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GENERALIZABILITY OF THE PROPOSITIONAL AND PREDICATE CALCULI TO INFINITE-VALUED CALCULI

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INTRODUCTION: A logical calculus which is to be applied in the development of axiomatic systems must provide an adequate vocabulary of defined logical terms which, when incorporated in suitable theorems, will permit the explication of the needed logical concepts. Some authors, e.g. Woodger [7] and Carnap [1], have utilized for this purpose modifications of Whitehead and Russell's [6] extensive list of defined terms. An infinite-valued logic intended for similar applications must likewise have an adequate vocabulary of complex logical terms. Such a vocabulary may be readily devised by generalizing the definitions of **PM** [6] (or some other sufficiently complex two-valued logical system) provided that the theorems in which the defined terms occur can all be generalized to theorems in the new logic. The question thus arises as to whether there is any infinite-valued logic in which every theorem of **PM** has a valid generalization. More precisely: Is there an infinite-valued logic into which **PM** generalizes in the sense defined below in 1.18? The principle theses of the present paper answer this question in the affirmative for the propositional and first-order pure functional calculi in terms of an infinite-valued logic in which two binary connectives represent each binary connective of **PM** (equivalence excluded). The question of the generalizability of that part of **PM** which utilizes higher functional calculi is left open.

1. GENERALIZING AND DEGNERATING CALCULI.

1.1 Explication. A *language* is taken to be a set of *wff* defined by a given set of *symbols* and *rules of formation*.

1.2 Explication. The symbols of a language constitute a mutually exclusive set of sets of *symbol tokens* in that language. A *token of* a symbol α is an element of the set α.

1.3 Definition. A *token* in $\mathcal{A}$ is a symbol token in language $\mathcal{A}$ or an unbroken, finite, ordered sequence the terms of which are symbol tokens in $\mathcal{A}$.

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1.4 Definition, A token in $\mathscr{A}$ is *similar* to another token in $\mathscr{A}$ iff, for each symbol token of $\mathscr{A}$ in the one there is a token of the same symbol of $\mathscr{A}$ in the other.

1.5 Definition. A *formula* in $\mathscr{A}$ is a set each element of which is a token in $\mathscr{A}$ which is similar in $\mathscr{A}$ to every other element of the set and not similar in $\mathscr{A}$ to any token in $\mathscr{A}$ which is not an element of the set. It follows that the symbols of $\mathscr{A}$ constitute a subset of the formulae in $\mathscr{A}$. A *token of* a formula is an element of the set which constitutes the formula.

Note: When no ambiguity is likely to result repeated references to the language, $\mathscr{A}$ in the preceding definitions, are generally omitted.

1.6 Definition. A token x is *part of* a token y iff x is identical with y, a term of y or a subsequence of y.

1.7 Definition. A token *x occurs at position n* in a token y iff x is part of y and the first term of x is the nth term of y.

1.8 Definition. A formula α *occurs at position n* in a formula β iff a token of α occurs at position n in each token of β.

1.9 Definition. A formula α *occurs* in β iff for some n α occurs at position n in β.

1.10 Definition. The *occurrence* of α in β at position n is the set γ of tokens of α such that each member of γ occurs at position n in a token of β.

1.11 Definition. A formula α is the result of *replacing* the occurrences of formula δ at positions $n_1, n_2, \ldots, n_k$ in β by occurrences of γ iff there are tokens x of α, y of β, $v_1, v_2, \ldots, v_k$ of γ, and $w_1, w_2, \ldots, w_k$ of δ such that for $1 \leq i \leq k$ each v_i occurs at position m_i in x and each w_i occurs at position n_i of y and those parts of x of which no part of any v_i is a part are similar, in the order in which they occur to those parts of y of which no part of any w_i is a part.

1.12 Definition. A formula α *consists* of $\beta_1, \beta_2, \ldots, \beta_n$ iff the β_i are occurrences of formulae in α such that each token of α is the sum, in the sense of **PM*160** and/or ***161**, of n tokens one selected from each of the β_i in the order listed.

1.13 Definition. Formulas α and β occur in *corresponding positions* in γ and δ iff there is a number n such that α occurs at position n in γ and β occurs at position n in δ.

