y).$

But in

$\vdash (x \stackrel{\equiv}{\equiv} y) \;\&\; (y \stackrel{\equiv}{\equiv} z) \prec (x \stackrel{\equiv}{\equiv} z),$

the premisses cannot be weakened to extensional identity.

$\vdash \Box(x \equiv y) \equiv (x \stackrel{\equiv}{\equiv} y).$

Proof: (i) $x \stackrel{\equiv}{\equiv} y \supset \Box(x \equiv y)$
For: $x \stackrel{\equiv}{\equiv} y \supset . (p \stackrel{\equiv}{\equiv} p \supset . \Box(x \equiv p) \equiv \Box(x \equiv p)) \supset . \Box(x \equiv x) \supset \Box(x \equiv y)$
from D21'
Result by **S5** and $\vdash x \stackrel{\equiv}{\equiv} x$

$\vdash (\mathrm{x} \stackrel{\equiv}{\equiv} \mathrm{y}) \equiv \Box(\mathrm{x} \stackrel{\equiv}{\equiv} \mathrm{y})$

Proof: $\vdash \Box(x \stackrel{\equiv}{\equiv} y) \equiv \Box\Box(x \equiv y)$
$\equiv \Box(x \equiv y)$

$\vdash (\mathrm{A}p)(\mathbf{E}(p) \equiv p) \supset . \sim(\mathrm{A}p)(x \equiv p)$

Proof: $\vdash x \equiv p \supset . ((p \equiv p) \supset . \mathbf{E}(p) \equiv \mathbf{E}(p)) \supset . \mathbf{E}(x) \equiv \mathbf{E}(p)$; from D21'.
$\vdash (\mathrm{A}p)(\mathbf{E}(p) \equiv p) \supset . (\mathrm{A}p)(x \equiv p \supset . \mathbf{E}(x) \equiv p)$
$\supset . (x \equiv \sim\mathbf{E}(x) \supset . \mathbf{E}(x) \equiv \sim\mathbf{E}(x))$
$\supset . \sim(x \equiv \mathbf{E}(x))$
$\supset . \sim(\mathrm{A}p)(x \equiv p)$

$\Vdash \sim\mathbf{E}(p) \supset . (\Sigma x) \sim(x \equiv p)$

Proof: $\vdash x \equiv p \supset . \mathbf{E}(x) \supset \mathbf{E}(p)$; as in previous theorem
$\vdash \sim\mathbf{E}(p) \supset . (\Sigma x)\mathbf{E}(x) \supset (\Sigma x)\sim(x \equiv p)$
$\Vdash (\Sigma x)\mathbf{E}(x) \supset . \sim\mathbf{E}(p) \supset (\Sigma x) \sim(x \equiv p)$
Result using +**R.6** and R +**R.4**

T5. *If all primitive predicates (including* '**E**'*) of* +**R*** *are extensional then*

(i) extensional identity satisfies $\equiv 2$.
(ii) strict identity satisfies $\stackrel{\equiv}{\equiv} 2$.
(iii) $(x \stackrel{\equiv}{\equiv} y) \equiv (\mathrm{A}f)(f(x) \equiv f(y))$.

Proof: If (ii) holds then (iii) follows (as above in T9). If (i) holds then (ii) follows as indicated above. It remains to establish (i). The proof is by induction over the number of occurrences of primitive symbols '$\sim$', '$\supset$', 'π', and 'A' in wff A. For a primitive predicate 'f', by hypothesis, $\vdash \text{ext}(f)$. Hence

$$\vdash x \equiv y \supset . f(x) \supset f(y)$$

This provides the basis of the induction. In the inductive step these cases are considered.

(i) A is of the form $\sim A_1$. Then B is of the form $\sim B_1$. By induction hypothesis, $\vdash x \equiv y \supset . A_1 \supset B_1$, with proviso ($\alpha$), and also therefore, by symmetry, $\vdash x \equiv y \supset . B_1 \supset A_1$, with proviso ($\alpha$). Hence $\vdash x \equiv y \supset . \sim A_1 \supset \sim B_1$, with proviso ($\alpha$).
(ii) A is of the form $A_1 \supset A_2$. Then B is of the form $B_1 \supset B_2$ where one of B_1 and B_2 is the same as one of A_1 and A_2. By induction hypothesis

$$\left.\begin{array}{l} \vdash x \equiv y \supset . A_1 \supset B_1 \\ \vdash x \equiv y \supset . B_1 \supset A_1 \\ \vdash x \equiv y \supset . A_2 \supset B_2 \\ \vdash x \equiv y \supset . B_2 \supset A_2 \end{array}\right\} \text{ with proviso } (\alpha)$$

Hence $\vdash x \equiv y \supset . (A_1 \supset B_1) \supset (A_2 \supset B_2)$, with proviso ($\alpha$), by propositional calculus and combining the provisos.
(iii) A is of the form $(\text{A}u)A_1$ (or $(\pi x)A_1$). By induction hypothesis, $\vdash x \equiv y \supset . A_1 \supset B_1$, with proviso ($\alpha$).

$\vdash(\text{A}u)(x \equiv y \supset . A_1 \supset B_1)$, with proviso ($\alpha$) for A_1 and B_1.
$\vdash x \equiv y \supset . (\text{A}u)A_1 \supset (\text{A}u)B_1$, with proviso ($\alpha$) for A_1 and B_1 and with u distinct from x and y, i.e.
$\vdash x \equiv y \supset . (\text{A}u)A_1 \supset (\text{A}u)B_1$, with proviso ($\alpha$) for A and B.

The case for 'π' is similar.

T6. *If all primitive predicates of* +**R*** *are* em, *i.e. extensional or modal, then*

(i) *strict identity satisfies* $\equiv\!\!\!\equiv 2$.
(ii) $(x \equiv\!\!\!\equiv y) \equiv (\text{A}f)(f(x) \equiv f(y))$.

Proof: (ii) follows from (i) (as in T9 above). Proof of (i) is analogous to proof of (i) in T5, except that induction is over the number of occurrences of '$\sim$', '$\supset$', 'A', 'π' and '$\Box$', and that induction starts from

$$\left.\begin{array}{l} \vdash \text{em}(f) \\ \vdash x \equiv\!\!\!\equiv y \supset . f(x) \equiv f(y) \end{array}\right\} \text{ where '}f\text{' is a primitive predicate}$$

Theorems 5 and 6 point up a defect in the introduction of identity, as in **S5R*** by schemes like $\equiv 1$ and $\equiv 2$, or $\equiv 1$ and $\equiv\!\!\!\equiv 2$. For these hold *only* provided that the interpretations are restricted in the way the conditions of T5 and T6 require. The schemes narrow, without warrant and without making it explicit, the class of interpretations of the primitive predicates.

The theory of identity presented has one serious deficiency, in the division of predicates into extensional and non-extensional. For under the proposal predicates like 'is possibly tall' come out as extensional! There are various ways out of this difficulty. A first is to treat an individual predicate as extensional only if it has no non-extensional parts, and to assume that all primitive individual predicates are extensional. But it has all the defects of the introduction of identity in **S5R***, and really shelves the problem; for the problem simply reappears at the interpretational level. A more satisfactory way out starts afresh: sentence predicates 'Φ', 'Ψ', etc. are added to **S52R***. Then extensionality of sentence predicates is defined as before (following Russell).

$$\text{ext}\ \Phi \equiv_{Df} (\mathrm{A}p,q)(p \equiv q \supset .\ \Phi(p) \equiv \Phi(q));$$

and extensionality of individual predicates is then defined thus:

$$\text{ext}(f) \equiv_{Df} \sim (\mathrm{S}\Phi)(\mathrm{S}g)[(\pi x)(f(x) \equiv \Phi g(x))\ \&\ \sim \text{ext}\ \Phi].$$

In other words, an individual predicate is extensional if it does not decompose into an expression containing a non-extensional sentence predicate. Under this definition all purely non-contingent properties are non-extensional.

