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EXPRESSIBILITY IN TYPE THEORY

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§1. *Introduction*

This paper is based on a conception of mathematics not as a system of statements about "mathematical objects" but as a system of derived rules of inference which may be applied to physical objects. It aims to build a foundation for mathematics based on elementary rules of logic which is independent, so far as possible, of ontological presuppositions. The concept of expressibility is introduced here mainly for use in constructing such a foundation. It is a generalization and adaptation to type theory of the "strong Γ-consistency" defined by Henkin in [4].

Let $\mathfrak{L}$ be a simple theory of types. Let Γ be a set of individual or predicate constants in $\mathfrak{L}$ such that all members of Γ belong to the same type. Let γ be a predicate in $\mathfrak{L}$ such that $\gamma(\mathbf{a})$ is wf if $\mathbf{a} \in \Gamma$ (where $\mathbf{a}$ is used autonymously). We shall say that Γ is *expressible by* γ in $\mathfrak{L}$ if there exists a complete and consistent extension $\mathfrak{M}$ of $\mathfrak{L}$ such that, for every constant $\mathbf{b}$ of the relevant type, $\gamma(\mathbf{b})$ is valid in $\mathfrak{M}$ iff, for some $\mathbf{a} \in \Gamma$, the wff $\mathbf{b} = \mathbf{a}$ is valid in $\mathfrak{M}$.

It will be shown that a simple theory of types can serve as a satisfactory foundation for mathematics if and only if certain meta-predicates (i.e. predicates defined in the metatheory) are expressible by predicates in the system. Incidentally, we shall find that certain important problems of consistency, e.g. ω-consistency of number theory, consistency of choice axioms, are reducible to problems of expressibility.

Let us say that a system $\mathfrak{L}$ of type theory is *adequate for mathematics* iff

(1) the set of wffs of $\mathfrak{L}$ is recursive, its set of theorems recursively enumerable,

(2) there is a designated class of individual constants called *names* (for objects) such that

(a) if $\mathbf{a}$, $\mathbf{b}$ are typographically distinct names, then $\vdash \mathbf{a} \neq \mathbf{b}$ in $\mathfrak{L}$,

(b) if $A(x_1, \ldots, x_n)$ is a wff in which the individual variables $x_1, \ldots, x_n$ occur free such that

$$\vdash \exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n) \text{ in } \mathfrak{L},$$

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then there are names $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ such that $A(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ is not refutable in $\mathfrak{L}$,

(3) every mathematical theorem is a theorem of $\mathfrak{L}$ in the sense that both the theorem and its proof can be translated into $\mathfrak{L}$ and the translation of the proof is a proof in $\mathfrak{L}$.

The first and third conditions present no difficulty in principle, which makes the second decisive. Sub-condition (2b) is equivalent (*cf.* **507T 508T** below) to each of the following:

(i) the metapredicate 'name' is expressible in $\mathfrak{L}$ by V, where V is the universal predicate of individuals,

(ii) $\mathfrak{L}$ is strongly Γ-consistent (where Γ is the set of names) in the sense of Henkin[11].

We shall construct a system $\mathfrak{L}\mu(\infty)$ in which there are infinitely many names and in which the axioms of choice are theorems. This system will be proved consistent by fairly elementary reasoning. It will be shown that the following statements are equivalent:

(a) $\mathfrak{L}\mu(\infty)$ satisfies condition (2b) of adequacy,

(b) it is possible to construct an extension of $\mathfrak{L}\mu(\infty)$ which contains an ω-consistent number theory,

(c) the metapredicate Γ_{fin} (defined immediately below) is expressible in $\mathfrak{L}\mu(\infty)$.

We shall say that a predicate constant P is Γ_{fin} iff either (1) P is $\lambda x(x \neq x)$ or (2) P has the form $\lambda x(x = \mathfrak{a}_1 \vee \ldots \vee x = \mathfrak{a}_n)$ where x is an individual variable and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are individual constants. Obviously if Γ_{fin} is expressible by a predicate γ, then $\gamma(P)$ can properly be interpreted by "P is finite."

In §6 we shall define a predicate "fin" and prove that if Γ_{fin} is expressible in our system by any predicate γ, then (1) it is expressible in this system by fin, and (2) the wff $\gamma = \mathsf{fin}$ is consistent in every extension of that system in which Γ_{fin} is expressible by γ. Thus the choice of a definition of finiteness, and that of an axiom of infinity, is determined by the requirements of adequacy.

If the equivalent conditions (a)—(c) are fulfilled, then we can construct an extension of $\mathfrak{L}\mu(\infty)$ which is adequate at least for all mathematical theories which can be finitely axiomatized, if not for all mathematics (cf. §9).

§§2-4 are devoted to the construction of $\mathfrak{L}\mu(\infty)$ and related systems. Expressibility of metapredicates in these systems is discussed in §§5-8. The remainder of the present introductory section is concerned with motivation and presuppositions. All our results are deduced by syntactical procedures without reference to semantics. Semantical considerations do, however, dominate the motivation of our work. A logistic system is designed for making inferences about objects. This motivates condition (2) of the definition of adequacy. In a typical application the objects which constitute the range of discourse would be some well-defined class of physical measurements or observable phenomena. In the general theory the only assumptions made about the objects are (1) that they are distinct one from another, (2) that we can assign a name to each biuniquely, and (3) any one of various alternative hypotheses (§4) as to how many objects are in the range of discourse. Beyond this we make no ontological commitments. We do not even make any assumptions concerning existence of sets. The role of set theory in our work is analogous to that of geometry in analysis. The heuristic nature of intuitive set theory makes it useful in the informal or semi-formal presentation of an argument. We use it only for this purpose. In this paper "set" is a synonym for "predicate", $x\varepsilon P$ an abbreviation for $P(x)$. Let us say that a symbol $\mathscr{J}$ *has a denotation* iff it refers to an entity E of which we can have knowledge independently of the language in which $\mathscr{J}$ is an expression. The names are the only symbols that we assume to have a denotation. It is only by inference that denotations can be assigned to other symbols, and it is not required that any of the latter have denotations. They may be nothing more than marks which serve to facilitate certain calculations. In physics, functions which take complex values are used where the imaginary part of a value of the function has no physical interpretation. The usefulness of this procedure is presumably due to the advantages of working with an algebraically closed field. Similarly in a logistic system there is much to be gained by closure of the system with respect to the logical operations. If there are some individual or predicate constants to which no denotation can reasonably be assigned, the benefits of logical closure may still justify their use.

In order that one may be able to assign appropriate meanings to sentences which contain quantifiers, it is necessary to define the ranges of variables. We shall define, for each variable, a syntactical range and a semantic range. The syntactical ranges are composed of expressions b such that from $\forall xF(x)$ one can infer $F(\mathsf{b})$. In particular, the syntactical range of an individual variable is composed of the individual constants. If F is a predicate variable such that $F(x_1, \ldots, x_n)$ is wf, its syntactical range is composed of those wffs in which the variables $x_1, \ldots, x_n$, and no others, occur free.

The semantic range of an individual variable must be composed of the objects, if the system is to serve its purpose. Then a wff of the form $\exists xP(x)$ can be interpreted by "there is an object which is P". This interpretation will be free from contradiction if the system satisfies condition (2) of the definition of adequacy. (Here "contradiction" is to be understood in a broad sense, according to which a system which is consistent but not ω-consistent would be contradictory).

The definition of the semantic range of a predicate variable is connected with the meanings assigned to sentences which contain quantified predicate variables. If b is the name of an object, we propose to interpret $\exists FF(\mathsf{b})$ by "there is something which can truthfully be said about b." But here "something" and "said" need to be made more precise. What can be said about the object depends not only on the object but also on the language. Let $\mathfrak{L}$ be the language we are using, and let $\exists FF(\mathsf{b})$ be understood to mean "there is a true statement about **b** which can be written in the language $\mathfrak{L}$." This leads us to identify the semantic range of a predicate variable with its syntactical range.

Thus interpreted, type theory is not a pure object language. It contains talk not only about objects, but also about talk, referring to some of its own expressions. This interpretation allows us to disentangle type theory from the ontological quandaries with which it has been associated in the literature, such as misgivings about impredicative definitions and about whether the system has a standard model.

One may regard the predicate calculi of higher orders as extensions of the first-order calculus constructed by introducing into the system certain forms of expression already present, at least potentially, in the metatheory. But only certain kinds of predicates can thus be introduced. A predicate can be introduced only if (1) it is uniform as to type (in the sense of **501D** below), and (2) its introduction is compatible with the axioms of extensionality. We shall see that in a system with a finite set of names all metapredicates that satisfy conditions (1) and (2) are expressible, hence can be introduced. But if there are infinitely many names, then there are predicates which do satisfy these conditions but cannot be introduced. We can define in the metatheory a function which enumerates all predicates of any given type, but in consequence of Cantor's theorem no such function is expressible in the system[12]. We shall find other metapredicates for which it is not easy to determine whether or not they are expressible, which raises the question whether the powers of expression of type theory are sufficient to make it adequate for mathematics. That is the problem of this paper.

§2. *Construction of the theory*

In §1 we mentioned the predicate calculi of higher orders as extensions of the first-order calculus. Here we use a short cut, starting with a pure predicate calculus of order ω which will be designated by the symbol $\mathfrak{F}\omega$. This will be extended by postulates, some of which will introduce individual and predicate constants. The symbols of $\mathfrak{F}\omega$ include (1) logical constants, i.e. quantifiers and truth-functional symbols, (2) individual and predicate variables[21]. Each variable is composed of a letter with a superscript which shows the type to which the variable belongs. Individual variables have the form $\dot{x}$, in which the dot is the type symbol. All higher type symbols are constructed by the rule: if $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$ are type symbols, then $(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)$ is a type symbol[22]. The atomic wffs of the system have the form $\overset{(t_1,\ldots,t_n)}{x}\overset{t_1}{x}\ldots\overset{t_n}{x}$, where $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$ are type symbols. All other wffs are formed from atomic wffs and logical constants in the usual way. The symbols described above are the only ones needed for $\mathfrak{F}\omega$, parentheses or dots being avoidable by use of a parenthesis-free form of notation. For working purposes, however, we shall use dots to show the scopes of logical constants.

Type superscripts can usually be omitted in the working notation. A formula in working notation may be a schema of which the instances are wffs. Thus the instances of the schema Fx have the form $\overset{(t)}{F}\overset{t}{x}$ where t stands for an arbitrary type symbol. Every such schema has a *basic* instance written by placing the individual superscript wherever the rules permit. All other instances are *type elevations* of the basic instance. For perspicuity we shall, so far as possible, follow the practice, wherever two variables in a formula must be of distinct types, of writing them in distinct alphabets or distinct parts of an alphabet. In particular, dyadic predicates will be symbolized by Greek letters. Where type superscripts are needed at all, it is usually sufficient to write them in one or two places in a formula (e.g. in $F\dot{x}$ the type of F is uniquely determined). In the working notation, atomic wffs composed of two variables will be written in the form $x\varepsilon F$ (in place of Fx), those composed of three variables in the form $x\alpha y$ (in place of αxy)[23].

The axioms of $\mathfrak{F}\omega$ are

(1) all substitution instances of tautologies

(2a) all wffs of the form

$$\forall xA \rightarrow A(y)$$

where $A(y)$ results from the wff A by substitution of y for x (subject to proper safeguards against confusion of free and bound variables)[24].

(2b) all wffs of the form

$$\forall FA \rightarrow A(B(x_{i1} \ldots x_{in}))$$

where F is an n-adic predicate variable having exactly m occurrences in A, each of these occurrences being in one of the atomic wffs $Fx_{i1} \ldots x_{in}$ ($1 \leq i \leq m$), $B(x_{11} \ldots x_{1n})$ is a wff in which there are free occurrences of $x_{11}, \ldots, x_{1n}$, and if $1 < i \leq m$ then $B(x_{i1} \ldots x_{in})$ results from $B(x_{11} \ldots x_{1n})$ by substitution of $x_{i1} \ldots x_{in}$ for $x_{11} \ldots x_{1n}$, $A(B(x_{i1} \ldots x_{in}))$ results from A by the m substitutions of $B(x_{i1} \ldots x_{in})$ for $Fx_{i1} \ldots x_{in}$[25], (subject to proper safeguards against confusions or collisions of variables[26]).

(3) the usual axioms of extensionality. For monadic predicates F, G these have the form

$$\forall x.\ x\varepsilon F \longleftrightarrow x\varepsilon G. \rightarrow \forall X.\ F\varepsilon X \rightarrow G\varepsilon X.$$

Later, when we introduce individual and predicate constants into the system, we shall allow the y in (2a) to be a constant of the same type as x.

The rules of inference of $\mathfrak{F}\omega$ are (1) *modus ponens* and (2) if x is a variable which does not occur free in A

$$A \rightarrow B \vdash A \rightarrow \forall xB.$$

We do not need any axiom of reducibility or *Mengenbildungsaxiom* because:

201T[27] *If A is a wff in which the variables $x_1, \ldots, x_n$ occur free, then*

$$\exists F \forall x_1 \ldots \forall x_n.\ Fx_1 \ldots x_n \longleftrightarrow A$$

Proof. Each instance of

$$\forall F \exists x_1 \ldots \exists x_n.\ Fx_1 \ldots x_n \leftrightarrow A. \rightarrow \exists x_1 \ldots \exists x_n.\ A \leftrightarrow A$$

is an instance of the axiom schema (2b). The consequent in this formula is refutable. Hence the result by contraposition.