1.14 Definition. A token is *wf in* $\mathscr{A}$ iff it is a token of $\mathscr{A}$ constructed in accordance with the rules of formation of $\mathscr{A}$. A formula$_{\mathscr{A}}$ is *wf* in $\mathscr{A}$ iff its tokens of $\mathscr{A}$ are *wf*. A *wff* of $\mathscr{A}$ is a formula$_{\mathscr{A}}$ which is *wf* in $\mathscr{A}$.

1.15 Definition. A language $\mathscr{B}$ is a *generalization* of a language $\mathscr{A}$ iff there is a mapping of $\mathscr{A}$ onto $\mathscr{B}$ such that: (i) every symbol in $\mathscr{B}$ is an image of one and only one symbol in $\mathscr{A}$ and (ii) for every wff Q of $\mathscr{B}$ there is one and only one wff P of $\mathscr{A}$ such that each symbol in $\mathscr{B}$ which occurs in Q is an image of a symbol in $\mathscr{A}$ occurring in a corresponding position in P.

1.16 Definition. A *calculus* **X** in a language $\mathcal{A}$ is a set of wff of $\mathcal{A}$, the *theorems* of **X**, which is completely specified by (i) a finite set of schemes the union of the instances of which is a subset of theorems, the *axioms* of **X**, and (ii) one or more *rules of inference* of **X** whereby new theorems of **X** can be obtained from established theorems.

1.17 Definition. A calculus **Y** in language $\mathcal{B}$ *degenerates* into a calculus **X** in language $\mathcal{A}$ iff $\mathcal{B}$ is a generalization of $\mathcal{A}$ by a mapping such that for every theorem Q of **Y** there is a theorem P of **X** such that each symbol which occurs in Q is an image of a symbol occurring in a corresponding position in P.

1.18 Definition. A calculus **X** in language $\mathcal{A}$ *generalizes* into a calculus **Y** in language $\mathcal{B}$ iff $\mathcal{B}$ is a generalization of $\mathcal{A}$ and by a mapping such that for every theorem P of **X** there is a theorem Q of **Y** such that each symbol which occurs in Q is an image of a symbol occurring in a corresponding position in P.

2. CHARACTERIZATION OF THE LANGUAGES.

The languages and the calculi which are being compared are, for clarity as well as brevity, displayed wherever practical in parallel columns with the generalized language or calculus on the right. Statements about the languages or the calculi are frequently arranged in the form of single sentences with such alternate or additional words as apply specifically to the generalized language or calculus appearing within parentheses.

2.1 Designation. Prototype tokens of the *logical symbols* of language $\mathcal{S}$ ($\mathcal{S}_{\mathcal{G}}$) are listed below followed by the terms used to designate the symbol occurrences. The same terms written with initial capitals are used to designate the symbols themselves. Thus Implicator is a symbol of $\mathcal{S}$. Its tokens resemble the horseshoe-shaped mark appearing below. Each of its occurrences in a formula is called an implicator. In fact, Implicator is the union of the set of implicators.

	$\mathcal{S}$		$\mathcal{S}_{\mathcal{G}}$
$\sim$	negator	$\neg$	negator
.	dot	.	dot
$\supset$	implicator	$\rightarrow$	strong implicator
		$-\!\!3$	weak implicator
$\vee$	disjunctor	$\vee$	strong disjunctor
		$+$	weak disjunctor
&	conjunctor	$\times$	strong conjunctor
		$\wedge$	weak conjunctor

The logical symbols of $\mathcal{S}$ ($\mathcal{S}_{\mathcal{G}}$) other than Dot and Negator constitute the binary connectives of $\mathcal{S}$ ($\mathcal{S}_{\mathcal{G}}$).

2.2 Designation. The *nonlogical symbols* of $\mathcal{S}$ ($\mathcal{S}_{\mathcal{G}}$) are an unspecified number of *propositional variables* (the same in both languages).