Equivalence and identity of propositions Certain criteria for the equivalence and identity of contents of sentences, whether the sentences are statement-capable or not, are expressible in the formalism developed. Two equivalence relations, which are here symbolised '$p \dot{\equiv} q$' and '$p \dot{\bar{\equiv}} q$', are, in fact, ejected by D21' and D22'. These expressions express only very weak equivalence relations between propositions: for they amount, respectively, to material and strict equivalence.

$$\vdash (p \dot{\equiv} q) \equiv (p \equiv q).$$

Proof: (i) $\vdash p \dot{\equiv} q \supset .(p \equiv p \supset .(p \equiv p) \supset (p \equiv p)) \supset .\ (p \equiv p) \equiv (p \equiv q)$; from D21'

$\therefore \vdash p \dot{\equiv} q \supset .\ p \equiv q$

(ii) $p \equiv q,\ (\mathrm{A}p')(\mathrm{A}q')(p' \equiv q' \supset .\ f(p') \equiv f(q')) \vdash f(p) \equiv f(q)$

$p \equiv q \vdash \text{ext}(f) \supset .\ f(p) \equiv f(q)$

$p \equiv q \vdash (\mathrm{A}f)(\text{ext}(f) \supset .\ f(p) \equiv f(q))$; choosing f appropriately.

$\therefore \vdash p \equiv q \supset .\ p \dot{\equiv} q$; by deduction theorem.

$\vdash (p \dot{\bar{\equiv}} q) \equiv (p \bar{\equiv} q)$

Proof: (i) $\vdash p \dot{\bar{\equiv}} q \supset .(p \bar{\equiv} p \supset .(p \bar{\equiv} p) \equiv (p \bar{\equiv} p)) \supset .(p \bar{\equiv} p) \supset .\ p \bar{\equiv} q$; from D22'

$\vdash p \dot{\bar{\equiv}} q \supset .\ p \bar{\equiv} q$

(ii) $p \bar{\equiv} q,\ \text{em}(f) \vdash f(p) \equiv f(q)$

$p \bar{\equiv} q \supset .\ p \dot{\bar{\equiv}} q$; by deduction theorem as in previous proof.

The definitions of identity of individuals are not *nearly* strong enough, then, to provide plausible definitions of identity of propositions. To approximate to such definitions +**R*** would have to be explicitly enriched by certain pure (i.e. quotation-free) non-em functors, and in particular by 'K_x'

i.e. 'x knows that' and '$\mathbf{As}_x$' i.e. 'x asserts that'. A much better approximation to propositional identity could then be made, with the definition

D5': $p = q \equiv_{Df} (\mathrm{A}f)(f(p) \equiv f(q))$.

Without the additional non-em functors, however, there is no guarantee that $(p = q)$ does not collapse to $(p \overset{=}{\equiv} q)$: compare T6.

Another statemental relation of much importance, statemental equivalence, can also be roughly explained.

D6': $p(x) \cong q(y) \equiv_{Df} (p(x) = q(x))$ & $(x \equiv y)$, where '$p(x)$' indicates that individual expression 'x' occurs in sentence 'p'.

For example: Scott is the author of Marmion $\cong$ the author of Waverley wrote Marmion. Statemental equivalence, though much stronger than material equivalence; may not preserve modal properties. Hence the rejections

$$*\, p(x) \cong q(y) \supset . \Box p(x) \supset \Box q(y)$$
$$*\, p(x) \cong q(y) \supset . p(x) \overset{=}{\equiv} q(y)$$

Equivalence and identity of properties In an analogous way various equivalence relations can be defined for property expressions. Consider, in particular

D23. $f \stackrel{\vee}{-} g \equiv_{Df} (\forall x)(f(x) \equiv g(x))$; actual co-extensiveness
D24. $f \equiv g \equiv_{Df} (\pi x)(f(x) \equiv g(x))$
D25. $f \overset{=}{\equiv} g \equiv_{Df} (\pi x)(f(x) \overset{=}{\equiv} g(x))$; logical co-extensiveness

But these relations by no means capture, even approximately, property identity in all the usually intended senses. An often preferred explication of property identity is provided by

D7': $f = g \equiv_{Df} (\pi x)(f(x) = g(x))$,

a definition which is well-formed with respect of $+\mathbf{R}^*$ only given a further extension of the formation rules. To obtain further definitions and to link these with D23-D25 and D7' a still higher-order predicate logic is wanted. Then compare

D8': $f \dot{\equiv} g \equiv_{Df} (\mathrm{A}h)(\mathrm{ext}(h) \supset . h(f) \supset h(g))$
D9': $f \approx g \equiv_{Df} (\mathrm{A}h)(h(f) \equiv h(g))$

Existence of propositions Various criteria for the existence of propositions may be compared:

CP1: $\mathrm{E}!p \equiv {\sim}(p \supset p)$

Thus $\vdash (\mathrm{A}p) \sim \mathrm{E}!p$, i.e. no proposition exists.
CP1 is at best a *criterion* for the existence of propositions: it is not intended to characterize proposition existence, but only to be strictly equivalent to a characterisation. In the case of CP1 such a characterisation might be a physical locatability one--propositions (properties) like other individuals exist iff they are physically locatable-or a connected deter-

minacy requirement--propositions, like other individual items, exist iff they are determinate in all extensional respects.
Under the first interpretation CP1 would be replaced by

CP1': $\text{Prop}(p) \supset .\ E!p \equiv {\sim}(p \supset p)$

Then $\vdash (A\underset{\sim}{p}).{\sim} E!\underset{\sim}{p} \equiv {\sim}(\underset{\sim}{p} \supset \underset{\sim}{p})$; so $(A\underset{\sim}{p})\ E!\underset{\sim}{p}$

CP2: $\mathsf{E}(p) \equiv p$,

i.e. a proposition exists iff it is true, a correspondence thesis. It follows:

$\mathsf{E}(t)\ \&\ {\sim}\mathsf{E}(f)$

i.e. the true exists but the false does not.

CP3: $\mathsf{E}(p) \equiv\ \Diamond p$

i.e. a proposition exists iff it is consistent.

CP4: $\mathsf{E}(p) \equiv (p \supset p)$

Thus $\vdash (Ap)\mathsf{E}(p)$, i.e. all propositions exist. Under the first interpretation

CP4': $\text{Prop}(p) \supset .\ \mathsf{E}(p) \equiv p \supset p$
$\vdash (A\underset{\sim}{p})\mathsf{E}(\underset{\sim}{p})$

CP5': $E!(p) \equiv \text{prop}(p)$
Thus $\vdash E!(p) \equiv .\ \mathsf{T}p \vee \mathsf{F}p$

Since $[\text{Prop}(p) \supset (p \supset p)]$ but not conversely, CP5' is stronger than CP4. Under CP4 contents of statement-incapable sentences exist; e.g. the content of the sole Cretan assertion that all Cretan statements are false exists under CP4 though not under CP5'. Thus CP4, since it admits paradoxical contents, is really too liberal. On the other hand, if neuter propositions were admitted (as existing) CP5' would be too restrictive.
There remain further criteria which cannot be symbolised without the introduction of more non-em functors, e.g. the implausible

CP6': $\mathsf{E}(p)$ iff p is or has been entertained.