One could use **201T** as an axiom schema and dispense with (2b). We have chosen to use (2b) in order to emphasize that $\mathfrak{F}\omega$ is based on pure logic without existential postulates. By the definition of the semantic range of a predicate variable given in §1, any wff in which $x_1, \ldots, x_n$ occur free is an instance of $Fx_1 \ldots x_n$.

The expression "$\Vdash A$ in $\mathfrak{L}$" will be used as an abbreviation for "A is not refutable in $\mathfrak{L}$." The following rules for operating with the symbol $\Vdash$ are easy to verify.

202T Let A, B be arbitrary wffs. Then

(1) $\Vdash A \vee B$ *iff either* $\Vdash A$ *or* $\Vdash B$,

(2) *if* $\Vdash A$ *and* $\vdash A \rightarrow B$, *then* $\Vdash B$,

(3) *if* $\vdash A$ *and* $\Vdash A \rightarrow B$, *then* $\Vdash B$,

(4) $\Vdash A$ *iff either* $\Vdash A \wedge B$ *or* $\Vdash A \wedge \neg B$,

(5) $\Vdash A \rightarrow B$ *iff* $\vdash A$ *implies* $\Vdash B$.

203D Let $\mathfrak{L}$ be the system $\mathfrak{F}\omega$ or an extension thereof. We shall say that $\mathfrak{M}$ is a *simple extension* of $\mathfrak{L}$ if $\mathfrak{M}$ is the same as $\mathfrak{L}$ or is generated from $\mathfrak{L}$ by introducing postulates of the form $A(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$, where (1) A is a wff of $\mathfrak{L}$, (2) $x_1, \ldots, x_n$ is a complete list of the variables occurring free in A, (3) $A(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ is the expression which results when $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are substituted respectively for $x_1, \ldots, x_n$ in all free occurrences of the latter in A, all type superscripts remaining unchanged by the substitution, (4) the new symbols $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are not in the vocabulary of $\mathfrak{L}$ and in $\mathfrak{M}$ they are individual or predicate constants.

All systems constructed in this paper are simple extensions of $\mathfrak{F}\omega$.

If the A of **203D** is a closed wff of $\mathfrak{L}$, no new symbols are introduced by the postulate.

204D In this case we shall say that the postulate is *isophasic*. $\mathfrak{M}$ is an *isophasic extension* of $\mathfrak{L}$ if it is generated from $\mathfrak{L}$ entirely by isophasic postulates.

205T *Let* $\mathfrak{L}$ *be a simple extension of* $\mathfrak{F}\omega$ *and* $\mathfrak{M}$ *a simple extension of* $\mathfrak{L}$. *Let* B *be an arbitrary wff of* $\mathfrak{M}$.

We shall use these notations: if C is a wff of $\mathfrak{M}$, let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be the new constants (i.e. constants which belong to the vocabulary of $\mathfrak{M}$ but not to that of $\mathfrak{L}$) which occur in C or B or both; let $y_1, \ldots, y_n$ be variables which do not occur in C or B and are of the same types, respectively, as $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$; let $C(y)$, $B(y)$ stand for the expressions resulting when $y_1, \ldots, y_n$ are substituted respectively for $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in C, B.

Then $\vdash B$ *in* $\mathfrak{M}$ *iff there exists a conjunction* C *of postulates generating* $\mathfrak{M}$ *from* $\mathfrak{L}$ *such that*

(1) $\vdash \forall y_1 \ldots \forall y_n . C(y) \rightarrow B(y)$ *in* $\mathfrak{L}$.

Proof. Assume $\vdash B$ in $\mathfrak{M}$ and let Π be a formal proof of B. Let C be the conjunction of those postulates generating $\mathfrak{M}$ from $\mathfrak{L}$ which occur in Π. Let

$\mathfrak{a}_1, \ldots, \mathfrak{a}_p$ be the new constants occurring in Π. Let $\Pi(y)$ be the result of substituting the variables $y_1, \ldots, y_p$ for these constants in Π. Then $C(y)$, $\Pi(y)$ is a proof in $\mathfrak{L}$ of $B(y)$ from the hypothesis $C(y)$.

We assert now that $\vdash C(y) \rightarrow B(y)$ in $\mathfrak{L}$. To prove this it is sufficient, by the deduction theorem, to show that the rule of generalization has not been applied in $\Pi(y)$ to any variable occurring free in $C(y)$. By **203D** no variable occurs free in C, so the only variables occurring free in $C(y)$ are $y_1, \ldots, y_p$. But these occur only where they have been substituted for constants in Π, so the rule of generalization has not been applied to them. The assertion follows. Now generalize on $y_1, \ldots, y_p$ and (1) is proved.

The converse is obvious. This result is also useful in the contrapositive form:

206T $\nvdash$ *B in* $\mathfrak{M}$ *iff* $\nvdash \exists y_1 \ldots \exists y_n.\ C(y) \wedge B(y)$ *in* $\mathfrak{L}$.

207T *If B is wf is* $\mathfrak{L}$, *then* $\vdash B$ *in* $\mathfrak{M}$ *iff there exists a conjunction C of postulates generating* $\mathfrak{M}$ *from* $\mathfrak{L}$ *such that*

$$\vdash \exists y_1 \ldots \exists y_n C(y) \rightarrow B \text{ in } \mathfrak{L}.$$

Proof. The hypothesis implies that none of the variables $y_1, \ldots, y_n$ occurs in $B(y)$. Hence the result from **205T**.

From **206T** it is easy to infer that

208T $\mathfrak{M}$ *is a consistent system iff, for every (finite) conjunction C of postulates generating* $\mathfrak{M}$ *from* $\mathfrak{L}$,

$$\nvdash \exists y_1 \ldots \exists y_n\ C(y) \text{ in } \mathfrak{L}.$$

209D We shall say that $\mathfrak{M}$ is a *conservative extension* of $\mathfrak{L}$ iff every wff of $\mathfrak{L}$ which is provable in $\mathfrak{M}$ is also provable in $\mathfrak{L}$.

From **207T** it follows that

210T *If* $\mathfrak{M}$ *is a simple extension of* $\mathfrak{L}$, *then* $\mathfrak{M}$ *is conservative iff, for every conjunction C of postulates generating* $\mathfrak{M}$ *from* $\mathfrak{L}$,

$$\vdash \exists y_1 \ldots \exists y_n\ C(y) \text{ in } \mathfrak{L}.$$

211T *If* $\mathfrak{M}$ *is a conservative extension of* $\mathfrak{L}$, *then* $\mathfrak{M}$ *is consistent iff* $\mathfrak{L}$ *is consistent.*

212T *If* $\mathfrak{M}_1$ *is a conservative extension of* $\mathfrak{L}$ *and* $\mathfrak{M}_2$ *is a conservative extension of* $\mathfrak{M}_1$, *then* $\mathfrak{M}_2$ *is a conservative extension of* $\mathfrak{L}$.

§3. *Introducing constants*

We are now ready to construct extensions of $\mathfrak{F}\omega$ having a vocabulary of individual and predicate constants. There are two conditions we wish these extensions to satisfy:

301D (1) they must be conservative in the sense of **209D**, and (2) they must be *quantificationally closed* in this sense: let x be a variable and A a wff in which x occurs free; if $\vdash \exists x A$, then there is a constant b such that $\vdash A(\mathsf{b})$ where $A(\mathsf{b})$ is the expression obtained by substituting b for x in A, and if for every constant **b** of the relevant type $\vdash A(\mathbf{b})$, then $\vdash \forall x A$. The first of these conditions will assure that we are not smuggling in any existential assumptions, and the second that sentences containing quantifiers are provable if and only if they are true according to the syntactical definitions of the ranges of variables in §1. The semantic definition of the range of individual variables will be dealt with later.

302D Let A be a monadic wff (i.e. a wff in which exactly one variable occurs free) and x the free variable in A. The expression $y\mathring{\varepsilon}A$ shall stand for the schema $\forall x.\ A \rightarrow A(y)$. This expression may be used if y is a constant of the same type as x or a variable of this type which is free at all places where it is substituted for x.

Let $\mathfrak{L}0$ be an arbitrary isophasic extension of $\mathfrak{F}\omega$. For each monadic wff A of $\mathfrak{L}0$ we now introduce a constant ℩(A) which is to be of the same type as the free variable in A. We shall write ℩A for ℩(A) wherever A is used as a syntactical variable for a wff. If P is a predicate constant we write ℩P for ℩$(x\varepsilon P)$. These constants, which we call ℩- constants, are introduced by postulating, for every pair A, B of monadic wffs which are of the same type,

303P ℩$A\mathring{\varepsilon}A \wedge$ ℩$B\mathring{\varepsilon}B \wedge\colon \forall y.\ y\mathring{\varepsilon}A \longleftrightarrow y\mathring{\varepsilon}B. \rightarrow$ ℩$A =$ ℩B

Let $\mathfrak{L}1$ be the extension generated by these postulates. In this system there are monadic wffs which are not wf in $\mathfrak{L}0$ because they contain occurrences of one or more ℩-constants. We now make an extension $\mathfrak{L}2$ of $\mathfrak{L}1$ by introducing a constant ℩A for every monadic wff A which is wf in $\mathfrak{L}1$ but not in $\mathfrak{L}0$. For every such A coupled with every B of the same type as A, there is a postulate having the form of **303P** generating $\mathfrak{L}2$. Repeating this procedure, we construct an infinite sequence of extensions.

Let $\mathfrak{L}$℩ be the extension of $\mathfrak{L}0$ generated by all the postulates which generate members of this sequence. An expression is wf in $\mathfrak{L}$℩ iff it is wf in $\mathfrak{L}n$ for some non-negative integer n.

Our ℩-operator is partly analogous to Hilbert's ε-function[31] and to the functions $\iota_{\alpha(o\alpha)}$ in Church's formulation of type theory[32]. On the other hand it must be emphasized that the mark ℩ standing alone is not a predicate of $\mathfrak{L}$℩. Nor is any expression of the form ℩$(x\varepsilon F)$ where x and F are free variables wf in $\mathfrak{L}$℩, since the ℩-operator can be applied only to monadic wffs. In consequence of this restriction it is possible to prove, without appeal to any axiom of choice, that

304T *$\mathfrak{L}$℩ is a conservative extension of* $\mathfrak{L}0$.

Proof.

305D Let $\phi\varepsilon\,\mathrm{sf}$ (to be read "ϕ is a *selective function*") stand for

$$\forall x\forall F.\ x\phi F \rightarrow x\mathring{\varepsilon}F. \wedge \forall x\forall F\forall y\forall G\colon\!.\ x\phi F \wedge y\phi G \wedge \forall z.\ z\mathring{\varepsilon}F \longleftrightarrow z\mathring{\varepsilon}G\colon \rightarrow x = y$$

Note that the domain of a selective function thus defined is not restricted to non-empty predicates.

We shall use the lemma: *if ϕ is a selective function and G is a predicate of the same type as the members of the domain of ϕ, then there is an extension ϕ' of ϕ such that*

$$\phi' \varepsilon \text{ sf} \wedge \exists x\ x\phi' G$$

The proof is elementary.

If B is a wff of $\mathfrak{L}$℩, the expression $\mathfrak{L}(B)$ will denote the extension of $\mathfrak{L}0$ generated by those ℩-postulates (**303P**) in which every ℩-constant occurring has at least one occurrence in B. We shall prove first that $\mathfrak{L}(B)$ is a conservative extension of $\mathfrak{L}0$.

Let ℩$B_1, \ldots,$ ℩B_p be a complete list of the ℩-constants which occur in B. In consequence of the way in which the ℩-constants are written the phrase "occurs in" expresses a transitive relation between ℩-constants. Hence any ℩-constant which occurs in one of the ℩B_i ($1 \leq i \leq p$) *is* one of the ℩B_i. We may assume that these constants are ordered by their subscripts so that first come all those which are wf in $\mathfrak{L}0$, next those that are wf in $\mathfrak{L}1$ but not in $\mathfrak{L}0$, and so on. This assures that if ℩B_j occurs in ℩B_i then $j \leq i$.

We now construct a sequence of systems $\mathfrak{M}_0 = \mathfrak{L}0, \mathfrak{M}_1, \ldots, \mathfrak{M}_p$ such that if $1 \leq i \leq p$, $\mathfrak{M}_i$ is the extension of $\mathfrak{M}_{i-1}$ generated by postulates which introduce ℩B_i and an auxiliary constant ϕ_i which can be discarded later. These postulates have the form

(1) $\phi_i \supseteq \phi_k \wedge \phi_i \varepsilon \text{ sf}$
(2) $\forall F\colon \forall x.\ x\varepsilon F \longleftrightarrow B_i(x). \rightarrow$ ℩$B_i\ \phi_i F$

where ϕ_k is defined as follows: if there is a B_j of the same type as B_i with $j < i$, let k be the largest of the subscripts j which satisfy this condition, and let ϕ_k be the selective function introduced by the postulates which generate $\mathfrak{M}_k$; if there is no such j, let $k = 0$ and let ϕ_0 be the null relation.

We assert that the postulates (1) and (2) generate a conservative extension of $\mathfrak{M}_{i-1}$. If $k \neq 0$, ϕ_k is a selective function in consequence of the postulates generating $\mathfrak{M}_k$; if $k = 0$, ϕ_k is a selective function since the null-relation satisfies **305D**. By **201T** and extensionality there is exactly one F such that $\forall x.\ x\varepsilon F \longleftrightarrow B_i(x)$, and by the lemma there is a selective function which is an extension of ϕ_k and includes this F in its domain. The assertion follows by **210T**, hence by **212T** $\mathfrak{M}_p$ is a conservative extension of $\mathfrak{L}0$. Clearly the theorems of $\mathfrak{L}(B)$ are theorems of $\mathfrak{M}_p$, so $\mathfrak{L}(B)$ is a conservative extension of $\mathfrak{L}0$.