2.3 Definition. The *rules of formation* of $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$ are the following:

(i) Propositional variables are wf and constitute the *elementary wffs.*
(ii) A formula consisting of a negator followed by a wff occurrence is wf and constitutes the *negation of* the wff.
(iii) A formula consisting of a dot followed by two wff occurrences separated by a (strong/weak) implicator is wf and constitutes the (*strong/weak*) *implication of* the wff occurring second *by* the wff occurring first.
(iv) A formula consisting of a dot followed by two wff occurrences separated by a (strong/weak) disjunctor is wf and constitutes the (*strong/weak*) *disjunction of* the wff occurring first *with* the wff occurring second.
(v) A formula consisting of a dot followed by two wff occurrences separated by a (strong/weak) conjunctor is wf and constitutes the (*strong/weak*) *conjunction of* the wff occurring first *with* the wff occurring second.

2.4 Definition. The term *junction* will be used as a metalanguage variable ranging over the designata of the six terms: (strong/weak) disjunction, (strong/weak) conjunction. As with other variables, when junction is used more than once in a single sentence form the intended sentences are obtained by substituting the same term for each such use.

2.5 Definition. (i) If P is a wff of $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$ then the *junctior of* P is P. (ii) If $P_1, P_2, \ldots, P_n$ are wff of $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$ then the *junction of* $P_1, P_2, \ldots, P_n$ is the junction of P_1 with the junction of $P_2, \ldots, P_n$.

2.6 Designation. The *logical symbols* of language $\mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}\mathcal{G}})$ are the logical symbols of $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$ plus the following

$\mathcal{S}_{\mathcal{J}}$		$\mathcal{S}_{\mathcal{J}\mathcal{G}}$
$\exists$ existential quantifier	$\vee$	[strong] existential operator
$\forall$ universal operator	$\wedge$	[weak] universal operator

Note: Since each of the operators is represented by a single operator in $\mathcal{S}_{\mathcal{J}\mathcal{G}}$ the modifiers, enclosed in brackets above, are generally omitted.

2.7 Designation. The *nonlogical symbols* of language $\mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}\mathcal{G}})$ are:

(i) the propositional variables of $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$;
(ii) a set of *individual variables*;
(iii) for each positive integer n less than some unspecified number, a set of *n-place predicate variables*.

2.8 Definition: A formula in $\mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}\mathcal{G}})$ is an *existential quantifier* or a *universal quantifier*, respectively, iff it consists of an existential operator followed by an occurrence of an individual variable.

2.9 Definition. The rules of formation for $\mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}\mathcal{G}})$ are the following:

(i) to (v): a reiteration of 2.3 (i) to (v);
(vi) A formula which consists of the occurrence of an n-place predicate variable followed by n occurrences of individual variables is wf and is an elementary *wff.*

(vii) A formula consisting of the occurrence of an existential or universal quantifier followed by the occurrence of a wff is wf.

2.10 Definition. An occurrence β is *the scope of* α in a wff γ of $\mathcal{S}_{\mathfrak{J}}(\mathcal{S}_{\mathfrak{J}\mathfrak{G}})$ iff α is an occurrence of a universal or an existential quantifier and γ consists of α followed by β.

2.11 Definition. A *free occurrence* of an individual variable α in a formula β is an occurrence of α in some formula γ which occurs in β but not in a quantifier nor in the scope any quantifier in which β occurs.

2.12 Definition. A *bound occurrence* of an individual variable α in a formula β is an occurrence of α in the scope of a quantifier which occurs in β and in which α occurs.

2.13 Definition. An individual variable α *occurs free* in a formula β iff there is a free occurrence of α in β.

2.14 Definition. An individual variable α *occurs bound* in a formula β iff there is a bound occurrence of α in β.

2.15 Definition. The relation *represents* performs a many-one mapping of the symbols of $\mathcal{S}$ and $\mathcal{S}_{\mathfrak{J}}$ onto the symbols of $\mathcal{S}_{\mathfrak{G}}$ and $\mathcal{S}_{\mathfrak{J}\mathfrak{G}}$, respectively, in the following way:

(i) Each nonlogical symbol of $\mathcal{S}_{\mathfrak{G}}$ or $\mathcal{S}_{\mathfrak{J}\mathfrak{G}}$ *represents* a typographically similar symbol of $\mathcal{S}$ or, respectively, $\mathcal{S}_{\mathfrak{G}}$;
(ii) Negator *represents* Negator;
(iii) Strong Implicator and Weak Implicator *represent* Implicàtor;
(iv) Strong Disjunctor and Weak Disjunctor *represent* Disjunctor;
(v) Strong Conjunctor and Weak Conjunctor *represent* Conjunctor;
(vi) Dot *represents* Dot;
(vii) [Strong] Existential Operator *represents* Existential Operator;
(ix) [Weak] Universal Operator represents Universal Operator.