For under CP6' the existence of a proposition is a time-dependent, and perhaps person-dependent, contingent matter. Improvements on the psychologistic CP6' like
CP7': $\mathsf{E}(p)$ iff p can be entertained
CP8': $\mathsf{E}(p) \equiv \Diamond(\Sigma x)(\mathsf{B}_x(p) \vee \mathsf{B}_x({\sim}p))$, i.e. existent contents are the possible objects of belief or disbelief, reduce to CP4, CP8' just like CP4 (under the first interpretation) allows for the existence of non-propositional "beliefs" and "thoughts".

The chief competing criteria are CP1 and CP5': for CP2 and CP3 can be knocked out on various counts. First, 'E', if not an extensional predicate, is at least referentially transparent. If 'E' is extensional CP3 is eliminated automatically. Even if extensionality is disqualified as a requirement, transparency as summed up in

(1): $p(x) \cong q(y) \supset . \mathsf{E}(p(x)) \equiv \mathsf{E}(q(y))$

holds. CP3 is eliminated since it fails to satisfy (1). Secondly, the existence of a proposition is not merely a contingent matter; a proposition if it exists necessarily exists, i.e.

(2): $\mathsf{E}(p) \supset \Box \mathsf{E}(p)$

CP2 is disqualified as it fails to satisfy (2). Thirdly, criteria CP2 and CP3 have to be rejected if

(3): $\mathsf{E}(p) \supset \mathsf{E}(\sim p)$,

and therefore

(3'): $\mathsf{E}(p) \equiv \mathsf{E}(\sim p)$[15]

is correct. Given CP7' or CP8', (3') is immediate. If (3) is rejected peculiarities appear. For instance, under CP2 $\mathsf{E}(p \vee \sim p)$ is true but either p or $\sim p$ does not exist. Then existent propositions are related to subpropositions which do not exist. Therefore, on a component theory of propositions, some existent propositions are in the extraordinary position of having components which do not exist. This argument can also be worked against CP5'; but it can be halted by discarding component theories.

Only CP1 and CP5' and their equivalents are left standing undiminished. Choice between CP1 and CP5' presents a characteristic conflict issue. Whether CP1 or CP5' or some other criterion is selected depends both on whether a reduction of propositions, say to positive and negative facts or to mental items or to sentences, is attempted, and on whether criteria for the existence of propositions are assimilated to those for the existence of observable or locatable items or, as a case of criteria for the existence of abstract items, are given an independent status. Arguments for either choice can be contrived. For instance in favour of CP5' it can be argued: all propositions are possible objects of belief or disbelief; possible objects of belief or disbelief exist; therefore all propositions exist. But the argument begs the question, because the dispute as to whether propositions exist reappears in the issue as to whether possible objects of belief or disbelief exist. It is tempting to argue that merely possible objects of this sort do not exist, since they fail to satisfy expected criteria for the existence of objects. In favour of CP1 it may be argued: only physically locatable items really exist; propositions are not physically locatable; therefore propositions do not really exist. But the argument is inconclusive too inasmuch as it appeals to a criterion for existence which the other rejects. Philosophical usage on the whole lends support to CP5' but that is not decisive. For reasons for supporting CP1 appear more cogent; and some of the philosophical reasons supporting CP5' rest on such discreditable assumptions as that we cannot talk about what does not exist. The dispute as to whether and which propositions exist is, however, ,a philsophical conflict issue, in Wisdom's sense[16]; and in the end a decision is not so urgent.

Since CP5' cannot be expressed in +R*, CP4 is used as an approximation. When both CP1 and the over-liberal CP4 are adopted as definitions it follows:

$\vdash \mathsf{E}(p) \equiv \sim \mathrm{E}!p$; $\vdash \mathsf{E}(p) \supset \mathrm{Prop}(p)$

$\vdash p \equiv q \supset. \mathsf{E}(p) \equiv \mathsf{E}(q)$; $\vdash p \equiv q \supset. \mathrm{E}!p \equiv \mathrm{E}!q$

$\vdash \mathsf{E}(p) \equiv \mathsf{E}(q)$; $\vdash \mathrm{E}!p \equiv \mathrm{E}!q$

$\vdash \mathsf{E}(p) \equiv \mathsf{E}(\sim p)$;

$\vdash \mathsf{E}(p \vee q) \equiv \mathsf{E}(p) \vee \mathsf{E}(q) \equiv \mathsf{E}(p \,\&\, q) \equiv \mathsf{E}(p) \,\&\, \mathsf{E}(q)$

$\vdash \mathrm{E}!(p \vee q) \equiv \mathrm{E}!p \vee \mathrm{E}!q \equiv \mathrm{E}!(p \,\&\, q) \equiv \mathrm{E}!p \,\&\, \mathrm{E}!q$

$\vdash \mathsf{E}(t\!\!\!/) \,\&\, \mathsf{E}(f\!\!\!/)$; $\vdash \sim \mathrm{E}!t\!\!\!/ \,\&\, \sim \mathrm{E}!f\!\!\!/$

If existential quantifiers are defined:

$(\exists p)A(p) \equiv_{Df} (\mathrm{S}p)(A(p) \,\&\, \mathrm{E}!p)$

$(\mathrm{E}p)A(p) \equiv_{Df} (\mathrm{S}p)(A(p) \,\&\, \mathsf{E}(p))$

then $\vdash (\mathrm{E}p)A(p) \equiv (\mathrm{S}p)A(p)$; and $\vdash \sim (\exists p)A(p)$.
Thus all inferences for 'S' remain valid for 'E', but many are rejected for '∃', in particular

$$* \qquad A(q) \dashv (\exists p)A(p)$$

On the existence of properties Among various ways of defining the existence of properties in +R*, or in **S5R***, a first that stands out is

P1. $\mathsf{E}(f) \equiv_{Df} (\Sigma x)(f(x) \,\&\, \mathsf{E}(x))$ (the *instantial* criterion),

i.e. a property exists iff something exists which has it.

$\vdash \mathsf{E}(f) \equiv (\exists x)f(x)$

If temporal variables are introduced P1 can be improved upon with

P1.1 $\mathsf{E}(f) \equiv_{Df} (\Sigma x)(\Sigma t)(f(x;t) \,\&\, \mathsf{E}(x;t))$,

and several interesting theses can be symbolised, e.g. the sempiternal hypothesis for individuals: $(\pi x)((\Sigma t)\mathrm{E}(x;t) \supset (\pi t)\mathsf{E}(x;t))$, a statement which is plainly not a theorem. In contrast a sempiternal hypothesis for properties:

$$(\mathrm{A}f)((\Sigma t)\mathsf{E}(f;t) \supset (\pi t)\mathsf{E}(f;t))$$

(and also for propositions) is a little more tempting. This hypothesis can be proved by using the definition:

$$\mathsf{E}(f;t) \equiv_{Df} \mathsf{E}(f) \,\&\, (t = t)$$

For $\vdash (\Sigma t)\mathsf{E}(f;t) \supset. (\Sigma t)((t = t) \,\&\, \mathsf{E}(f))$
$\supset. \mathsf{E}(f)$
$\supset. (\pi t)(\mathsf{E}(f) \,\&\, . \, t = t)$
$\supset. (\pi t)\mathsf{E}(f;t)$.