Finally to prove the theorem, let B be a wff of $\mathfrak{L}0$, let Π be a proof of B in $\mathfrak{L}$℩ and let $\mathfrak{L}(\Pi)$ be the extension of $\mathfrak{L}0$ generated by those ℩-postulates in which all ℩-constants occurring have occurrence in Π. Then Π is a proof in $\mathfrak{L}(\Pi)$ and $\mathfrak{L}(\Pi)$ is conservative by the above reasoning. So B is provable in $\mathfrak{L}0$.

This theorem shows that the postulates generating $\mathfrak{L}$℩ from $\mathfrak{L}0$ are nothing more than implicit definitions of the ℩-constants. If $\mathfrak{L}0$ is $\mathfrak{F}\omega$, any inference made by means of $\mathfrak{L}$℩ is made by pure logic. Incidentally, by introducing the ℩-constants we have formalized what is known as "natural deduction." From $\exists xA$ one can always infer "℩A is such an x."

306T $\mathfrak{L}\mathbf{\iota}$ *is quantificationally closed in the sense of* **301D**.

Proof. This can be proved by reasoning similar to that used in [6][33] with respect to Hilbert's ε-function.

This theorem holds in all isophasic extensions of $\mathfrak{L}\mathbf{\iota}$, in particular in the system $\mathfrak{L}\mu$ which we shall now construct.

The motive for constructing $\mathfrak{L}\mu$ is that we need a system in which the $\mathbf{\iota}$-operator is expressed by predicates in the system so that we can use the axioms of choice. The simplest way to do this would be to define μ as an abbreviation for $\mathfrak{L}\mathbf{\iota}$

$$\mathbf{\iota}(\phi\,\varepsilon\,\mathrm{sf} \wedge \forall F \exists x\; x\phi F)$$

and generate $\mathfrak{L}\mu$ as an extension of $\mathfrak{L}\mathbf{\iota}$ by postulating for each monadic predicate constant P of $\mathfrak{L}\mathbf{\iota}$

307P $$\mathbf{\iota}P\;\mu\,P.$$

It will be more convenient, however, to use a stronger set of postulates.

308D Let ξ wo F be an abbreviation for

$$\forall x \forall y\colon x,y\varepsilon F \rightarrow .\; x\xi y \vee y\xi x \vee x = y\colon \wedge \forall G \therefore \forall x.\; x\varepsilon G \rightarrow x\varepsilon F. \rightarrow \exists x\colon x\mathring{\varepsilon}G \wedge \forall y.\; y\xi x \rightarrow \neg y\mathring{\varepsilon}G$$

to be read "ξ is a *well-ordering* of F[34]."

Let ρ be an abbreviation for

$$\mathbf{\iota}(\forall F\;\; \xi\,\mathrm{wo}\,F)$$

so that for each type $\boldsymbol{t}$ there is a ρ of type $(\boldsymbol{tt})$ which is a well-ordering of the universal set of type $(\boldsymbol{t})$. Then $\mathfrak{L}\mu$ can be defined as the extension of $\mathfrak{L}\mathbf{\iota}$ generated by postulating for each $\mathbf{\iota}$-constant $\mathbf{\iota}A$

309P $\forall y.\; y\,\rho\,\mathbf{\iota}A \rightarrow \neg y\mathring{\varepsilon}A$

In consequence of these postulates, $\mathfrak{L}\mu$ has the convenient property that for each non-empty predicate P, $\mathbf{\iota}P$ is the ρ-first member of P. This permits us to define a set of selective functions μ by the schema

$$\forall x \forall F\colon .\; x\mu F \longleftrightarrow\colon x\mathring{\varepsilon}F \wedge \forall y.\; y\rho x \rightarrow \neg y\mathring{\varepsilon}F$$

and

310T *the schema* **307P** *is then deducible from* **309P**.

311T $\mathfrak{L}\mu$ *is a conservative extension of* $\mathfrak{L}0$ *iff the instances of the schema*

312P $$\exists\phi\,.\,\phi\varepsilon\,\mathrm{sf} \wedge \forall F \exists x\; x\phi F$$

are axioms (*or theorems*) *of* $\mathfrak{L}0$.

Proof. Assume **312P**. Since these are choice axioms the well-ordering theorem $\exists\xi\forall F\xi$ wo F holds in $\mathfrak{L}0$. It is easy to see that $\mathfrak{L}\mu$ is the same as the extension of $\mathfrak{L}0$ generated by the postulates

(1) $\forall F\, \rho\, \mathrm{wo}\, F$

(2) for each monadic wff A

$$℩A \mathring{\varepsilon} A \wedge \forall y.\, y \rho ℩A \rightarrow \neg y \mathring{\varepsilon} A$$

We may assume that this extension is constructed by stages, first postulating all instances of (1) and those instances of (2) in which A is wf in $\mathfrak{L}0$, next those instances of (2) in which A is wf in $\mathfrak{L}1$ but not in $\mathfrak{L}0$, and so on, thus introducing all **℩**-constants.

Each ρ not only occurs in those places where it appears above as ρ, but also in one instance of (2) where it occurs as the right argument of a ρ of higher type.

Let B be an arbitrary wff of $\mathfrak{L}\mu$, let $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$ be the types of the **℩**-constants occurring in B. Let $\mathfrak{L}\mu(B)$ be the extension of $\mathfrak{L}0$ generated by (i) those instances of (1) in which ρ belongs to one of the types $(\boldsymbol{t}_1\boldsymbol{t}_1), \ldots,$ $(\boldsymbol{t}_n\boldsymbol{t}_n)$, (ii) those instances of (2) in which $℩A$ is one of the **℩**-constants occurring in B. We may suppose that $\mathfrak{L}\mu(B)$ is generated by a finite sequence of extensions of which each introduces just one new constant. It is then easy to verify that, in consequence of the well-ordering theorem and **210T**, each of these extensions is conservative.

It follows by **212T** that $\mathfrak{L}\mu(B)$ is a conservative extension of $\mathfrak{L}0$ and by an argument similar to the last part of the proof of **304T**, that $\mathfrak{L}\mu$ is likewise conservative.

To prove the second part: the axioms of choice are deducible from **307P** by quantificational closure (**306T**).

Thus the system $\mathfrak{L}\mu$ has a set of axioms which have prefixes containing existential quantifiers. But this need not raise any ontological ghosts. There can be no question about the existence of selective functions which have in their domain all predicates of a given type; for such functions are definable in the metatheory of $\mathfrak{L}℩$. The choice axioms merely state that such functions are included among the predicates of the object language. We shall have more to say about this in §5.

313T *Let B be a wff of $\mathfrak{L}℩$, hence also of $\mathfrak{L}\mu$. Let C be the conjunction of the postulates which generate the system $\mathfrak{L}(B)$ (the system $\mathfrak{L}\mu(B)$) defined in the proof of* **304T** (**311T**). *Then, using the notation of* **205T**,

$$\vdash B \textit{ in } \mathfrak{L}℩ (\textit{in } \mathfrak{L}\mu)$$

if and only if

$$\vdash C(y) \rightarrow B(y) \textit{ in } \mathfrak{L}0.$$

Proof. It is sufficient to show (1) that $\mathfrak{L}℩$ is a conservative extension of $\mathfrak{L}(B)$ and (2) that $\mathfrak{L}\mu$ is a conservative extension of $\mathfrak{L}\mu(B)$.

To prove (1), let A be an arbitrary wff of $\mathfrak{L}℩$. Construct the system $\mathfrak{L}(B \wedge A)$ by the method of **304T**, introducing first those constants which occur in B. By this procedure one can show that $\mathfrak{L}(B \wedge A)$ is a conservative extension of $\mathfrak{L}(B)$, from which (1) can be inferred as in the last part of the proof of **304T**.

Similarly (2) can be proved by adaptation of the reasoning of **311T**.

314D Let us say that a wff A of $\mathfrak{L}\mathbf{1}$ is *basic* iff there exists a wff B of $\mathfrak{L}0$ such that $\vdash A \longleftrightarrow B$ in $\mathfrak{L}\mathbf{1}$.

We have used the expression $\mathfrak{L}0$ to denote an arbitrary isophasic extension of $\mathfrak{F}\omega$. It follows that $\mathfrak{L}\mathbf{1}$, which we have defined as the extension of $\mathfrak{L}0$ generated by **303P**, likewise denotes an arbitrary member of a class of systems. Let $\{\mathfrak{L}\mathbf{1}\}$ denote this class. It is easy to see that

315T $\{\mathfrak{L}\mathbf{1}\}$ *is closed with respect to extensions generated by basic postulates.*

$\mathfrak{L}\mu$ is not in the class $\{\mathfrak{L}\mathbf{1}\}$, since **309P** are not basic. The class $\{\mathfrak{L}\mu\}$ can be defined similarly.

To complete the vocabulary of our systems we mention here a few notations that will be used in the sequel. The customary expressions of set theory can be defined in obvious ways.

If A is a wff and $x_1, \ldots, x_n$ is a complete list of its free variables,

$$\lambda x_1 \ldots x_n(A) =_{def} \mathbf{ɿ}(\forall x_1 \ldots \forall x_n.\ F(x_1, \ldots, x_n) \longleftrightarrow A).$$

316D The concept of a function is introduced by

$$\begin{aligned} 1-s &=_{def} \lambda\xi(\forall x \forall y \forall z\colon y\xi x \wedge z\xi x. \rightarrow y = z) \\ s-1 &=_{def} \lambda\xi(\forall x \forall y \forall z\colon x\xi y \wedge x\xi z. \rightarrow y = z) \\ 1-1 &=_{def} 1-s \cap s-1 \end{aligned}$$

Terms containing free variables are used in this paper in a few places. Such expressions, which have the form $\phi`x$ can be introduced by postulating

$$\phi\varepsilon 1-s \rightarrow \forall x.\exists y\ \ y\phi x \rightarrow \phi`x\phi x$$

It is not difficult to show that such postulates are conservative.

§4. *Names for objects.*

As stated in §1, each object is to have exactly one name, so if **a**,**b** are distinct names we must have $\vdash \mathbf{a} \neq \mathbf{b}$. We shall use Arabic numerals as names, placing a dot over each to distinguish the name-numerals from the "proper numerals" which will be defined in §7. If the range of discourse is finite, the names will be $\dot{0}, \ldots, \dot{n}$ for the appropriate choice of n; if it is infinite, the names will be $\dot{0}, \dot{1}, \dot{2}, \ldots$

It would be natural to introduce these as new constants by postulating $\dot{m} \neq \dot{n}$ for each pair of distinct names, but it is more convenient to define them as abbreviations for certain expressions already present in $\mathfrak{L}\mathbf{1}$.

401D Accordingly we define

$$\begin{aligned} \dot{0} &=_{def} \mathbf{ɿ}(x = x) \\ \dot{1} &=_{def} \mathbf{ɿ}(x \neq \dot{0}) \\ \dot{2} &=_{def} \mathbf{ɿ}(x \neq \dot{0} \wedge x \neq \dot{1}) \\ &\ldots \end{aligned}$$

If the range of discourse is infinite, we postulate

402P $$\dot{1} \neq \dot{0},\ \dot{2} \neq \dot{0},\ \ldots$$

These will be called the *least number postulates*. From them it can be inferred that $\dot{m} \neq \dot{n}$ for all pairs of distinct names.

Let $\mathfrak{L}\mathbf{1}(\infty)$ ($\mathfrak{L}\mu(\infty)$) be the extension of $\mathfrak{L}\mathbf{1}(\mathfrak{L}\mu)$ generated by **402P**. If the range of discourse has exactly n members we postulate the first n - 1 wffs of **402P** and

403P $$\dot{n} = \dot{0}.$$

Then it is easy to prove for all numerals m such that $m > n$ that $\dot{m} = \dot{0}$. In this case the only names are $\dot{0}, \ldots, (n \dot{-} 1)$. The systems thus generated will be called $\mathfrak{L}\mathbf{1}(n)$, $\mathfrak{L}\mu(n)$.

404T *For each positive integer n, every closed wff of* $\mathfrak{F}\omega$ *is decidable in* $\mathfrak{L}\mathbf{1}(n)$, *so* $\mathfrak{L}\mathbf{1}(n)$ *is consistent.*

Proof. Every wff of the form $\forall x\, \dot{x}\varepsilon F$ is equivalent in $\mathfrak{L}\mathbf{1}(n)$ to the conjunction

$$\dot{0}\varepsilon F \wedge \ldots \wedge (n \dot{-} 1)\varepsilon F.$$

Every wff of the form $\forall F\, F\varepsilon A$, where F is of type (•), is equivalent in $\mathfrak{L}\mathbf{1}(n)$ to a conjunction of the form

$$P_1\varepsilon A \wedge \ldots \wedge P_q\varepsilon A,$$

where $q = 2^n$ and each P_i has one of the forms $\lambda x(x \neq x)$, $\lambda x(x = \dot{m}_1 \vee \ldots \vee x = \dot{m}_k)$, each $\dot{m}_i$ being a name in $\mathfrak{L}\mathbf{1}(n)$.

By a generalization of the two preceding statements it can be shown that every closed wff of $\mathfrak{F}\omega$ is equivalent in $\mathfrak{L}\mathbf{1}(n)$ to a wff in which every variable occurring is bound by λ and the only non-logical constants occurring are lambda-constants, names and =. These wffs are clearly decidable in $\mathfrak{L}\mathbf{1}(n)$. The result follows.