2.16 Definition. A formula β in $\mathcal{S}_{\mathfrak{G}}$ or in $\mathcal{S}_{\mathfrak{J}\mathfrak{G}}$ is said to be a *generalizatum of* a formula α in $\mathcal{S}$ or in $\mathcal{S}_{\mathfrak{J}}$ iff each symbol occurring in β represents a symbol occurring in α in a corresponding position.

2.17 Theorem. *$\mathcal{S}_{\mathfrak{G}}$ is a generalization of $\mathcal{S}$ and $\mathcal{S}_{\mathfrak{J}\mathfrak{G}}$ is a generalization of $\mathcal{S}_{\mathfrak{J}}$.*

Proof: Represents, as defined in 2.15, satisfies the requirements of 1.15(i) while generalizatum of, as defined in 2.16, together with the rules of formation for $\mathcal{S}$ and $\mathcal{S}_{\mathfrak{G}}$ as set forth in 2.3 and those for $\mathcal{S}_{\mathfrak{J}}$ and $\mathcal{S}_{\mathfrak{J}\mathfrak{G}}$ as set forth in 2.9 satisfy the requirements of 1.15(ii).

2.18 Conventions. In such metalanguage formulations as axiom and theorem schemes the symbols P, Q, and R with and without subscripts and the formulae $F(x)$, $F(y)$, $G(x)$, are used as metavariables, i.e. variables of the metalanguage, ranging over wff of $\mathcal{S}$, $\mathcal{S}_{\mathfrak{J}}$, $\mathcal{S}_{\mathfrak{G}}$, and $\mathcal{S}_{\mathfrak{J}\mathfrak{G}}$ while the symbols x, y, and z with and without subscripts are used as metavariables ranging over individual variables. The logical symbols of $\mathcal{S}$, etc. are designated by symbols of like design, however, a pair of successive dots is sometimes designated by a colon and a sequence of n dots by a single dot with the numeral n as a superscript.

2.19 Definition. A wff P is a *junction* iff there are wffs Q and R such that P is the junction of Q with R.

3. THE CALCULI.

3.1 Definition. The *axioms* of calculus $\mathfrak{L}(\mathfrak{L}_{\mathfrak{G}})$ are all of the wffs of $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$ which are instances of the axiom schemes A1 through A8b (GA1 through GA8b).

3.2 Definition. The *axioms* of calculus $\mathfrak{L}_{\mathfrak{J}}(\mathfrak{L}_{\mathfrak{J}\mathfrak{G}})$ are all of the wffs of $\mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}\mathcal{G}})$ which are instances of the axiom schemes A1 through A14b (GA1 through GA14b).

Note: The axiom schemes are selected and arranged so that the instances of each scheme in the right hand column are generlizata of the instances of the scheme appearing opposite it in the left hand column. The axioms are not all independent and those schemes in the left hand column marked with asterisks are readily derivable from the others.

3.3 Definition. The *rule of inference* for calculus $\mathfrak{L}(\mathfrak{L}_{\mathfrak{G}})$ is the following:

R1 (GR1). If P is a theorem and the implication (weak implication) of Q by P is a theorem, then Q is a theorem.

3.4 Definition. The *rules of inference* for calculus $\mathfrak{L}_{\mathfrak{J}}(\mathfrak{L}_{\mathfrak{J}\mathfrak{G}})$ are the following:

R1 (GR1). If P is a theorem and the implication (weak implication) of Q by P is a theorem, then Q is a theorem.

R2 (GR2). If P is a theorem then any wff of $\mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}\mathcal{G}})$ consisting of an occurrence of a universal quantifier followed by an occurrence of P is a theorem.