But the definition and result are most unsatisfactory. In analogous ways it can be shown that a property, if it exists, exists everywhere and that a property if it exists at some space-time point exists throughout space-time.

The resulting claims are downright misleading. For redness does not exist everywhere; and certainly both redness and blueness, or redness and non-redness, do not exist everywhere in space-time. The sempiternal hypothesis for properties only gets its plausibility through confusion with an atemporal hypothesis: that the existence of properties is *independent* of time. This thesis is already exhibited by P1.1. Opposing P1 is

P2. $\exists (f) \equiv_{Df} (\pi x)(f(x) \supset \mathsf{E}(x))\ \&\ (\Sigma x) f(x)$,

i.e. a property exists iff some item has it and everything that has it exists. The Σ-clause in P2 is essential as otherwise all inconsistent properties would exist, though many consistent ones would not: indeed the most common existent properties would be inconsistent ones. P2 would be more convincing if 'π' and 'Σ' were replaced, respectively, by existential analogues '$\forall$' and '$\exists$': but then the definition resulting is logically equivalent to P1. As it stands P2 is not at all satisfactory, despite the attractiveness of its analogue for classes. For if, as seems reasonable, and as follows from Meinong's principle of independence of existence from so-being, non-existent items such as unicorns can have properties then few properties have universal existence; and ultimately only one property, existence, and certain of its compounds. Under this criterion common properties such as redness and hardness do not exist! This provides one of the usual reasons to preferring P1 to P2 or to

P3. $E!f \equiv \sim(\pi x)(f(x) \supset f(x))$

under which $\vdash (Af) \sim E!f$, i.e. no properties exist.
However usual reasons for insisting that P3 is false rest on the assumption that we can only talk significantly about what exists, an assumption which rests in turn on the reference theory of meaning. Once the reference theory is abandoned, so are many of the reasons for rejecting P3. This includes objections to P3 such as that it rules out as false all sorts of statements many people (ordinary speakers) want to put up as true, e.g. 'There actually are properties which ...', 'There exists a property, hardness, which some material objects have'. Insofar as such people want to assert more than that some material objects have a property, hardness, insofar as they want to assert also that the property of hardness exists, they are making ontological claims; and the truth of such assertions cannot be warranted just by appeal to reference-theory-loaded common usage. It is the reference theory too that is at the back of the assumption that one cannot consistently assert that no properties exist, any more than one can consistently say that some things do not exist. However although it is impossible to state consistently that some actual property (proposition) does not exist, there are no such difficulties in stating that no properties exist or that no actual properties exist.

In recent discussions of the existence of properties, especially as concerns the acceptability of standard higher-order quantification theory, an all-or-none doctrine prevails: either no properties exist so quantification theory must, in accordance with the reference theory and its consequence, Quine's criterion of ontological commitment, remain a first-order theory, or else properties—all properties—exist. But the doctrine that when some properties exist all do has its source in the pervasive reference theory of meaning. For consider a typical argument: If it can be asserted that some properties exist then the language system concerned must contain the equivalent of property variables. But if property variables are admitted the corresponding universal concept, that of properties, must be admitted[17]. Alternatively, if some properties exist the linguistic admission that they do carries commitment to the universal notion, of all properties. But (linguistic) admission of the universal concept, of properties, entails commitment to the existence of *all* properties. This last crucial step makes direct application, however, of the reference theory: otherwise why should property discourse entail property existence? Of course it does not: one can talk about properties, use predicate variables, and quantify non-existentially over properties, without being committed ontologically to the existence of properties. And though maybe not true, it is certainly consistent to claim that some but not all properties exist. The erroneous all-or-none doctrine repudiates outright, however, P1 and P2 and leads to criteria strictly equivalent either to P3 or to

P4. $\mathsf{E}_2(f) \equiv_{Df} (\pi x)(f(x) \supset f(x))$,

under which all properties exist.[18] Under P4 not only non-instantiated properties exist: even inconsistent properties such as round-squareness which could not be actually instantiated exist. Even non-existence exists! Yet how can it? Under P4 rampant Platonism would flourish without the Platonic details of this existence of ideas. For under P4 the existence of properties becomes a merely *formal* matter: there appears to be no adequate filling out of how properties exist, what their existence is like, why they exist, and so on.

P3 and P4 break all links between the existence of properties and the existence of property-instances. So an actual property can in no way be a matter of its (actual, or actual and possible) instances. But this does not imply that a property may not be a matter of its instances, for example in the sense of being an abstraction from its instances. Moveover the existence of instances of a given property is not really sufficient for ascribing existence to an abstraction from these and other instances.

The widely favoured instantiation criterion evades these sorts of objections to P4, and leads to such interesting results, as:

$\not\vdash \mathsf{E}(\mathsf{E})$, i.e. existence exists

For: $\vdash \mathsf{E}(\mathsf{E}) \equiv (\Sigma x)\mathsf{E}(x) \equiv (\exists x)\mathsf{E}(x)$

$\vdash \Box \sim \mathsf{E}(\sim \mathsf{E})$,

i.e. necessarily non-existence does not exist, where $(\sim f)u \equiv_{Df} \sim f(u)$. [Scope does not raise a problem, yet. But for further developments this definition of predicate negation is unsatisfactory.] Hence

$\vdash (\mathsf{S}f) \sim \mathsf{E}(f)$.

In addition

$\Vdash \exists(\mathsf{E})$ and $\Vdash \sim \exists(\sim \mathsf{E})$

For $\exists(\sim \mathsf{E}) \equiv (\pi x)(\sim \mathsf{E}(x) \supset \mathsf{E}(x))\ \&\ (\Sigma x) \sim \mathsf{E}(x)$
$\supset. (\pi x)\mathsf{E}(x)$

In contrast $\exists(\exists)$ and $\exists(\sim \exists)$ are not well-defined.

Nonetheless the instantiation definition faces a number of difficulties too. First, it destroys the powerful formal analogy between properties and propositions, rejecting construal of propositions as zero-place relations. If the analogy held the non-existence of properties would stand in much the position of the non-existence of propositions, and P3 for example would result. Secondly, the instantiation definition opposes plausible compounding principles for property existence. For example, f (e.g. existence) may exist though $\sim f$ does not; f (e.g. redness) and g (e.g. non-redness) may both exist though the compound f and g does not; and $f \vee g$ may exist though neither f nor g exists. Yet how can the compounding of two existent items result in nothing actual (and no energy release); and how can an existent entity have non-existent disjunctive components? Satisfying expected compounding conditions leads back to P3 or P4. Thirdly, the instantiation definition neglects the fact that properties are abstractions (from their instances) and accordingly are indeterminate in various respects. But such indeterminacy is, in the case of individual items such as Pegasus and the present king of France, quite enough for the ascription of non-existence. Thus the instantial criterion must suppose that properties are quite different kinds of items from concrete individuals, and are not susceptible to the same ontological assessment. But why, when a salient feature of a property, as distinct from its manifold of property-instances, is that it is a single individual item? P3 once again escapes this problem; for it can be seen as a strict consequence of certain definitions of individual existence, in particular those which imply that an item does not exist if it is indeterminate. Fourthly, if cardinal numbers are properties of manifolds[19] then, on the instantiation definition, numbers exist iff the requisite manifolds exist. Thus if no n-manifolds exist, for n large, the cardinal number n does not exist. The cardinal numbers cease to exist after a certain number. Numbers larger than this number will be merely possible properties. The parts of mathematics where these numbers are studied, like much of mathematics, are simply concerned with possibilia. To avoid the status-damaging conclusion that much of mathematics is not about actual matters at all, a more liberal criterion than P1 is often suggested, namely the criterion