From here through **411T** it will be assumed that $\mathfrak{L}0$ is $\mathfrak{F}\omega$. Then the axioms of $\mathfrak{L}\mathbf{1}(\infty)$ are the axioms of $\mathfrak{F}\omega$, the **1**-postulates (**303P**) and the least-number postulates. Since every theorem of $\mathfrak{L}\mathbf{1}(\infty)$ is deducible from a finite subset of these,

405T *every theorem of* $\mathfrak{L}\mathbf{1}(\infty)$ *is provable in* $\mathfrak{L}\mathbf{1}(n)$ *for sufficiently large n.*

It follows that

406T *every wff which is consistent in infinitely many of the finite systems is consistent in* $\mathfrak{L}\mathbf{1}(\infty)$.

The choice axioms **312P** are provable in each of the finite systems, so by **404T, 406T** they are consistent in $\mathfrak{L}\mathbf{1}(\infty)$, from which we infer

407T $\mathfrak{L}\mathbf{1}(\infty)$ *is consistent and by* **311T** *so is* $\mathfrak{L}\mu(\infty)$.

It is easy to find a wff A such that A is provable in infinitely many of the finite systems and $\neg A$ is also provable in infinitely many of these. So by **406T** and **407T**:

408T *Both* $\mathfrak{L}\mathbf{1}(\infty)$ *and* $\mathfrak{L}\mu(\infty)$ *are incomplete.*

409D Let us say that a wff A is an *infinity formula* if A is refutable in each of the finite systems but not in $\mathfrak{L}\mathbf{1}(\infty)$. Then

410T *A is an infinity formula iff* (1) $\nvdash A$ *in* $\mathfrak{L}\mathbf{1}(\infty)$, *and* (2) *each of the least-number postulates is a consequence of A.*

By **405T**:

411T *No infinity formula can be proved in* $\mathfrak{L}\mathbf{1}(\infty)$ *and it is easy to see that this holds also for* $\mathfrak{L}\mu(\infty)$.

In order to satisfy condition (3) of adequacy (§1), $\mathfrak{L}\mathbf{1}(\infty)$ must be extended by postulating an infinity formula. In consequence of some results of Trahtenbrot [9] there is no weakest (or strongest) infinity formula, which would seem to make it difficult to justify selection of any particular postulate. This difficulty is resolved in §6.

§5. *Expressibility of metapredicates*

501D A predicate Γ in the metatheory of $\mathfrak{L}\mathbf{1}$(and of its isophasic extensions) will be called a *metapredicate* iff: (1) the meaning of the statement $\Gamma(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$, where $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are $\mathbf{1}$-constants, is defined when these constants are used autonymously; and (2) Γ is uniform as to type in the sense that, whenever $\Gamma(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ and $\Gamma(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ are true, then, for $1 \leq i \leq n$, $\mathfrak{a}_i$ and $\mathfrak{b}_i$ are of the same type.

It will be convenient to write $\overset{n}{\mathfrak{a}}\epsilon\Gamma$ for $\Gamma(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$, where use of the symbol ϵ (not ε) will serve as a reminder that the predicate is defined in the metatheory. We shall also write $\overset{n}{\mathfrak{a}}\varepsilon\gamma$ for $\gamma(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ where γ is an n-adic predicate in $\mathfrak{L}\mathbf{1}$ and the $\mathfrak{a}_i$ are used in the language $\bar{\mathfrak{L}}\mathbf{1}$.

Let us say that an n-ad of $\mathbf{1}$-constants $(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ is relevant for Γ iff, for any $\overset{n}{\mathfrak{a}}$ such that $\overset{n}{\mathfrak{a}}\epsilon\Gamma$, $\mathfrak{a}_i$ and $\mathfrak{b}_i$ $(1 \leq i \leq n)$ are of the same type.

502D Let $\mathfrak{L}$ be an isophasic extension of $\mathfrak{L}\mathbf{1}$, Γ a metapredicate, γ a predicate in $\mathfrak{L}\mathbf{1}$. We shall say that γ *expresses* Γ in $\mathfrak{L}$ iff

(1) $\mathfrak{L}$ is consistent,
(2) for each $\overset{n}{\mathfrak{a}}$, if $\overset{n}{\mathfrak{a}}\epsilon\Gamma$, then $\vdash \overset{n}{\mathfrak{a}}\varepsilon\gamma$ in $\mathfrak{L}$,
(3) for each relevant $\overset{n}{\mathfrak{b}}$ there exists $\overset{n}{\mathfrak{a}}\epsilon\Gamma$ such that

$$\vdash \overset{n}{\mathfrak{b}}\varepsilon\gamma \rightarrow \overset{n}{\mathfrak{b}} = \overset{n}{\mathfrak{a}} \text{ in } \mathfrak{L}.$$

503D We shall say that Γ is *expressible by* γ in $\mathfrak{L}$ iff $\mathfrak{L}$ has an extension in which γ expresses Γ. (We do not require that such an extension be constructible in the sense that one can formulate an effective set of rules for writing the postulates which generate it.)

504T *If γ expresses Γ in $\mathfrak{L}$, then γ expresses Γ in every consistent isophasic extension of $\mathfrak{L}$.*

505T *If Γ is expressible by γ in $\mathfrak{M}$ and $\mathfrak{M}$ is an isophasic extension of $\mathfrak{L}$, then Γ is expressible by γ in $\mathfrak{L}$.*

For a monadic metapredicate, the definition of expressibility in §1 is equivalent to **503D**. This follows from Lindenbaum's theorem[51] and **504T**.

506T *Let* $\mathfrak{M}$ *be the extension of* $\mathfrak{L}$ *generated by postulating* $\overset{n}{\mathfrak{a}}\varepsilon\gamma$ *for each* $\overset{n}{\mathfrak{a}}\epsilon\Gamma$. *Then the following are equivalent*:

(1) Γ *is expressible by* γ *in* $\mathfrak{L}$,

(2) *for every finite collection of relevant n-ads* $\overset{n}{\mathfrak{b}}_1, \ldots, \overset{n}{\mathfrak{b}}_p$ *there exist* $\overset{n}{\mathfrak{a}}_1, \ldots, \overset{n}{\mathfrak{a}}_p\epsilon\Gamma$ *such that*

$$\vdash \overset{n}{\mathfrak{b}}_1\varepsilon\gamma \to \overset{n}{\mathfrak{b}}_1 = \overset{n}{\mathfrak{a}}_1 \,.\, \wedge \ldots \wedge \,.\, \overset{n}{\mathfrak{b}}_p\varepsilon\gamma \to \overset{n}{\mathfrak{b}}_p = \overset{n}{\mathfrak{a}}_p \textit{ in } \mathfrak{M},$$

(3) *for every finite collection* $B_1, \ldots, B_p$ *of wffs of the same type as* γ, *there exist* $\overset{n}{\mathfrak{a}}_1, \ldots, \overset{n}{\mathfrak{a}}_p\epsilon\Gamma$ *such that*

$$\vdash \exists\overset{n}{x}.\ \overset{n}{x}\varepsilon\gamma \wedge \overset{n}{x}\varepsilon B_1. \to \overset{n}{\mathfrak{a}}_1\varepsilon B_1{:}\ \wedge \ldots \wedge{:}\ \exists\overset{n}{x}.\ \overset{n}{x}\varepsilon\gamma \wedge \overset{n}{x}\varepsilon B_p. \to \overset{n}{\mathfrak{a}}_p\varepsilon B_p \textit{ in } \mathfrak{M}.$$

Proof. Assume (3). Let us say that a wff B is relevant iff it is of the same type as γ. It will be shown first that there exists a consistent extension $\mathfrak{N}$ of $\mathfrak{M}$ such that, for every relevant B,

(4) *if* $\vdash\overset{n}{\mathfrak{a}}\not\varepsilon B$ *in* $\mathfrak{N}$ *for every* $\overset{n}{\mathfrak{a}}\epsilon\Gamma$, *then* $\vdash\forall\overset{n}{x}.\ \overset{n}{x}\varepsilon\gamma \to \overset{n}{x}\not\varepsilon B$ *in* $\mathfrak{N}$.

To see this, observe that from

(a) $\vdash \exists\overset{n}{x}.\ \overset{n}{x}\varepsilon\gamma\wedge\overset{n}{x}\varepsilon B_1. \to \overset{n}{\mathfrak{a}}_1\varepsilon B_1{:}\ \wedge{:}\ \exists\overset{n}{x}.\ \overset{n}{x}\varepsilon\gamma\wedge\overset{n}{x}\varepsilon B_2. \to \overset{n}{\mathfrak{a}}_2\varepsilon B_2$

and

(b) $\vdash\overset{n}{\mathfrak{a}}_1\not\varepsilon B_1$

one can infer

(c) $\vdash \forall x.\ \overset{n}{x}\varepsilon\gamma \to \overset{n}{x}\not\varepsilon B_1. \wedge{:}\ \exists\overset{n}{x}.\ \overset{n}{x}\varepsilon\gamma\wedge\overset{n}{x}\varepsilon B_2. \to \overset{n}{\mathfrak{a}}_2\varepsilon B_2{:}$

This argument applies also where (a) holds for $B_1, \ldots, B_p$ ($p > 2$) and (b) holds for any proper subset of $B_1, \ldots, B_p$. Thus we can generate an extension $\mathfrak{N}_1$ of $\mathfrak{M}$ by postulating $\forall\overset{n}{x}.\ \overset{n}{x}\varepsilon\gamma \to \overset{n}{x}\not\varepsilon B$ for every relevant B such that $\vdash\overset{n}{\mathfrak{a}}\not\varepsilon B$ in $\mathfrak{M}$ for every $\overset{n}{\mathfrak{a}}\epsilon\Gamma$, and condition (3) will be preserved in $\mathfrak{N}_1$. If there are any relevant B such that $\vdash\overset{n}{\mathfrak{a}}\not\varepsilon B$ in $\mathfrak{N}_1$ for all $\overset{n}{\mathfrak{a}}\epsilon\Gamma$ but not $\vdash\forall\overset{n}{x}.\ \overset{n}{x}\varepsilon\gamma \to \overset{n}{x}\varepsilon B$ in $\mathfrak{N}_1$, let $\mathfrak{N}_2$ be generated from $\mathfrak{N}_1$ in the same way, and so on, infinitely many times if necessary, till we have an extension $\mathfrak{N}$ which satisfies (4).

This done, let $\overset{n}{\mathfrak{b}}_1\,\overset{n}{\mathfrak{b}}_2, \ldots$ be an enumeration of the relevant n-ads, by a Gödel numbering or some lexicographic rule. Now postulate, for each $\overset{n}{\mathfrak{b}}_i$,

(5) $\overset{n}{\mathfrak{b}}_i\varepsilon\gamma \to \overset{n}{\mathfrak{b}}_i = \overset{n}{\mathfrak{a}}_i$,

where the $\overset{n}{\mathfrak{a}}_i$ are determined by the following rule: if there exists at least one $\overset{n}{\mathfrak{a}}\epsilon\Gamma$ such that

$$\vdash \overset{n}{\mathfrak{b}}_i = \overset{n}{\mathfrak{a}} \wedge .\overset{n}{\mathfrak{b}}_1\varepsilon\gamma \to \overset{n}{\mathfrak{b}}_1 = \overset{n}{\mathfrak{a}}_1. \wedge \ldots \wedge .\overset{n}{\mathfrak{b}}_{i-1}\,\varepsilon\gamma \to \overset{n}{\mathfrak{b}}_{i-1} = \overset{n}{\mathfrak{a}}_{i-1} \textit{ in } \mathfrak{N},$$

then let $\overset{n}{\mathfrak{a}}_i$ be the first of these in the enumeration of the relevant n-ads; if no such $\overset{n}{\mathfrak{a}}\epsilon\Gamma$ exists, let $\overset{n}{\mathfrak{a}}_i$ be the first of all the $\overset{n}{\mathfrak{a}}\epsilon\Gamma$ (or an arbitrary $\overset{n}{\mathfrak{a}}\epsilon\Gamma$).

Thus the postulates are uniquely defined. It remains to be shown that they are consistent. We prove this by course-of-values induction on the sequence $\overset{n}{\mathfrak{b}}_1, \overset{n}{\mathfrak{b}}_2, \ldots$. Assume that the conjunction of the first p members of the sequence of postulates (5) is consistent. Then by rule (4) of **202T**, either

$$\vdash \overset{n}{\mathfrak{b}}_{p+1} \notin \gamma \wedge .\overset{n}{\mathfrak{b}}_1 \varepsilon \gamma \rightarrow \overset{n}{\mathfrak{b}}_1 = \overset{n}{\mathfrak{a}}_1 . \wedge \ldots \wedge .\overset{n}{\mathfrak{b}}_p \varepsilon \gamma \rightarrow \overset{n}{\mathfrak{b}}_p = \overset{n}{\mathfrak{a}}_p \; in \; \mathfrak{N}$$

or

$$\vdash \mathfrak{b}_{p+1} \varepsilon \gamma \wedge .\overset{n}{\mathfrak{b}}_1 \varepsilon \gamma \rightarrow \overset{n}{\mathfrak{b}}_1 = \overset{n}{\mathfrak{a}}_1 . \wedge \ldots \wedge .\overset{n}{\mathfrak{b}}_p \varepsilon \gamma \rightarrow \overset{n}{\mathfrak{b}}_p = \overset{n}{\mathfrak{a}}_p \; in \; \mathfrak{N}.$$

If the former, then clearly the $(p+1)$th postulate is simultaneously consistent with the first p postulates. If the latter, then

$$\vdash \exists \overset{n}{x} :. \overset{n}{x} \varepsilon \gamma \wedge : \overset{n}{x} = \overset{n}{\mathfrak{b}}_{p+1} \wedge .\overset{n}{\mathfrak{b}}_1 \varepsilon \gamma \rightarrow \overset{n}{\mathfrak{b}}_1 = \overset{n}{\mathfrak{a}}_1 . \wedge \ldots \wedge .\overset{n}{\mathfrak{b}}_p \varepsilon \gamma \rightarrow \overset{n}{\mathfrak{b}}_p = \overset{n}{\mathfrak{a}}_p \; in \; \mathfrak{N}$$

and the expression following the first conjunction sign in this formula is a relevant wff. It follows by (4) in its contrapositive form that there exists $\overset{n}{\mathfrak{a}} \epsilon \Gamma$ such that $\overset{n}{\mathfrak{b}}_{p+1} = \overset{n}{\mathfrak{a}}$ is simultaneously consistent with the first p postulates. This proves that the postulates (5) are consistent.