R3 (GR3). Let P_i be, for $1 \leqq i \leqq n$, the result of replacing all occurrences of the individual variable x in a wff P of $\mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}\mathcal{G}})$ by occurrences of the variable y_i, and let $y_1, y_2, \ldots, y_n$ be all of the individual variable occurring free in P, then, if $\exists xP$ $(\forall xP)$ is a theorem, there is an i, $1 \leqq i \leqq n$, such that P_i is a theorem.

3.5 Designation. The axiom schemes refered to in definitions 3.1 and 3.2 are listed below:

Axiom Schemes for $\mathfrak{L}$ and $\mathfrak{L}_{\mathfrak{J}}$		Axiom Schemes for $\mathfrak{L}_{\mathfrak{G}}$ and $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$	
A1	$.P \supset .Q \supset P$	GA1	$.P \rightarrow .Q \rightarrow P$
A2	$:P \supset Q \supset :Q \supset R \supset .P \supset R$	GA2	$:P \rightarrow Q \rightarrow :Q \rightarrow R \rightarrow .P \rightarrow R$
A3a	$:P \vee Q \supset :P \supset Q \supset Q$	GA3a	$:P \vee Q \rightarrow :P \rightarrow Q \rightarrow Q$
A3b	$.:P \supset Q \supset Q \supset .Q \vee P$	GA3b	$.:P \rightarrow Q \rightarrow Q \rightarrow .Q \vee P$

A4	$:\sim P \supset \sim Q \supset .Q \supset P$	GA4	$:\neg P \to \neg Q \to .Q \to P$
*A5a	$:P \supset Q \supset .\sim P \vee Q$	GA5a	$:P ⥽ Q \to .\neg P \vee Q$
*A5b	$:\sim P \vee Q \supset .P \supset Q$	GA5b	$:\neg P \vee Q \to .P ⥽ Q$
*A6a	$:P \vee Q \supset .\sim P \supset Q$	GA6a	$:P + Q \to .\neg P \to Q$
*A6b	$:\sim P \supset Q \supset .P \vee Q$	GA6b	$:\neg P \to Q \to .P + Q$
A7a	$:P \,\&\, Q \supset \sim .\sim P \vee \sim Q$	GA7a	$:P \wedge Q \to \neg.\neg P \vee \neg Q$
A7b	$.\sim .\sim P \vee \sim Q \supset .P \,\&\, Q$	GA7b	$.\neg.\neg P \vee \neg Q \to .P \wedge Q$
*A8a	$:P \,\&\, Q \supset \sim .P \supset \sim Q$	GA8a	$:P \times Q \to \neg.P \to \neg Q$
*A8b	$.\sim .P \supset \sim Q \supset .P \,\&\, Q$	GA8b	$.\neg P \to \neg Q \to .P \times Q$

Axiom Schemes for $\mathfrak{L}_{\mathfrak{I}}$ Axiom Schemes for $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$

A9 $.F(y) \supset \exists x F(x)$ GA9 $.F(y) \to \bigvee x F(x)$

where $F(y)$ is the result of replacing all free occurrences of x in $F(x)$ by occurrences of y and no free occurrence of x in $F(x)$ is in the scope of any quantifier in which y occurs.

A10 $.\exists x F(x) \supset \exists y F(y)$ GA10 $.\bigvee x F(x) \to \bigvee y F(y)$

where $F(y)$ is the result of replacing all free occurrences of x in $F(x)$ by occurrences of y and $F(x)$ is the result of replacing all free occurrences of y in $F(y)$ by occurrences of x.

A11 $.\forall x\ .P \supset F(x) \supset .P \supset \forall x F(x)$ GA11 $.\bigwedge x\ .P \to F(x) \to .P \to \bigwedge x F(x)$

where x does not occur free in P.

*A12 $:P \supset \exists x F(x) \supset \exists x\ .P \supset F(x)$ GA12 $:P \to \bigvee x(x) \to \bigvee x\ .P \to F(x)$

where x does not occur free in P.