P5. $\mathsf{E}_3(f) \equiv_{Df} \Diamond(f)$,

i.e. all & only consistent properties exist. Further motives emerge in reconstructing theories of universals for introducing instead of P1 either P5 or

P6. $\mathsf{E}_4(f) \equiv_{Df} \Diamond\, \mathsf{E}(f)$

P6 is stronger than P5, since $\vdash \Diamond\, \mathsf{E}(f) \supset \Diamond(f)$. Possibility of properties differs however, from possibility of individuals. In contrast with the anticipated connexion $\Diamond(x) \equiv \Diamond \mathsf{E}(x)$, the relation $\Diamond(f) \equiv \Diamond \mathsf{E}(f)$ is rejected. For:

$$\Vdash \Diamond(\sim \mathsf{E}),$$

i.e. non-existence is a possible property, since

$$\Diamond(\sim \mathsf{E}) \equiv \Diamond(\Sigma x)\sim \mathsf{E}(x).$$

Hence

$$\Vdash \Diamond(\sim \mathsf{E})\ \&\ \sim\Diamond \mathsf{E}(\sim \mathsf{E});$$

and hence

$$*\Diamond(f) \supset \Diamond \mathsf{E}(f);$$

$$*\Diamond(f) \equiv \Diamond \mathsf{E}(f)$$

Under P5 and P6 a property may exist even when it has no existent instances. Indeed non-existence exists under P5. Thus many of the difficulties of P4 break out again. With respect to mathematics the Hilbert-Poincare equation of existence with consistency or with possible existence is rightly rejected by neo-intuitionism, since consistency is not sufficient for correctness: and outside mathematics the equation seems indefensible. Certainly existence of individuals and attributes should entail consistency: but consistency does not entail existence, i.e.

$\vdash \mathsf{E}(u) \supset \Diamond(u);$
$* \Diamond(u) \supset \mathsf{E}(u)$, u an attribute or individual variable

Some consistent mathematical items, i.e. geometric points and rigid bodies, are individuals; but they fail to satisfy criteria for the existence of individuals. They have, if you like, mathematical existence, but this just amounts to possible existence. No, an imagined chair, an angel, a 7-D space, or a god does not exist just because it is possible that it should; and likewise angelicness, divinity, possibility and non-denumerability do not exist *just* because it is possible that they should (if it is). So P4, P5 and P6 have to go. P5 and P6 also run foul of a plausible extensionality principle. For

$$\vdash f \equiv g \supset .\ \mathsf{E}(f) \equiv \mathsf{E}(g),$$

since $\vdash f \overset{\equiv}{\equiv} g \supset .\ (\pi x)(\mathsf{E}(x) \supset .\ f(x) \equiv g(x))$
$\supset .\ (\pi x)(\mathsf{E}(x)\ \&\ f(x) \equiv \mathsf{E}(x)\ \&\ g(x))$
$\supset .\ (\Sigma x)(\mathsf{E}(x)\ \&\ f(x)) \equiv (\Sigma x)(\mathsf{E}(x)\ \&\ g(x))$

But though

$\vdash f \overset{\equiv}{\equiv} g \supset .\ \Diamond(f) \equiv \Diamond(g)$	$\vdash f \overset{\equiv}{\equiv} g \supset .\ \Diamond \mathsf{E}(f) \equiv \mathsf{E}(g)$
$*f \equiv g \supset .\ \Diamond(f) \supset \Diamond(g)$	$*f \equiv g \supset .\ \Diamond \mathsf{E}(f) \supset \Diamond \mathsf{E}(g)$

At least with cardinal numbers the retreat to P5 or P6 is premature. For if cardinals are tied to paradigm sets their existence can be ensured under P1 given an appropriate criterion for the existence of sets, e.g. given

S1. $\mathsf{E}(\hat{w}) \equiv_{Df} (\pi x)(x \ \varepsilon\ \hat{w} \supset .\ \mathsf{E}(x))\ \&\ (\Sigma x)(x\ \varepsilon\ \hat{w}).$

For then all non-null subsets of an existent set exist, and therefore since some sets exist sets of arbitrarily large membership size exist. Finally since sets represent manifolds manifolds of arbitrarily large size exist. Nonetheless there are good nominalistic grounds for preferring a less liberal criterion under which at most manifolds of individuals exist, and so under which transfinite cardinals do not exist. Opposing P1-P6 is

P7'. $\sim ¢ (\mathsf{E}(f))$,

not so much a further criterion as a basis for rejection of criteria. Now '$\mathsf{E}(f)$' may be statement-incapable either because (i) it has some contextual defect, e.g. it is viciously self-referential, or because (ii) it is non-significant. As (i) is viable only in isolated cases, support for P7 must be founded on (ii). Limited support for (ii) derives from Russell's type theory under which such expressions as '$\mathsf{E}(\mathsf{E})$' would be discarded as non-significant. Two sketchy points. First, many expressions of the form '$\mathsf{E}(f)$' would not be rejected. Second, the case for Russell's particular significance theory is not strong[20], and there is a much stronger case for a significance theory under which all expressions '$\mathsf{E}(u)$', where 'u' is a designating expression, are significant[20]. This last point also serves to knock out support for (ii) from verifiability or confirmability theses[21]. In any case the reasons for rejecting as false verifiability and confirmability theories of sentence significance are well-known. The points indicated can be so elaborated as to provide good grounds for reckoning P7' false.

Associated with predicates 'E' and '$\exists$' are existential quantifiers defined thus:

$$(\mathrm{E}f)\, f(x) \equiv_{Df} (\mathrm{S}f)(f(x)\ \&\ \mathsf{E}(f))$$
$$(\exists f)\, f(x) \equiv_{Df} (\mathrm{S}f)(f(x)\ \&\ \exists(f))$$

It follows:

$$\vdash (\mathrm{E}f)f(x) \supset (\mathrm{S}f)(\exists x)f(x);$$
$$\vdash (\mathrm{E}f)f(x) \supset (\exists x)\mathsf{E}(x)$$
$$\vdash \mathsf{E}(f) \supset (\exists x)\mathsf{E}(x)$$

More important are the non-theorems:

* $(\mathrm{E}f)\, f(x) \equiv \mathsf{E}(x)$[22];
* $(\mathrm{E}f)\, f(x) \supset \mathsf{E}(x)$
* $(\mathrm{E}f)\, f(x) \equiv (\mathrm{S}f)(f(x)\ \&\ \mathsf{E}(x))$
* $\mathsf{E}(f)\ \&\ f(x) \supset \mathsf{E}(x)$

To vindicate these rejections consider a typical counterexample to the last: caninity exists, since some dogs exist, and Cerberus is canine, but Cerberus does not exist. The other rejections follow from this last rejection.

Under other criteria for existence the picture is different. For

$$\vdash \mathsf{E}(x) \equiv (\exists f)f(x)$$[22].