It follows that (3) implies (1). It is almost obvious that (1) implies (2). Assume (2) and consider first the case where Γ and γ are monadic. Let $B_i, \ldots, B_p$ be relevant wffs. We shall write B_i for $\lambda x(B_i(x))$ $(1 \leq i \leq p)$. By (2), there are $\mathfrak{a}_1, \ldots, \mathfrak{a}_p \epsilon \Gamma$ such that

(6) $\vdash \iota(\gamma \cap B_1) \varepsilon \gamma \rightarrow \mathfrak{a}_1 = \iota(\gamma \cap B_1) . \wedge \ldots \wedge . \iota(\gamma \cap B_p) \varepsilon \gamma \rightarrow \mathfrak{a}_p = \iota(\gamma \cap B_p)$

From each member of this conjunction we can deduce

$$\iota(\gamma \cap B_i) \varepsilon \gamma \wedge \iota(\gamma \cap B_i) \varepsilon B_i \rightarrow \mathfrak{a}_i \varepsilon B_i,$$

and, since for any monadic P, $\vdash \iota P \varepsilon P \longleftrightarrow \exists x \; x \varepsilon P$.

$$\exists x . \; x \varepsilon \gamma \wedge x \varepsilon B_i . \rightarrow \mathfrak{a}_i \varepsilon B_i$$

Apply this to (6), using rule (2) of **202T**, and it follows in this special case (Γ and γ monadic) that (2) implies (3).

To extend this to the general case it is sufficient to exhibit, for an arbitrary n-adic predicate P, an n-ad of constants $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$ such that

$$\vdash \overset{n}{\mathfrak{c}} \varepsilon P \longleftrightarrow \exists \overset{n}{x} \; \overset{n}{x} \varepsilon P$$

We show a method of doing this by illustrating it for the case $n = 2$ where, if α is the dyadic predicate under consideration, the required constants are

$$\iota(\exists z \; x \alpha z) \text{ and } \iota(\iota(\exists z \; x \alpha z) \alpha \; y).$$

The proof is now easy to complete.

It follows that,

507T *in order that* Γ *be expressible by* γ *in* $\mathfrak{L}$, *it is sufficient that*

(1) $\vdash \overset{n}{\mathfrak{a}} \varepsilon \gamma$ *for each* $\overset{n}{\mathfrak{a}} \epsilon \Gamma$,

and

(2) *the equivalent conditions* (2), (3) *of the preceding theorem are satisfied.*

It follows also from **506T** that,

508T *if* Γ *is a monadic metapredicate of individual constants, then* Γ *is expressible in* $\mathfrak{L}$ *by* V (*where* $\mathsf{V} = \lambda x(x = x)$) *iff* $\mathfrak{L}$ *is strongly* Γ-*consistent in the sense of Henkin*[52].

509T *If* (1) Γ *is expressible by* γ *in* $\mathfrak{L}$,
(2) A *is a wff such that* $\Vdash A$ *in every extension* $\mathfrak{M}$ *of* $\mathfrak{L}$ *in which* γ *expresses* Γ,
then (3) $\vdash A$ *in every such extension* $\mathfrak{M}$ *of* $\mathfrak{L}$, *and* (4) Γ *is expressible by* γ *in the extension of* $\mathfrak{L}$ *generated by postulating* A.

Proof. By (1) there is an extension $\mathfrak{M}$ of $\mathfrak{L}$ in which γ expresses Γ. By (2) and **504T**, $\Vdash A$ in every consistent extension of $\mathfrak{M}$, hence $\vdash A$ in $\mathfrak{M}$, from which (4) follows by **505T**.

510T *In the finite systems* $\mathfrak{L}\mathbf{1}(n)$, $\mathfrak{L}\mu(n)$ *all metapredicates are expressible.*

Proof. There is a finite subset $A = \{\mathbf{a}_1, \ldots \mathbf{a}_p\}$ of Γ such that $p \leq n$ and A is maximal with respect to the property that for all pairs $\mathbf{a}_i$, $\mathbf{a}_j \epsilon A$ such that $i \neq j$ the wffs $\mathbf{a}_i \neq \mathbf{a}_j$ are simultaneously consistent in $\mathfrak{L}\mathbf{1}(n)$. It can be verified that Γ is expressible in $\mathfrak{L}\mathbf{1}(n)$ by $\lambda x(x = \mathbf{a}_1 \vee \ldots \vee x = \mathbf{a}_p)$. This applies also to $\mathfrak{L}\mu(n)$.

511D For each type $\boldsymbol{t}$, let $\Gamma_{\mathbf{1}t}$ be the metapredicate composed of all dyads of the form $\mathbf{1}P$, P where P is of type $(\boldsymbol{t})$.

512T *For each type* $\boldsymbol{t}$, $\Gamma_{\mathbf{1}t}$ *is expressed by the* μ *of type* $(\boldsymbol{t}(\boldsymbol{t}))$ *in* $\mathfrak{L}\mu$. *It is expressible by* μ *in every extension of* $\mathfrak{L}\mathbf{1}$ *in which the relevant axiom of choice is not refutable.*

Proof. The first part is a consequence of **310T** and the fact that each μ is single-valued. The second part follows by **311T**.

This theorem shows how a question of consistency, such as consistency of an axiom of choice in a particular system, can be stated as a question of expressibility. It also shows that an axiom of choice is correctly interpreted as a statement to the effect that any universal selective function (of the relevant type) definable in the metalanguage is expressible in the object language.

513D Let $\Gamma_{\dot{n}}$ be the metapredicate composed of the names. From **507T** it is clear that condition (2b) of the definition of adequacy given in §1 can be stated in the form "$\Gamma_{\dot{n}}$ is expressible by V in $\mathfrak{L}$."

It can be proved that $\Gamma_{\dot{n}}$ is expressible by V in $\mathfrak{L}\mathbf{1}$, but this proof is not of much use to us since it does not hold for $\mathfrak{L}\mu$. It will be seen later that expressibility of $\Gamma_{\dot{n}}$ by V in $\mathfrak{L}\mu$ entails the consistency of number theory embedded in $\mathfrak{L}\mu$.

Consider an enumeration, based on some system of Gödel numbering or lexicographic ordering, of all the predicates of a given type $\boldsymbol{t}$ in $\mathfrak{L}\mathbf{1}$. Let Γ be defined by "$\Gamma(P,\dot{n})$ iff P is the nth predicate in the enumeration." Let Γ be expressed by γ in $\mathfrak{L}\mathbf{1}(n)$. Since $\vdash \dot{0} = \dot{n}$ in $\mathfrak{L}\mathbf{1}(n)$, there is more than one predicate P such that $\vdash P\gamma\dot{0}$ in $\mathfrak{L}1(n)$. In $\mathfrak{L}1(\infty)$, however, if Γ were expressed by γ then γ would be a mapping of a set of individuals onto the universal set of type $(\boldsymbol{t})$, which is impossible. So Γ is not expressible in $\mathfrak{L}1(\infty)$. The non-denumerable sets of mathematics are represented in $\mathfrak{L}1(\infty)$ and its extensions by predicates of which no enumeration is expressible in the system.

§6. *Expressibility of "finite"*

This concept has shown itself to be surprisingly elusive, so it may help to ask the question: what do we really mean by "finite" when the word is used informally? Let us say that a set has property A if we can write a complete list of names of its members, and that it has property B if we can specify a procedure by which, given any list of names, one can name a member of the set which is distinct from each of the objects named on the list. If a set has property A we do not hesitate to call it finite, and if it has property B we do not hesitate to call it infinite. There are, of course, many sets which we call finite even though we cannot list their members, e.g. the set of bacterial organisms in a shovelful of garden soil. But it seems reasonable to suppose that it is only because we lack the necessary information that we cannot list the members of such a set.

Property A is the basis of the definition in §1 of the metapredicate Γ_{fin}. This definition is the basic instance of a typically ambiguous schema. We shall be concerned here mainly with the basic instance. The definition of Γ_{fin} cannot be translated literally into the formal language, for this would require powers of self-reference which $\mathfrak{L}1$ and $\mathfrak{L}\mu$ do not have. In order to find a predicate to express Γ_{fin} one must look for some property which is possessed by the members of Γ_{fin} and their synonyms in $\mathfrak{L}1$ and by no other constants in $\mathfrak{L}1$ and can be described in that language. In searching for a proof that a particular predicate applies to all finite sets it is natural to try induction. In our search for a definition of finiteness in $\mathfrak{L}1$ we start by sketching a general theory of mathematical induction[61]. In the definitions which follow it is to be understood that, for some type $\boldsymbol{t}$, α is a predicate of type $(\boldsymbol{t}(\boldsymbol{t}))$ and P, Q, R are of type $(\boldsymbol{t})$.

601D We shall say that P is *α-hereditary* (in symbols, $P\varepsilon\alpha$-hered) iff

$$\forall F\colon F \subseteq P \rightarrow \forall x.\, x\alpha F \rightarrow x\varepsilon P.$$

602T *If X is a family of α-hered sets, then the intersection of the members of X is α-hered.*

603D We shall say that Q is *α-connected* to R (in symbols, $Q\,\alpha$-con R) iff

(1) $Q \supseteq R$,

and

(2) $\forall F\colon R \subseteq F \wedge F \subset Q. \rightarrow \exists x.\, x\alpha F \wedge x \not\varepsilon F \wedge x\varepsilon Q$

604T $\vdash Q\,\alpha\text{-con}\,R \rightarrow \forall F\colon F\varepsilon\alpha\text{-hered} \rightarrow .\ R \subseteq F \rightarrow Q \subseteq F$ *in* $\mathfrak{L}0$, i.e. *if Q α-con R, then Q is contained in every α-hered overset of R.*

Proof. Let F be an α-hered overset of R, Q α-con R. Then $R \subseteq F \cap Q$. If we also had

(1) $F \cap Q \subset Q$

then, since Q α-con R, we would have

$$\exists x. x\alpha(F \cap Q) \wedge x \notin (F \cap Q) \wedge x \varepsilon Q,$$

hence

$$\exists x. x\alpha(F \cap Q) \wedge x \notin F,$$

and F would not be α-hered. So (1) is false and the result follows.

605T *Suppose there exists a set Q which is both α-hereditary and α-connected to R. Then by* **604T**, *Q is the intersection of all α-hereditary oversets of R, and it is the union of all sets α-connected to R. Hence Q is uniquely determined by α and R.*

606D Such a set Q, if it exists, will be called the *α-completion* of R. The relation α will be called *inductively complete* iff every set has an α-completion.

607T *In order that α be inductively complete, the following condition is sufficient and necessary: If R is any set and Q is the intersection of all α-hereditary oversets of R, then*

(1) $\forall F\colon R \subseteq F \wedge F \subset Q. \rightarrow \exists x. x\alpha F \wedge x \notin F.$

Proof. Assume the condition holds. By **602T**, Q is α-hered. Hence

(2) $F \subseteq Q \wedge x\alpha F. \rightarrow x \varepsilon Q,$

which, in conjunction with (1), implies Q α-con R. This proves sufficiency. The necessity is obvious.

608T *If α is isotonic in the sense that*

$$\forall F \forall G\colon F \subseteq G \rightarrow \forall x. x\alpha F \rightarrow x\alpha G$$

then α is inductively complete.

Proof. The hypothesis implies that a set F is α-hered iff $\forall x. x\alpha F \rightarrow x\varepsilon F$, and the result follows by **607T**.

609D Given a relation ξ of type (*tt*), let $\xi\varepsilon$ be defined by

$$\xi\varepsilon =_{def} \lambda x F(\exists y. y\varepsilon F \wedge x \xi y).$$

Thus $\xi\varepsilon$ is the relative product of ξ and the relation ε and is of type (*t*(*t*)).

610T *If α is existential in the sense that, for some relation ξ, $\alpha = \xi\varepsilon$, then α is isotonic, hence (by the preceding theorem) α is inductively complete.*

611T *If $\alpha = \xi\varepsilon$, then P is α-hered iff*

$$\forall x\ \forall y\colon x\varepsilon P \wedge y \xi x. \rightarrow y \varepsilon P,$$

i.e. P is α-hered *iff, in the usual terminology, it is closed with respect to* ξ.

612T *The definitions and theorems* **601-611** *can be modified so as to apply to a triadic relation* α *of type* (***tt***(***tt***)) *and dyadic relations* ζ, η, θ *of type* (***tt***) *as follows*:

ζ *is* α-hered *iff*

$$\forall\xi : \xi \subseteq \zeta \rightarrow \forall x \forall y.\ \alpha(x,y,\xi) \rightarrow x\zeta y,$$

and θ α-con η *iff*

(1) $\theta \supseteq \eta$
(2) $\forall\xi$: $\eta \subseteq \xi \wedge \xi \subset \theta. \rightarrow \exists x \exists y.\ \alpha(x,y,\xi) \wedge \neg x\xi y \wedge x\theta y$.

The necessary modifications of the remaining definitions and theorems can easily be found by the reader. Similarly for n-adic α where $n \geq 3$.

Every proof by induction in mathematics is an application of **604T** or one of its extensions to n-adic relations ($n \geq 2$). To obtain the principle of finite induction on the natural numbers, let σ be Peano's successor function and let $\mathcal{N}$ be the set of natural numbers. The induction postulate can be stated in the form: $\mathcal{N}$ *is* $\sigma\varepsilon$-*connected to* $\{0\}$. Then by **604T**, if $0\varepsilon P$ and P is $\sigma\varepsilon$-hereditary, $\mathcal{N} \subseteq P$.

To obtain the principle of transfinite induction, let a set S be well-ordered by a relation ρ. Let ϕ be defined by: $x\phi R$ iff (1) $R \subset S$ and (2) x is the ρ-first member of $S - R$. Then S is ϕ-connected to each of its subsets — in particular the null-set. So by **604T**, if P is ϕ-hereditary, then $S \subseteq P$.

Every entity defined by induction in mathematics is, for some α, the α-completion of some set or relation. In particular, primitive recursion is a form of α-completion.

This completes the sketch of induction theory. It will now be used in finding an expression for Γ_{fin}.

614D Let τ be the relation defined by

$$\tau =_{def} \lambda FG(\exists x \forall y\colon y\varepsilon F \wedge y \not\varepsilon G. \rightarrow y = x)$$

i.e. $F\tau G$ iff there is at most one member of F which is not in G.

614D fin is the $\tau\varepsilon$-completion of $\{\Lambda\}$, i.e. of the family whose sole member is the null-set.

By **610T**, fin exists. By **605T**, fin is the union of all families of sets $\tau\varepsilon$-connected to $\{\Lambda\}$ and is itself $\tau\varepsilon$-connected to $\{\Lambda\}$. It is also $\tau\varepsilon$-hereditary, i.e. τ-closed.

615T *If* $F\varepsilon$fin *and* $G \subseteq F$, *then* $G\varepsilon$fin.

Proof. $G \subseteq F$ implies $G\tau F$. The result follows by **611T**.

The next theorem yields an alternative definition of fin.

616T *Let X be the family composed of those sets F such that every family of subsets of F has a maximal member. Then* X = fin.

This can be proved by showing that X is $\tau\varepsilon$-connected to $\{\Lambda\}$ and is $\tau\varepsilon$-hereditary.

617T *If* $P\epsilon\Gamma_{\mathsf{fin}}$, *then* $\vdash P\varepsilon\,\mathsf{fin}$ *in* $\mathfrak{L1}$; *that is to say,* fin *satisfies condition* (1) *of* **507T**.

Proof. If P is Λ the result is immediate. If P is $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$, then it is easy to see that the family composed of $\Lambda, \{\mathfrak{a}_1\}, \{\mathfrak{a}_1,\mathfrak{a}_2\}, \ldots, \{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$ is $\tau\varepsilon$-connected to $\{\Lambda\}$.

618T *If X is any predicate such that Γ_{fin} is expressible by X in $\mathfrak{L1}(\infty)$ and $\mathfrak{M}$ is an isophasic extension of $\mathfrak{L1}(\infty)$ in which X expresses Γ_{fin}, then* $\vdash X = \mathsf{fin}$ *in* $\mathfrak{M}$.

Proof. We show first that

(1) $\vdash X\varepsilon\tau\varepsilon\text{-hered}$ *in* $\mathfrak{M}$.

Suppose $R\epsilon\Gamma_{\mathsf{fin}}$ and Q is any predicate constant of type $(\cdot)$. From the definition of τ (**613D**) it is easy to deduce

$$\vdash Q\tau R \wedge Q \neq \Lambda. \rightarrow \exists x \therefore x\varepsilon Q \wedge \forall y{:}y\varepsilon Q \wedge y \not\varepsilon R. \rightarrow y = x.$$

Let $\mathsf{b} =_{def} \mathbf{1}(x\varepsilon Q \wedge \forall y{:}y\varepsilon Q \wedge y\not\varepsilon R. \rightarrow y = x)$. Then

$$\vdash Q = R \wedge Q \neq \Lambda. \rightarrow Q = R \cup \{\mathsf{b}\}.$$

If R is Λ, $R \cup \{\mathsf{b}\} = \{\mathsf{b}\}$. If R is $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$ then $R \cup \{\mathsf{b}\} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathsf{b}\}$. So in each case one can write an expression S such that $S\epsilon\Gamma_{\mathsf{fin}}$ and $\vdash Q\tau R \wedge Q \neq \Lambda. \rightarrow Q = S$. Since X expresses Γ_{fin} and $\Lambda\epsilon\Gamma_{\mathsf{fin}}$, it follows that

(2) $\vdash Q\tau R \rightarrow Q\varepsilon X$ in $\mathfrak{M}$.

Now let P,Q be two predicate constants of type $(\cdot)$. Since X expresses Γ_{fin}, there exists $R\epsilon\Gamma_{\mathsf{fin}}$ such that $\vdash P\varepsilon X \rightarrow P = R$ in $\mathfrak{M}$, hence $\vdash P\varepsilon X \rightarrow. Q\tau P \rightarrow Q\tau R$ and by (2) $\vdash P\varepsilon X \rightarrow. Q\tau P \rightarrow Q\varepsilon X$, from which (1) follows by quantificational closure and **611T**.

It is not difficult to show that

(3) $\vdash X\ \tau\varepsilon\text{-con}\ \{\Lambda\}$ *in* $\mathfrak{M}$.

The result follows from (1) and (3).

From this theorem it is easy to infer that

619T *If Γ_{fin} is expressible by X in $\mathfrak{L1}(\infty)$ or any isophasic extension of $\mathfrak{L1}(\infty)$, then it is expressible in this system by* fin, *and, (by* **509T**), *in the extension of this system generated by postulating* $X = \mathsf{fin}$.

So if the concept "finite" is expressible at all in a theory of types, we know how to express it.

620T *Let $\mathfrak{M}$ be an isophasic extension of $\mathfrak{L1}(\infty)$. If Γ_{fin} is expressible in $\mathfrak{M}$, then* $\not\vdash \mathsf{V}\not\varepsilon\,\mathsf{fin}$ *in* $\mathfrak{M}$.

Proof. Clearly, if $R\epsilon\Gamma_{\mathsf{fin}}$, then $\vdash \mathsf{V} \neq R$ in $\mathfrak{L1}(\infty)$, from which the result is easy to infer.

621D Let $\mathfrak{L}1$(inf) denote the extension of $\mathfrak{L}1$ generated by the postulate $V \notin$ fin. It is easy to see that this postulate is basic (**314D**), hence

622T $\mathfrak{L}1$(inf) *belongs to the class* $\{\mathfrak{L}1\}$.

Clearly $V \notin$fin, if consistent in $\mathfrak{L}1(\infty)$, is an infinity formula in the sense of **409D**, and, by **410T**,

623T $\mathfrak{L}1$(inf) *is an extension of* $\mathfrak{L}1(\infty)$.

In consequence of **620T** and **509T**,

624T *if* Γ_{fin} *is expressible in* $\mathfrak{L}1(\infty)$, *it is expressible in* $\mathfrak{L}1$(inf).

625T *Every statement made in* §6 *remains true if, for* $\mathfrak{L}1$, $\mathfrak{L}1(\infty)$, $\mathfrak{L}1$(inf) *we substitute* $\mathfrak{L}\mu$, $\mathfrak{L}\mu(\infty)$, $\mathfrak{L}\mu$(inf).

§7. *Number theory in* $\mathfrak{L}1$(inf)

In this section it will be shown that Γ_{fin} is expressible in $\mathfrak{L}1(\infty)$ iff number theory is ω-consistent in $\mathfrak{L}1$(inf). For this purpose it will be helpful to define the natural numbers as families of finite sets, using a definition equivalent to that of *Principia Mathematica*. The results of §6 make it possible to do this with minimal labor.

701D We define the relation θ as an abbreviation for

$$\lambda YX(\forall G\colon G\varepsilon Y \longleftrightarrow \exists F.\, F\varepsilon X \wedge F \subset G \wedge G\tau F),$$

where τ is defined by **613D**; that is to say, $Y\theta X$ iff Y is the family composed of all sets G such that, for some $F\varepsilon X$, G is composed of all the members of F and exactly one individual which does not belong to F.

702D Using 0 as an abbreviation for $\{\Lambda\}$, we define $\mathcal{N}$, the class of natural numbers, as the $\theta\varepsilon$-completion (**606D**, **609D**) of the class $\{0\}$.

Existence of $\mathcal{N}$ is confirmed by **610T**. By **605T** $\mathcal{N}$ is $\theta\,\varepsilon$-hereditary and $\theta\,\varepsilon$-connected to $\{0\}$.

703T *The relation* θ *has the following properties*:

(1) $\theta\,\varepsilon\, 1\text{-}s$ (**316D**)
(2) $\forall X \exists Y\ Y\theta X$
(3) $\forall X \forall Y.\ Y\theta X \rightarrow \Lambda \notin Y$
(4) $\forall X \forall Y\!:\!.\, Y\theta X \rightarrow : \exists G\ G\varepsilon Y \rightarrow \exists F.F\varepsilon X \wedge F \neq V.$

These are easily inferred from **701D**.

704T *If* $X\varepsilon\mathcal{N}$, *then*

(1) $X \subseteq$ fin
(2) $F\varepsilon X$ *iff* X *is the family of sets equipollent to* F
(3) $\exists F\ F\varepsilon X$ (*assuming* $V \notin$ fin)

Clearly 0 has these properties. Since $\mathcal{N}$ is $\theta\varepsilon$-connected to $\{0\}$ it is sufficient by **604T** to show that each of the three properties is $\theta\varepsilon$-hereditary. For (1) and (2) this is easily inferred from **701D**; for (3) it follows from (1), the infinity postulate $V \notin$fin and **703T**(4).

705T *Let κ be the restriction of θ to $\mathcal{N}$*, i.e. $\lambda YX(Y\theta X \wedge Y\varepsilon\mathcal{N} \wedge X\varepsilon\mathcal{N})$. *Then*

(a) $\kappa\varepsilon 1\text{-}1$
(b) $\forall X\colon X\varepsilon\mathcal{N} \rightarrow \exists Y.\ Y\kappa X \wedge Y \neq 0$
(c) $\mathcal{N}\kappa\varepsilon\text{-con}\,\{0\}$.

(a) follows from **703T**(1) and **704T** (2), (3); (b) from **703T** (2), (3); (c) from the definitions of $\mathcal{N}$ and κ.

These three properties of κ are equivalent to the properties of the successor function described by the Peano postulates. (Observe that (c) not only implies Peano's induction postulate in consequence of **604T**, but also implies that every number n distinct from zero is the successor of a number distinct from n.)

706D We have already defined zero. Let the positive integers be defined by

$$1 =_{def} \kappa\text{`}0,\ 2 =_{def} \kappa\text{`}1,\ \ldots$$

We shall call these the "proper numerals" to distinguish them from the name-numerals of §4. All the recursive functions and predicates of natural numbers are definable as $\mathbf{1}$-constants, using **606D**, **612T**. All theorems of number theory are theorems of $\mathfrak{L}\mathbf{1}(\text{inf})$.

707T $$\vdash \forall F\colon F\varepsilon\text{fin} \longleftrightarrow \exists X.\ X\varepsilon\mathcal{N} \wedge F\varepsilon X \textit{ in } \mathfrak{L}\mathbf{1}(\text{inf}).$$

This can be proved by showing that the predicate

$$\lambda F(\exists X.\ X\varepsilon\mathcal{N} \wedge F\varepsilon X)$$

is $\tau\varepsilon$-hereditary and $\tau\varepsilon$-connected to $\{\Lambda\}$.

708T *Given any constant P of type* $(\cdot)$ *and any proper numeral* $n > 0$,

$$\vdash P\varepsilon n \longleftrightarrow P = \{\mathbf{1}P,\ \mathbf{1}_1 P,\ \ldots,\ \mathbf{1}_{n-1}P\} \textit{ in } \mathfrak{L}\mathbf{1}(\text{inf})$$

where

$$\mathbf{1}_1 P =_{def} \mathbf{1}(x\varepsilon P \wedge x \neq \mathbf{1}P)$$
$$\cdots$$
$$\mathbf{1}_i P =_{def} \mathbf{1}(x\varepsilon P \wedge x \neq \mathbf{1}P \wedge \ldots \wedge x \neq \mathbf{1}_{i-1}P)$$
$$\cdots$$

Proof. The schema

$$\forall F{::}\ F\varepsilon n \longleftrightarrow \exists x_1 \ldots \exists x_n \forall y{:}.\ y\varepsilon F \longleftrightarrow{:}\ y = x_1 \vee \ldots \vee y = x_n. \wedge x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n$$

can be proved, by recursion, for each positive integer n.

709T *Let Γ_n be the metapredicate composed of the proper numerals. The following statements are equivalent*:

(1) Γ_n *is expressible by* $\mathcal{N}$ *in* $\mathfrak{L}\mathbf{1}(\text{inf})$
(2) Γ_{fin} *is expressible in* $\mathfrak{L}\mathbf{1}(\infty)$.