Proof: (i) $\mathsf{E}(x) \supset. \mathsf{E}(x) \ \& \ (\Sigma x)\mathsf{E}(x) \ \& \ (\pi x)(\mathsf{E}(x) \supset \mathsf{E}(x))$

$$\supset. (\mathrm{S}f)(f(x) \ \& \ (\Sigma x)f(x) \ \& \ (\pi x)(f(x) \supset \mathsf{E}(x)))$$

$$\supset. (\exists f)f(x)$$

(ii) $(\exists f)f(x) \supset. (\mathrm{S}f)(f(x) \ \& \ (\pi x)(f(x) \supset \mathsf{E}(x)))$

$$\supset. (\mathrm{S}f)(f(x) \ \& \ . f(x) \supset \mathsf{E}(x))$$

$$\supset. \mathsf{E}(x)$$

(iii) $\vdash \mathsf{E}(x) \equiv (\mathrm{S}f)(f(x) \ \& \ \mathsf{E}(x))$

Hence

$$\vdash (\exists f)f(x) \equiv (\mathrm{S}f)(f(x) \ \& \ \mathsf{E}(x)).$$

$$\vdash \exists (f) \ f(x) \supset. \mathsf{E}(x)$$

Now let us add to +**R*** the primitive predicate 'mot', read 'is moving', and the axiom:
+**R.6'**: $\vdash (\exists x) \text{ mot } (x) \ \& \ (\exists x) \sim \text{mot}(x)$, i.e. there exists something that is moving, and a thing at rest also exists. It is assumed that a suitable reference frame is selected. +**R.6'** renders +**R.6** superfluous. It follows from +**R.6'**:

$$\Vdash \mathsf{E}(\text{mot}) \ \& \ \mathsf{E}(\sim \text{mot}),$$

i.e. motion and rest both exist. The '-ness' transformation used could be made explicit: here it can be regarded as absorbed into parentheses. It also follows

$$\Vdash \sim (\mathrm{A}f) \ (\mathsf{E}(f) \equiv \sim \mathsf{E}(\sim f))$$

which contradicts one reformulation of Leonard's law[23]

L5.1 $(\mathrm{A}f)(\mathsf{E}(f) \equiv \sim \mathsf{E}(\sim f))$

Indeed

$$\Vdash \sim (\mathrm{AE}f)(\mathsf{E}(f) \equiv \sim \mathsf{E}(\sim f)),$$

where

$$(\mathrm{AE}f)B(f) \equiv_{Df} (\mathrm{A}f)(\mathsf{E}(f) \supset B(f)),$$

so contradicting

L5.2 $(\mathrm{AE}f)(\mathsf{E}(f) \equiv \sim \mathsf{E}(\sim f))$

since

$$\Vdash (\mathrm{S}f)\mathsf{E}(f).$$

Though

$$\Vdash (\mathrm{S}f)(\mathsf{E}(f) \equiv \sim \mathsf{E}(\sim f))$$

and

$$\Vdash (\mathrm{E}f)(\mathsf{E}(f) \equiv \sim \mathsf{E}(\sim f)),$$

taking 'E' or '$g \vee {\sim} g$' as 'f', the universal generalisations of these results are not plausible. On the other hand it seems that

$$[(\mathrm{A}f)(\exists(f) \equiv {\sim}\exists({\sim}f))]$$

though not a theorem holds for almost all properties - a further reason for discarding P2.

To put P2 back in the race predicates or predication links such as 'has' and 'is' or individual expressions must be taken to carry existential loading. But rather than imposing existential loading on the interpretation of the symbolism it is far better to make it explicit, e.g. in this way:

$$f(x^{\mathsf{E}}) \equiv_{Df} f(x) \;\&\; \mathsf{E}(x)$$
$$f^{\mathsf{E}}(x) \equiv_{Df} f(x) \;\&\; \mathsf{E}(f);$$
$$f^{\exists}(x) \equiv_{\overline{Df}} f(x) \;\&\; \exists(f)$$

Since

$$\vdash f^{\exists}(x^{\mathsf{E}}) \equiv f^{\exists}(x)$$

and

$$\vdash f^{\exists}(x) \supset f(x^{\mathsf{E}}),$$

there are two ways an exponent of P2 might try to build in existential commitment. But the stronger of these, '$f^{\exists}(x)$', cannot be used in P2 on pain of circularity, and adoption of the weaker '$f(x^{\mathsf{E}})$', in place of '$f(x)$' in P2, collapses P2 into P1; for

$$\begin{aligned}(\exists(f))^{\mathsf{E}} &\equiv (\pi x)(f(x^{\mathsf{E}}) \supset \mathsf{E}(x)) \;\&\; (\Sigma x)f(x^{\mathsf{E}})\\ &\equiv (\pi x)(f(x) \;\&\; \mathsf{E}(x) \supset.\ \mathsf{E}(x)) \;\&\; (\Sigma x)(f(x) \;\&\; \mathsf{E}(x))\\ &\equiv \mathsf{E}(f).\end{aligned}$$

Thus ensuing investigations concentrate chiefly on properties of E.

(α) $\quad \vdash (\forall x)(\mathsf{E}f)f(x),$

i.e. every existing item has some existent properties.

Proof: $\vdash (\forall x)(\mathrm{E}f)f(x) \equiv (\forall x)(\mathrm{S}f)(f(x) \;\&\; (\exists x)f(x))$
$\equiv.\ (\pi x)(\mathsf{E}(x) \supset.\ (\mathrm{S}f)(f(x) \;\&\; (\exists x)f(x))$

and

$$\begin{aligned}\vdash \mathsf{E}(x) &\supset.\ \mathsf{E}(x) \;\&\; (\Sigma x)(\mathsf{E}(x) \;\&\; \mathsf{E}(x))\\ &\supset.\ \mathsf{E}(x) \;\&\; (\exists x)\mathsf{E}(x)\\ &\supset.\ (\mathrm{S}f)(f(x) \;\&\; (\exists x)f(x))\end{aligned}$$

Hence

$$(\pi x)(\mathsf{E}(x) \supset (\mathrm{S}f)(f(x) \;\&\; (\exists x)f(x)))$$

To strengthen (α) to: there exists a property which every existent has, further compound predicates are defined:

$$(f \vee g)(u) \equiv_{Df} f(u) \vee g(u);$$
$$(f \;\&\; g)(u) \equiv_{Df} f(u) \;\&\; g(u)$$

$$(f \supset g)(u) \equiv_{Df} f(u) \supset g(u);$$
$$(f \equiv g)(u) \equiv_{Df} f(u) \equiv g(u)$$
$$\Box(f) \equiv_{Df} (\pi x)\Box f(x);$$

and similarly for polyadic predicates.

$$(f \dashv g)(u) \equiv f(u) \dashv g(u);$$
$$(f \mathrel{\overset{\equiv}{\equiv}} g)(u) \equiv_{Df} f(u) \mathrel{\overset{\equiv}{\equiv}} g(u)$$

Then

$$\Vdash \mathbf{E}(g \vee \sim g)$$
$$\vdash (\Sigma x)\mathbf{E}(x) \supset (\exists x)(g(x) \vee \sim g(x))$$
$$\Vdash (\mathrm{A}f)(\Box(f) \supset \mathbf{E}(f)),$$

i.e. all analytic properties exist. This result reveals another inadequacy of P1. It follows too

$$\Vdash (\mathrm{E}f)(\forall x)f(x),$$

i.e. there exists a property every existent has; and

$$\Vdash \Box(f) \supset (\forall x)f(x).$$

Furthermore an **S5** modal logic of properties follows: for

$$\Box(f) \equiv \sim\Diamond(\sim f);$$
$$\vdash \Box(f) \supset \mathbf{E}(f)$$
$$\Box(f \supset g) \supset. \Box(f) \supset \Box(g);$$
$$\Box(f) \supset \Box\Box(f)$$
$$\vdash \sim\Box(f) \supset \Box\sim\Box(f).$$

Many other results follow as well, e.g.