Proof. Assume (1) and let $\mathfrak{M}$ be an extension of $\mathfrak{L}\mathbf{1}(\text{inf})$ in which $\mathcal{N}$

expresses Γ_n. Let P be a predicate of type $(\cdot)$. By **502D** there exists a proper numeral n such that

$$\vdash \mathsf{1}(P\varepsilon X_\wedge X\varepsilon\mathcal{N})\,\varepsilon\mathcal{N} \to \mathsf{1}(P\varepsilon X_\wedge X\varepsilon\mathcal{N}) = n \text{ in } \mathfrak{M}.$$

hence by **707T** $\vdash P\varepsilon\mathsf{fin} \to P\varepsilon n$, and by **708T** there exists $R\epsilon\Gamma_{\mathsf{fin}}$ such that

$$\vdash P\varepsilon\mathsf{fin} \to P = R \text{ in } \mathfrak{M}$$

which proves, in conjunction with **617T**, that fin expresses Γ_{fin} in $\mathfrak{M}$. By **623T, 505T** this implies (2).

Assume (2). Then by **624T** Γ_{fin} is expressible by fin in $\mathfrak{L}\mathsf{1}(\mathsf{inf})$. Let $\mathfrak{N}$ be an extension of $\mathfrak{L}\mathsf{1}(\mathsf{inf})$ in which fin expresses Γ_{fin}. Let W be any predicate constant of type $((\cdot))$. By **704T**(3), **707T**,

$$\vdash W\varepsilon\mathcal{N} \to \mathsf{1}W\varepsilon\mathsf{fin} \text{ in } \mathfrak{L}\mathsf{1}(\mathsf{inf}).$$

Since fin expresses Γ_{fin} in $\mathfrak{N}$ it follows that there exists $R\epsilon\Gamma_{\mathsf{fin}}$ such that

$$\vdash W\varepsilon\mathcal{N} \to \mathsf{1}W = R \text{ in } \mathfrak{N}.$$

Now it is not difficult to prove that for each $R\epsilon\Gamma_{\mathsf{fin}}$ there exists a numeral n such that

$$\vdash R\varepsilon 0 \vee \ldots \vee R\varepsilon n,$$

so

$$\vdash W\varepsilon\mathcal{N} \to. \ \mathsf{1}W\varepsilon 0 \vee \ldots \vee \mathsf{1}W\varepsilon n \text{ in } \mathfrak{N},$$

hence by **704T**(2)

$$\vdash W\varepsilon\mathcal{N} \to. \ W = 0 \vee \ldots \vee W = n,$$

from which it is easy to infer that Γ_n is expressible by $\mathcal{N}$ in $\mathfrak{N}$, hence by **505T** in $\mathfrak{L}\mathsf{1}(\mathsf{inf})$.

710T Γ_n *is expressible by* $\mathcal{N}$ *in* $\mathfrak{L}\mathsf{1}(\mathsf{inf})$ *iff number theory is* ω*-consistent in* $\mathfrak{L}\mathsf{1}(\mathsf{inf})$.

Proof. Obviously $\vdash n\varepsilon\mathcal{N}$ in $\mathfrak{L}\mathsf{1}(\mathsf{inf})$ for each proper numeral n, hence Γ_n is expressible by $\mathcal{N}$ iff condition (2) of **507T** is satisfied, and this is clearly equivalent to ω-consistency.

711T *Every statement made in this section remains true if, for* $\mathfrak{L}\mathsf{1}$, $\mathfrak{L}\mathsf{1}(\infty)$, $\mathfrak{L}\mathsf{1}(\mathsf{inf})$ *we substitute* $\mathfrak{L}\mu$, $\mathfrak{L}\mu(\infty)$, $\mathfrak{L}\mu(\mathsf{inf})$.

§8. *Arithmetic on the name-numerals*

Since we can construct arithmetic on the proper numerals one might ask, why bother with arithmetic on the names? The answer is twofold. From a formal point of view, the conditions of adequacy set out in §1, in particular condition (2b), can be satisfied in an extension of $\mathfrak{L}\mu(\infty)$ only if we have at least the possibility of constructing an ω-consistent number theory on the names. This is proved by **806T** below. From the standpoint of motivation and semantics, the empirical objects which are discussed when mathematics is used in science are usually described in numerical terms, either by whole numbers or rational numbers, or by ordered finite sequences of such.

Up to now we have given parallel treatment to $\mathfrak{L}\mathbf{1}$ and $\mathfrak{L}\mu$. But here we reach a stage at which the two systems diverge. To save space we confine our attention here to the more important of the two, namely $\mathfrak{L}\mu$. We shall use the abbreviation $\alpha\rangle$‘**b** for $\lambda x(x\alpha\mathbf{b})$ where α is a dyadic predicate and **b** an individual or predicate constant of the relevant type. $\alpha\rangle$‘**b** may be verbalized by "the α-segment of **b**". We shall also use expressions of the form $\alpha\rangle$‘x where α or x or both are variables. For justification of this see the end of §3.

Let $\mathfrak{L}\nu$ be the extension of $\mathfrak{L}\mu(\mathrm{inf})$ generated by the postulate

801P $$\forall\dot{x}\ \rho\rangle‘x\ \varepsilon\ \mathrm{fin}$$

where ρ is the well-ordering of the individuals defined in §3. This postulate, together with the axioms of $\mathfrak{L}\mu(\mathrm{inf})$ implies that the well-ordering of the individuals is of type ω. Thus all of number theory is provable on the name-numerals in $\mathfrak{L}\nu$.

The members of $\mathcal{N}$ (**702D**), subsets of $\mathcal{N}$, etc. form a model for $\mathfrak{L}\nu$, which can be used to show that if the proper numerals are expressible by $\mathcal{N}$ in $\mathfrak{L}\mu(\mathrm{inf})$, then the name-numerals are expressible by V. This motivates the next two theorems.

802T *Assume that the axioms of* $\mathfrak{L}0$ *are the axioms of* $\mathfrak{F}\omega$ *and the axioms of choice. Let* Q *be a monadic predicate constant such that* $\vdash\exists x\ x\varepsilon Q$ *in* $\mathfrak{L}\mu$. *Then one can construct in the metatheory a function* Π_Q, *to be called projection on* Q, *which maps into itself the set of expressions composed of the wffs, variables and* **1**-*constants of* $\mathfrak{L}\mu$ *such that*

(1) *if* **a** *is an individual constant, then* Π_Q‘**a** *is an* **1**-*constant such that* $\vdash\Pi_Q$‘**a** $\varepsilon\ Q$ in $\mathfrak{L}\mu$,

(2) *if* B *is a theorem of* $\mathfrak{L}\mu$ *so is* Π_Q‘B.

Proof. We shall write Π for Π_Q. If x is a variable or **1**-constant, x' stands for Π‘x.

Let $(\boldsymbol{t})$ be the type of Q. If x is a variable of arbitrary type, let x' be the variable which results when each dot in the type superscript of x is replaced by $\boldsymbol{t}$.

We shall use the expression p‘ for "power set of," e.g. p‘Q denotes the power set of Q.

If x is a variable or **1**-constant, the expression Qx' is defined as follows:

if x is of type $\cdot$, Qx' stands for $x'\varepsilon Q$
if x is of type $(\cdot)$, Qx' stands for $x'\varepsilon$ p‘Q
if x is of type $(\cdot\cdot)$, Qx' stands for $x'\varepsilon$ p‘$(Q\times Q)$

and similarly for higher types.

$$Qx_1'\ldots x_n' \text{ stands for } Qx_1'\wedge\ldots\wedge Qx_n'.$$

If A is a wff of $\mathfrak{L}\mu$, the wff $\Pi`A$ is constructed in three steps, namely:

(a) Replace each variable x in A by x', wherever its occurrence is not contained in an $\mathbf{1}$-constant. Replace each $\mathbf{1}$-constant $\mathfrak{a}$ in A by the constant $\mathfrak{a}'$ which will be defined later. Let A' denote the resulting expression.

(b) For each expression of the form $\forall x'$ ($\exists x'$) occurring in A' and not contained in an $\mathbf{1}$-constant, substitute $\forall x'.\ Qx' \rightarrow (\exists x'.\ Qx' \wedge)$, the punctuation being modified as required to suit the context, e.g. $\forall x' \exists y' A'(x', y')$ becomes $\forall x'$: $Qx' \rightarrow \exists y'.\ Qy' \wedge A(x', y')$. Let A'' stand for the result of this operation.

(c) If A is a closed wff of $\mathfrak{L}\mu$, then $\Pi`A$ is $A''$. Otherwise let $x_1, \ldots, x_n$ be a complete list of the variables occurring in $A$. Then $\Pi`A$ is $Qx_1' \ldots x_n' \rightarrow A''$.

If T is a theorem of $\mathfrak{L}0$, then $\Pi`T$ is a theorem of $\mathfrak{L}\mu$. To prove this it can be verified (bearing in mind the hypothesis $\vdash \exists x\ x \varepsilon Q$) that (i) if $A$ is an axiom of $\mathfrak{L}0$, i.e. an axiom of $\mathfrak{F}\omega$ or an axiom of choice, then $\vdash \Pi`A$ in $\mathfrak{L}\mu$, and (ii) if B follows from A (from A and $A \rightarrow B$) by a rule of inference, then $\Pi`B$ is deducible from $\Pi`A$ (from $\Pi`A$ and $\Pi`(A \rightarrow B)$) in $\mathfrak{L}\mu$.

It remains to be shown how, for each $\mathbf{1}$-constant $\mathfrak{a}$ one can define $\mathfrak{a}'$ so as to satisfy (1) and (2). Let B be a theorem of $\mathfrak{L}\mu$. We may assume without loss of generality that B is a closed wff. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be the $\mathbf{1}$-constants occurring in B, and let $C(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ be the conjunction of the postulates which generate the system $\mathfrak{L}\mu(B)$ defined in the proof of **311T** from $\mathfrak{L}0$. Then by **313T**,

$$\vdash \forall x_1 \ldots \forall x_n.\ C(x_1, \ldots, x_n) \rightarrow B(x_1, \ldots, x_n) \text{ in } \mathfrak{L}0,$$

and by what we have just proved, the projection of this wff on Q, i.e.

$$\forall x_1' \ldots \forall x_n' \colon Qx_1' \ldots x_n' \rightarrow .\ C''(x_1', \ldots, x_n') \rightarrow B''(x_1', \ldots, x_n'),$$

must be a theorem of $\mathfrak{L}\mu$. This is equivalent to

$$\forall x_1' \ldots \forall x_n' \colon Qx_1' \ldots x_n' \wedge C''(x_1', \ldots, x_n').\ \rightarrow Qx_1' \ldots x_n' \wedge B''(x_1', \ldots, x_n').$$

From this it is easy to infer, since $\Pi`B$ is B'' (B being a closed wff), that conditions (1) and (2) will be satisfied if $\mathfrak{a}_1', \ldots, \mathfrak{a}_n'$ are so defined that

(3) $\vdash Q\mathfrak{a}_1' \ldots \mathfrak{a}_n' \wedge C''(\mathfrak{a}_1', \ldots, \mathfrak{a}_n')$ in $\mathfrak{L}\mu$.

In constructing $\mathfrak{L}\mu$ from $\mathfrak{L}0$, as in the proof of **311T**, each $\mathbf{1}$-constant is introduced by a postulate of the form

$$\mathbf{1}A \mathring{\varepsilon} A \wedge \forall y.\ y \rho \mathbf{1}A \rightarrow \neg y \mathring{\varepsilon} A,$$

which says that $\mathbf{1}A$ is the ρ-first x such that $x \mathring{\varepsilon} A$, where ρ is a well-ordering of V. Now suppose that ρ' is an arbitrary well-ordering of Q and $\mathbf{1}A$ is an individual constant. Let $\Pi`\mathbf{1}A$ be so defined that, if the intersection $Q \cap A''$ is non-empty, then $\Pi`\mathbf{1}A$ is the ρ'-first member of this intersection, and otherwise it is the ρ-first member of Q. In formal language let

$$\Pi`\mathbf{1}A = \mathbf{1}(Qx' \wedge x' \mathring{\varepsilon}(Q \cap A'') \wedge \forall y'.\ y' \rho' \mathbf{1}A \rightarrow \neg y' \mathring{\varepsilon}(Q \cap A''))$$

where we may take ρ' to be $\mathbf{1}(\xi \text{ wo } Q)$, or any other well-ordering of Q. This will satisfy (3) so far as concerns individual constants.

Now consider the **1**-constants of type $(\cdot\cdot)$, i.e. dyadic relations of individuals. These are introduced by postulates which say that $\mathbf{1}A$ is the ρ_1-first relation such that $\mathbf{1}A\xi A$ where ρ_1 is a well-ordering relation of type $((\cdot\cdot)\,(\cdot\cdot))$. The postulates of $\mathfrak{L}\mu$ require that the ρ of type $(\cdot\cdot)$ mentioned in the preceding paragraph be the ρ_1-first of the well-orderings of the individuals. Except for this condition, ρ_1 can be an arbitrary well-ordering of the universal set of type $((\cdot\cdot))$. So, having appropriately selected ρ' we can satisfy condition (3) by setting $\Pi`\rho_1 = \rho_1'$ equal to any well-ordering of $\mathsf{p}`(Q \times Q)$ such that ρ' is the ρ_1'-first member of the set of well-orderings of Q. The reader should have no difficulty finding the proper formal expression for a relation satisfying this condition.

Now to generalize. Let ρ be any of the well-ordering relations used to introduce **1**-constants in constructing $\mathfrak{L}\mu$. If there is no well-ordering of lower type in the field of ρ (as in the case where ρ is a well-ordering of individuals), then $\Pi`\rho$ can be an arbitrary well-ordering of the relevant set in the model (e.g. $Q$, $\mathsf{p}`Q$, $\mathsf{p}`(Q \times Q \times Q)$, etc.). If ρ does have such a relation of lower type in its field, then we assume that such a relation of lower type has already been selected. Let ρ_0' be the relation thus selected. Then ρ' must satisfy the condition that ρ_0' is the ρ'-first well-ordering of the relevant set in the model.

Thus Π can be constructed in various ways to satisfy (3). This completes the proof.

803T *Let* $\mathfrak{L}0$ *be as in the preceding theorem. Then there exists a projection* Π_N *on* $\mathcal{N}$(**702D**) *such that*

(1) *if* $\mathfrak{a}$ *is an individual constant,* $\vdash \Pi`\mathfrak{a}\,\varepsilon\,\mathcal{N}$ *in* $\mathfrak{L}\mu(\text{inf})$,
(2) *if* $\dot{n}$ *is a name and* n *the corresponding proper numeral, then* $\Pi_N`\dot{n}$ *is* n,
(3) *if* $\vdash B$ *in* $\mathfrak{L}\nu$, *then* $\vdash \Pi_N`B$ *in* $\mathfrak{L}\mu(\text{inf})$.

Proof. Let Π_N be constructed as a special case of the projection described in the preceding theorem, with Q specialized to $\mathcal{N}$ and the well-ordering of the individuals mapped on the natural ordering of the proper numerals. (1) and (2) are immediate. To prove (3) it is sufficient to supplement the proof of **802T** by showing that (i) the axiom of infinity $V\,\$\,\text{fin}$ maps on a theorem of $\mathfrak{L}\mu(\text{inf})$, which it does since $\vdash \mathcal{N}\$\text{fin}$, and (ii) **801P** maps on a theorem, which it does since it can be proved in $\mathfrak{L}\mu(\text{inf})$ that the natural ordering of the proper numerals has the property ascribed to ρ by **801P**.

804T *If* $\mathfrak{L}\mu(\text{inf})$ *is consistent, then* **801P** *is neither refutable nor provable in* $\mathfrak{L}\mu(\text{inf})$.

Proof. In consequence of **803T**, if the negation of **801P** were a theorem there would be a contradiction in $\mathfrak{L}\mu(\text{inf})$.

By constructing a projection on $\mathsf{p}`\mathcal{N}$ it can be seen that if **801P** were a theorem this would involve a contradiction in $\mathfrak{L}\mu(\mathrm{inf})$, since no ordering of $\mathrm{p}`\mathcal{N}$ of type ω can be expressible in $\mathfrak{L}\mu(\mathrm{inf})$.

805T *The following statements are equivalent*:

(1) Γ_{fin} *is expressible in* $\mathfrak{L}\mu(\infty)$.
(2) $\underline{\Gamma}_{\dot{n}}$ (513D) *is expressible by* $\vee$ *in* $\mathfrak{L}\nu$,
(3) Γ_{fin} *is expressible in* $\mathfrak{L}\nu$.

Proof. We shall use the projection defined in **803T**. Suppose $\Gamma_{\dot{n}}$ is not expressible by $\vee$ in $\mathfrak{L}\nu$. Then there are individual constants $\mathbf{a}_1, \ldots, \mathbf{a}_p$ such that, for all possible choices of names $\dot{n}_1, \ldots, \dot{n}_p$,

$$\vdash \mathbf{a}_1 \neq \dot{n}_1 \vee \ldots \vee \mathbf{a}_p \neq \dot{n}_p \text{ in } \mathfrak{L}\nu.$$

So by **803T**, for all possible choices of proper numerals $n_1, \ldots, n_p$,

$$\vdash \mathbf{a}_1' \neq n_1 \vee \ldots \vee \mathbf{a}_p' \neq n_p \text{ in } \mathfrak{L}\mu(\mathrm{inf}).$$

Since we also have, by **803T**,

$$\vdash \mathbf{a}_1', \ldots, \mathbf{a}_p' \,\varepsilon \mathcal{N} \text{ in } \mathfrak{L}\mu(\mathrm{inf}),$$

it follows that the metapredicate composed of the proper numerals is not expressible by $\mathcal{N}$ in $\mathfrak{L}\mu(\mathrm{inf})$, hence by **711T**, **709T** Γ_{fin} is not expressible in $\mathfrak{L}\mu(\infty)$.

This proves that (1) implies (2). Now assume (2) and let

$$\mathrm{R} =_{def} \lambda F(\exists x\ F \subseteq \rho\,\rangle`x)$$

i.e. R is the family of subsets of ρ-segments. It is easy to prove that R is $\tau\varepsilon$-hereditary and that $\{\Lambda\} \subseteq \mathrm{R}$, so

(a) $\vdash \mathrm{fin} \subseteq \mathrm{R}$ in $\mathfrak{L}\nu$.

We also have, for every name numeral $\neq \dot{0}$,

(b) $\vdash \rho\,\rangle`\dot{n} = \{\dot{0}, \ldots, (n \dot{-} 1)\}$ in $\mathfrak{L}\nu$

and

(c) $\rho\,\rangle`\dot{0} = \Lambda$ in $\mathfrak{L}\nu$

From (a)—(c) it follows that (2) implies (3). That (3) implies (1) is immediate by **505T**.

The next theorem shows that $\vee$ can express $\Gamma_{\dot{n}}$ in $\mathfrak{L}\mu(\infty)$ only if the well-ordering of the individuals is of order-type ω, that is to say:

806T *If* $\mathfrak{M}$ *is an extension of* $\mathfrak{L}\mu(\infty)$ *in which* $\vee$ *expresses* $\Gamma_{\dot{n}}$, *then* $\mathfrak{M}$ *is also an extension of* $\mathfrak{L}\nu$.

Proof. It is easy to see that the statements (a)—(c) in the proof of the preceding theorem are provable in $\mathfrak{L}\mu(\infty)$.

Since $\vee$ expresses $\Gamma_{\dot{n}}$ in $\mathfrak{M}$ there is, for every individual constant $\mathbf{a}$, a name $\dot{n}$ such that $\vdash \mathbf{a} = \dot{n}$ in $\mathfrak{M}$, hence,

$$\vdash \rho \rangle \text{`}\mathfrak{a} = \rho \rangle \text{`}\dot{n} \text{ in } \mathfrak{M},$$

from which it follows by (b), (c) and **618T** that

$$\vdash \rho \rangle \text{`}\mathfrak{a} \varepsilon \mathrm{fin},$$

and by quantificational closure

(1) $\vdash \forall x\ \rho \rangle \text{`}x \varepsilon \mathrm{fin}$ in $\mathfrak{M}$.

From the least-number postulates (**402P**), (b), (c) and quantificational closure it can be inferred that our hypothesis implies

$$\vdash \forall x \exists y\ y \not\subset\ \rho \rangle \text{`}x \text{ in } \mathfrak{M},$$

so $V \not\subset R$ where R is defined as in the preceding theorem, so by (a),

(2) $\vdash V \not\subset \mathrm{fin}$ in $\mathfrak{M}$.

The result follows from (1) and (2). From **805T** and **806T** it is not difficult to deduce the equivalence of the statements (a)—(c) on p. 258, §1.

§9. *Concluding remarks*

We have shown that the system $\mathfrak{Q}\nu$ satisfies condition (2) of adequacy if and only if Γ_{fin} is expressible in $\mathfrak{Q}\mu(\infty)$. That it satisfies condition (1) is clear from the way it is constructed. Condition (3) remains to be verified. We have shown how arithmetic can be developed in this system and it is easy to see that basic concepts and constructions of mathematics, such as groups and other algebraic structures, topological spaces, Lebesgue integrals, etc. can readily be identified with **1**-constants in $\mathfrak{Q}\nu$. More generally, let T be a mathematical theory which is finitely axiomatizable. Let $A(b_1, \ldots, b_n)$ be the conjunction of the axioms of T, where $b_1, \ldots, b_n$ are the undefined terms. Then we have in $\mathfrak{Q}\nu$ such **1**-constants as

$$\lambda x_1(\exists x_2 \ldots \exists x_n\, A(x_1, \ldots, x_n))$$

and $\lambda x_1 \ldots x_n(A(x_1, \ldots, x_n))$. Evidently T can be embedded in $\mathfrak{Q}\nu$. Its axioms are absorbed in the postulates which introduce the relevant **1**-constants.

It is conceivable, however, that there might be some special mathematical theories which cannot be finitely axiomatized even in a system of type theory, and which could be incorporated in an extension of $\mathfrak{Q}\nu$ only if we introduced new constants by infinite sets of postulates. In view of this it seems better not to be dogmatic in asserting that $\mathfrak{Q}\nu$ satisfies condition (3). The system does, however, seem adequate for all the better known and more important parts of mathematics.

Finally, we now review the question whether any ontological presuppositions have been used in constructing $\mathfrak{Q}\nu$ as an extension of $\mathfrak{F}\omega$. This extension is generated by

(1) the **1**-postulates (**303P**)
(2) the axioms of choice
(3) the postulates **309P**

(4) The infinity postulate $\exists F\ F \notin \mathsf{fin}$
(5) **801P**

The **1**-postulates, as we have seen, are nothing more than implicit definitions. They give names to things of which the existence is already proved within the system (*cf*. **304T, 210T**). On (2) we commented following **311T** and following **512T**. The comments made here on (1) apply also to (3) in consequence of **311T**. Postponing discussion of (4) for the moment and passing to (5), this postulate merely transposes to the individuals a system of relations, i.e. number theory, already constructed in $\mathfrak{L}\mu(\mathsf{inf})$ on predicates of type $((\cdot))$.

Now as to (4). We seem at first sight to have committed ourselves to the assumption that there exists, independently of the language $\mathfrak{L}\nu$, an infinite totality of some kind. And this is really rather paradoxical. The trend in physics today seems to favor a finite universe. In any case, the totality of human knowledge is necessarily finite. The end results of most if not all mathematical inferences, if they are statements about reality, are statements about finite configurations of things or events.

Why, then, do we use an infinite system? Presumably because the methods of classical mathematics require a field which is closed with respect to limit operations, even though our measuring instruments are not sensitive to indefinitely small space-time intervals. The use of infinite predicates, then, like that of certain complex-valued functions in physics (cf. §1), is to facilitate calculation. The calculations are reliable if Γ_{fin} is expressible in $\mathfrak{L}\mu(\infty)$, and if this is the case, then all our results could have been obtained, at least in principle, in one of the systems $\mathfrak{L}\mu(n)$ without assuming anything about expressibility of Γ_{fin}.

So if we really are assuming existence of an infinite totality, we must be guilty of a preposterous act of make-believe, and that merely for the sake of computational convenience. But there is really no need to interpret the infinity postulate this way. A predicate P is finite if there is a self-terminating algorithm which lists the objects x such that $x \varepsilon P$ and terminates itself when the list is complete. P is infinite if either (a) there is a listing procedure which terminates only when the operator gets tired or bored or runs out of paper, or (b) there is no listing procedure. Thus the infinite is essentially open-ended. "The completed infinite", like "the round square", is a contradiction in terms. The postulate $\mathsf{V} \notin \mathsf{fin}$ is a statement, not about the universe, but about the logistic system itself. It means, in effect "we are using an open-ended system."

NOTES

11. Henkin's definition of strong Γ-consistency in [4] must be adapted to the present context by specifying that the variable in $A_i(x)$ is an individual variable.

12. In consequence of our definition of the range of a predicate variable, and also of condition (1) of adequacy, we are compelled to give due recognition to the fact,

pointed out many years ago by Skolem, but frequently ignored in the literature, that a set may be enumerable in metalanguage without being enumerable in the object language. Of course all our predicates are enumerable in metalanguage. See end of § 5, below.

21. It is not easy to decide whether to use "predicate" as a primitive idea and define "function" in terms of predicate, as is done in this paper, or to follow the reverse procedure, as is done, for instance, in [1]. The choice made here, prompted in the first place by the conception of type theory as an extension of the first order predicate calculus, is also partly motivated by the thought that "predicate" is a more elementary and "function" a more sophisticated concept, as suggested by the history of the two.

22. *cf.* [5], pp. 129, 131. Our notation differs from the abbreviated form of [5], p. 131 in that we do not omit the outer pair of parentheses. In his fourth edition Ackermann abandoned his type symbols.

23. Perspicuity is the aim of these notations. The forms $x \varepsilon F$ and $y \alpha x$ imitate simple syntactical patterns common in natural language: noun-copula-adjective, as in "three is prime," and noun-verb-noun as in "four exceeds three."

24. For definition of confusions and collisions of variables see [7], p. 136.

25. This axiom schema is analogous to $*509_n$ on p. 297 of [2].

26. See note 24.

27. Reference numbers followed by **T** refer to theorems, by **D** to definitions, by **P** to postulates.

31. [6], vol. 2, pp. 9 ff.

32. [1], pp. 57 ff.

33. Vol. 2, p. 15.

34. *Cf.* the definition of well-ordering on p. 21 of [3].

51. [8], pp. 98, 366.

52. Subject to the modification specified in note 11.

61. Our method of approach to this problem has been inspired by that of Whitehead and Russell in [10], Part III, Section C. Our predicate "fin" (**614D** below) is equivalent to the analogue in $\mathfrak{N}$ of the "Cls induct" of [10], but the form of its definition is entirely different.

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