$$\vdash \mathbf{E}(f \vee g) \equiv \mathbf{E}(f) \vee \mathbf{E}(g);$$
$$\vdash \mathbf{E}(f \mathbin{\&} g) \supset \mathbf{E}(f) \mathbin{\&} \mathbf{E}(g)$$ [24]
$$\vdash \Box(f \supset g) \supset. \mathbf{E}(f \mathbin{\&} h) \supset \mathbf{E}(g \mathbin{\&} h)$$

More interesting than (α) is

(β): $$(\forall x)(\mathrm{E}f)(f \not\equiv \mathbf{E} \mathbin{\&} f(x)),$$

i.e. every existing item has some existent property other than existence, which follows once the property of motion or rest is shown to be distinct from existence: for then

$$\vdash \mathbf{E}(x) \supset. \mathbf{E}(m \vee \sim m) \mathbin{\&} (m \vee \sim m)(x) \mathbin{\&} (m \vee \sim m \not\equiv \mathbf{E})$$
$$\supset. (\mathrm{S}f)(\mathbf{E}(f) \mathbin{\&} f(x) \mathbin{\&} f \not\equiv \mathbf{E})$$
$$\supset. (\mathrm{E}f)(f \not\equiv \mathbf{E} \mathbin{\&} f(x)).$$

So

$$\vdash (\pi x)(\mathbf{E}(x) \supset (\mathrm{E}f)(f \not\equiv \mathbf{E} \mathbin{\&} f(x)))$$

Other properties can be used equally well in this argument, e.g. the property of being self-identical or the property of an individual of being an improper part of itself. For self-identity differs from existence, i.e.

$$\Vdash \mathsf{E} \not\equiv \mathsf{I},$$

where

$$\mathsf{I}(x) =_{Df} (x \equiv x).$$

For:

$$\Vdash (\Sigma x) \sim \mathsf{E}(x) \ \& \ x \equiv x.$$

Hence

$$\Vdash (\Sigma x) \sim (\mathsf{E}(x) \equiv \mathsf{I}(x));$$
$$\Vdash \sim (\mathsf{E} \equiv \mathsf{I}).$$

Since

$$\Vdash (\mathsf{E} \equiv \mathsf{I}) \supset (\mathsf{E} \equiv \mathsf{I}),$$
$$\Vdash (\mathsf{E} \not\equiv \mathsf{I}).$$

Since existence differs from self-identity, existence differs from identity, a point which may be defended alternatively using the following slick argument from Plato.[25] For some property, e.g. mot, both the property and its complement, ~mot, exist. But if existence were the same as the attribute of identity, *not* just self-identity, then since motion and rest exist, motion and rest are identical, indeed all existent properties are one, which is impossible on several counts, e.g. because

$$\Vdash \text{mot} \not\equiv \sim \text{mot}.$$

For

$$\Vdash \sim (\pi x)(\text{mot}(x) \not\equiv \sim \text{mot}(x)).$$

$\Vdash \sim (\mathsf{E} \equiv \sim \mathsf{E})$, i.e. existence differs from non-existence.

Proof: $\vdash f \equiv g \supset. \ \mathsf{E}(f) \supset \mathsf{E}(g)$
So $\vdash \mathsf{E}(f) \ \& \ \sim\mathsf{E}(g) \supset. \ \sim(f \equiv g)$
$\vdash \mathsf{E}(\mathsf{E}) \ \& \ \sim \mathsf{E}(\sim \mathsf{E}) \supset. \ \sim (\mathsf{E} \equiv \sim \mathsf{E}).$

Next

$\Vdash \text{mot} \not\equiv \mathsf{E}$, i.e. existence differs from motion.

To show this a slight detour is made so as to take in another argument of Plato's, Define '*f* blends with *g*' as '*f* and *g* are coinstantiable', more exactly

$$f \, \mathsf{B} \, g =_{Df} (\Sigma x)(f(x) \ \& \ g(x)).$$

Then $f \, \mathsf{B} \, \mathsf{E} \equiv \mathsf{E}(f);$
and $\Vdash \text{mot} \, \mathsf{B} \mathsf{E}$

i.e. motion blends with existence, because

$$\vdash \text{mot} \, \mathsf{B} \, \mathsf{E} \equiv \mathsf{E}(\text{mot}),$$

i.e. because motion blends with existence iff motion exists.

$\vdash \sim(\text{mot } \mathbf{B} \sim \text{mot})$, i.e. motion does not blend with rest.

Now existence differs from motion because existence blends with rest but motion does not blend with rest. More formally,

$$\begin{aligned} \Vdash \text{mot} \equiv \mathbf{E} \supset. & \ (\pi x)(\text{mot}(x) \equiv \mathbf{E}(x)) \\ \supset. & \ \text{mot } \mathbf{B} \sim \text{mot} \equiv \mathbf{E}\ \mathbf{B} \text{ mot} \\ \supset. & \ f. \end{aligned}$$

Hence $\vdash (\text{mot} \not\equiv \mathbf{E})$

On additional problems generated by relations There is the question of the best criterion for the existence of (binary) relations, both of relations-in-intension and of relations-in-extension. There are of course criteria for the existence of relations parallelling all those discussed in the case of properties, since one-place predicates may be obtained by identifying or fixing the places of two-place predicates. Thus *corresponding* to the philosophically rather unpopular P3 is the philosophically popular

R3. $\quad E!R \equiv \sim(\pi x)(\pi y)(xRy \supset xRy)$,

under which no relations exist; and corresponding to the instantial criterion P1 is the criterion

R1. $\quad \mathbf{E}(R) \equiv (\exists x)(\exists y)\ xRy$[26].

But the matter is not so simple; for also corresponding to R1 is the criterion

$R1_1$. $\quad \mathbf{E}_1(R) \equiv (\Sigma x)(\Sigma y)(xRy \ \& \ \mathbf{E}(x))$

- but why should x be favoured over y? - and the criterion

$R1_2$. $\quad \mathbf{E}_{11}(R) \equiv (\Sigma x)(\Sigma y)(xRy \ \& \ .\ \mathbf{E}(x) \vee \mathbf{E}(y))$.

Since criteria for the existence of relations ought to mesh with criteria for the existence of properties—especially if relations are just properties of ordered pairs, and since there is no clear formal distinction between one and two place predicates--further problems are raised for P1 and P2. How is P1, for example,--to be satisfactorily extended? $R1_2$ as well as classing as existent far too many suspect relations, falls foul of the notion that a relation is a property of ordered pairs. For an ordered pair exists presumably only if its elements exist, and the property of these exists, applying P1, iff an ordered pair exists. Thus this analysis of relations leads given P1 to R1. However, adopting the standpoint of P1, R1 rules out too much. For a relation-instance, in contrast to a property-instance, may exist even though one of its relata does not exist or does not now exist. Both intensional relations such as knowledge and temporal relations create special problems on this score: consider for example the relation of $(\text{great})^5$-grandfather. The question of the existence of relations raises, then, important philosophical issues as to whether relations exist independently or whether they somehow reduce to properties, or to properties and a few basic ordering relations such as temporal ordering or lexicographical ordering, or to membership relations, or to resemblance relations. Further issues arise since some relations are much less remote than others; thus abstruse relations and highly generic relations such as causation may be alleged to occupy an invidious position as regards ontic standing compared with more specific relations such as hitting and touching. In fact relations are often regarded, without any good reason, as much more remote than properties, even as merely logical constructions, and not as items whose instances can be perceived[27]—indeed as items *which do not exist* and which therefore, if they are to be spoken of, are, according to the reference theory, in need of reduction. Many of the philosophical issues are however bypassed by adoption of a criterion equivalent to R3, a criterion which meshes with and has the same support as P3; then too the reduction questions can be considered on their own without being so clouded by ontological issues.

Statements about attributes cannot be satisfactorily reduced to statements, expressible in +**R***, about actual attribute-instances. But can these statements be generally replaced by statements about a more comprehensive class of instance statements, say by statements about possible attribute-instances? Can all attribute talk be eliminated in the way these examples suggest, where "Punctuality is a virtue" is replaced by "For all possible x, if x is a punctual act then x is a virtuous act"[28] and "Redness exists" by "Some item which is red exists"? In certain cases such replacements--preserving logical equivalence, not synonymy--can, it seems, be made. To sharpen the problem a higher predicate logic without types is wanted, in which all statements about attributes can be expressed. Such logics not only have considerable intrinsic interest: they are vital for a full assessment of the logic and ontology of attributes.

NOTES

1. The system **R*** is presented in R. Routley, "Some things do not exist," *Notre Dame Journal of Formal Logic*, Vol. 7 (1966), pp. 251-276. (This article is referred to as *SE*).

2. For these, see Church's system $\mathbf{F}^2$ in A. Church, *Introduction to Mathematical Logic, Part I*, Princeton (1956), pp. 295-7. Much terminology and notation, in particular substitution notation, has been taken over from Church.

3. In place of **R.3C** this axiom-scheme may be adopted:

   (**A**): $(Sf)(\pi x_1, x_2, \ldots, x_n)(f(x_1, x_2, \ldots, x_n) \equiv A)$, where A is any wff containing distinct individual variables $x_1, x_2, \ldots, x_n$ and f is an n-adic predicate variable which does not occur free in A.

   That **R-3C** and (**A**) are equivalent in this setting is shown, in effect, by L. Henkin, "Banishing the rule of substitution for functional variables," *The Journal of Symbolic Logic*, Vol. 18 (1953), p. 201.

4. J. Łukasiewicz, "The Shortest Axiom for the Implicational Calculus of Propositions," *Proceedings of the Royal Irish Academy*, Vol. 52, A3 (1948), pp. 25-33.

5. See J. Łukasiewicz and A. Tarski, "Investigations into the sentential calculus," printed in A. Tarski, *Logic, Semantics & Metamathematics*, Oxford (1955), p. 55.

6. For details of **S5R*** see R. Routley, "Identity and existence in quantified modal logics," *Notre Dame Journal of Formal Logic*, vol. 10 (1969), pp. 113-149. (This article is subsequently referred to as *IE*).

7. For reasons explained in *IE*.

8. The proof is similar to that in *IE*.

9. Analytic assertion can be reduced to assertion of analytic statements, much as in system **T*** of *IE*, by using a Lewis formulation of the modal logic; and then assertion can be suppressed in the usual manner.

10. For more details of this many-one reduction of protothetic to extended propositional calculus, see S. Leśniewski, "Grundzüge eines neuen systems der Grundlagen der Mathematik," *Fundamenta Mathematicae*, vol. 14 (1929), pp. 1-81. See also, A. Church, *op. cit.*, pp. 152-154, pp. 306-307.

11. See, in effect, A. N. Prior, "A family of paradoxes," *Notre Dame Journal of Formal Logic*, Vol. 2 (1961), pp. 16-32.

12. This line of attack on the semantical paradoxes is not elaborated in any detail in the present paper. For a little more on this approach, see R. Routley, "On a Significance Theory," *Australasian Journal of Philosophy*, vol. 44 (1966), pp. 172-209.

13. Compare R. Carnap, *Meaning & Necessity*. Enlarged edition, Chicago (1956), p. 45.

14. For the intended interpretation of 'π', 'Σ' and '**E**', and, more generally, of the subsystem **S5R*** of +**R***, see *IE*.

15. A thesis proposed by V. Macrae.

16. See J. Wisdom, *Other Minds*, Blackwell, Oxford (1952), pp. 1-34, and especially pp. 2-4. Choice between criteria for the existence of attributes also typically raises philosophical conflict issues. These issues have been misrepresented by Carnap, *op. cit.*, pp. 206-221.

17. See, e.g. R. Carnap, *op. cit.*, pp. 43-44.

18. The predicate/property distinction, which parallels the sentence/proposition distinction, has been glossed over. For most of the discussion the second interpretation of predicate logics, where only property-specifying predicates are represented by predicate symbols, can be presupposed.

19. A thesis defended by R. Routley, "What numbers are," *Logique et Analyse*, No. 31 (1965).

20. These claims are defended in more detail in "On a significance theory," *op. cit.* See also S. Halldén, *The Logic of Nonsense*, Uppsala (1949), and Fred Sommers' articles, "The Ordinary Language Tree," *Mind* 68 (1959), p. 160-185, and "Types and Ontology," *Philosophical Review*, 72 (1963), pp. 327-363.

21. Carnap's reason for classing non-internal existence questions as lacking content: see R. Carnap, *op. cit.*, pp. 206-221.

22. Adoption of a formula of this form as a definition of '**E** (x)' was contemplated by H. S. Leonard, "The Logic of Existence," *Philosophical Studies*, VII, 4 (June

1956), pp. 49–64, and is taken for granted by A. N. Prior, *Time and Modality*, Oxford (1957), p. 31, and by many earlier writers. Quite apart from the circularity of the definition, since '**E**' is needed in defining the existential quantifier, the proposed definition has other major deficiencies: some of these are elaborated in R. Routley, "Exploring Meinong's Jungle: Items and Descriptions" (unpublished).

23. Leonard, *op. cit.*

24. For several further results of this sort see R. Carnap, *Introduction to Symbolic Logic and its applications*, Dover (1958), pp. 109–110.

25. Plato, *Sophist*, 255 C. Using an unrestricted predicate logic with unlimited quantifiers many of Plato's arguments in this part of the *Sophist* can be formalised, though not in a style of which Plato would approve.

26. Compare A. N. Whitehead and B. Russell, *Principia Mathematica*, Vol. I, Cambridge (1911), ***25.03**. But for the purpose for which the definition ***25.03** is intended, to provide a definition of the existence of relations-in-extension, it is inadequate. For how can a class or a relation-in-extension really exist if some of its elements do not exist? An alternative definition of the existence of relations-in-extension which avoids these problems is provided by

$$\mathbf{E}(R_{\mathbf{E}}) \equiv (\pi\, x, y)(xR_{\mathbf{E}}y \supset \cdot\ \mathbf{E}(x)\ \&\ \mathbf{E}(y))\ \&\ (\Sigma\, x, y)\, xRy$$

But there is more reason for saying that relations-in-extension, like classes and similar abstractions, do not exist at all.

27. See, e.g., the discussion of whether relations really exist and the connected issue as to whether they can be sensed in B. Russell, *Principles of Mathematics*, Allen & Unwin, London (1903), pp. 96–100; B. Russell, *The Philosophy of Leibniz*, Allen & Unwin, London (1900); J. N. Findlay, *Meinong's Theory of Objects and Values*, Oxford (1963), e.g. pp. 142–145; J. L. Austin, *Philosophical Papers*, edited J. O. Urmson and G. J. Warnock, Oxford (1961), pp. 18–22.

28. Compare G. Ryle, "Systematically misleading expressions," reprinted in *Logic & Language*, 1st series, edited A. Flew, Blackwells, Oxford (1951), pp. 20–1.

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