*A13	$:\exists x F(x) \,\&\, \exists x F(x) \supset$ $\exists x\ .F(x) \,\&\, F(x)$	GA13	$:\bigvee x F(x) \times \bigvee x F(x) \to$ $\bigvee x\ .F(x) \times F(x)$
A14a	$.\exists x P \supset \sim \forall x \sim P$	GA14a	$.\bigvee x P \to \neg \bigwedge x \neg P$
A14b	$.\sim \forall x \sim P \supset \exists x P$	GA14b	$.\neg \bigwedge x \neg P \to \bigvee x\ P$

3.6 Remark. A comparison of the axiom schemes given above for $\mathfrak{L}_{\mathfrak{G}}$ and $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$ with the axiom schemes and definitions of Rose and Rosser [5] and of Hay [4] reveals a correspondence between symbols indicated in the following table:

$\mathfrak{L}_{\mathfrak{G}}$ and $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$	$\to$	$⥽$	$\vee$	$+$	$\wedge$	$\times$	$\neg$	$\bigvee$	$\bigwedge$
Rose and Rosser	C	AN	A	B	K	L	N		
Hay	$\supset$		$\vee$	$+$	$\wedge$	$.$	$\sim$	$\exists$	( )

Our Strong Implicator ($⥽$), for which the cited authors have no special symbol, corresponds to the C' of Dienes [3]. The rules of formation of the Polish notation which is used by Rose and Rosser require that the capital letters representing binary connectives occur in the positions assigned to dots in $\mathcal{S}_{\mathcal{G}}$ and $\mathcal{S}_{\mathcal{IG}}$.

3.7 Citation. References to [6], [2], [5] and [4] will be cited by use of the authors' original numbers with the bold prefixes **PM**, **C**, **R** and **H**, respectively.

3.8 Theorem. *Calculi* $\mathfrak{L}$ *and* $\mathfrak{L}_{\mathfrak{G}}$ *each contain the statement calculus* **R**2 *and* **R**3.

Proof: A1, A2, A4, **R**1 (GA1, GA2, GA4, **GR**1) correspond to RA1, RA2, RA4, **R** Rule C, respectively, while a theorem corresponding to RA3 is derivable from A2, A3a and A3b by **R**1 (from GA2, GA3a and GA3b by **GR**1). Because of this metatheorem, theorems of **R**2 and **R**3 will frequently be cited in proof of theorems of $\mathfrak{L}(\mathfrak{L}_{\mathfrak{G}})$ and $\mathfrak{L}_{\mathfrak{F}}(\mathfrak{L}_{\mathfrak{F}\mathfrak{G}})$.

3.9 Theorem. *Calculus* $\mathfrak{L}$ *contains the classical two-valued calculus.*

Proof: The three Łukasiewicz axiom schemes for the two-valued propositional calculus are (1) **R**3.32, (2) A1, (3) a theorem scheme derivable from A6b, A2 and **R**2.16.

3.10 Theorem. *Conversely, calculus* $\mathfrak{L}$ *is contained in the classical two-valued calculus.*

Proof: The axiom schemes of $\mathfrak{L}$ can be derived from any of the equivalent sets of axioms of the classical two-valued propositional calculus, e.g. the axiom schemes A1 to A8b correspond, respectively, to the following theorems of **PM**: ***2.02**, ***2.06**, ***1.62**, a theorem derivable from ***2.06**, ***2.68**, ***1.4**, ***2.17**, ***1.01**, ***1.01**, ***2.53**, ***2.54**, ***3.1**, ***3.11**, ***4.63** and ***4.63**.

3.11 Theorem. *Calculus* $\mathfrak{L}_{\mathfrak{G}}$ *is contained in the infinite-valued statement calculus of* **R**2.3.

Proof: All of the axioms of $\mathfrak{L}_{\mathfrak{G}}$ correspond to axioms or definitions of **R** excepting GA3a,b which follow from **R**A3 and the definition of A, and GA5a,b which introduce a new connective. Complete correspondence is obtained if a new symbol say D is introduced into **R** which is defined as AN and hence corresponds to our Strong Implicator.

3.12 Theorem. *Calculus* $\mathfrak{L}_{\mathfrak{F}}$ *contains the pure first-order functional two-valued calculus* CF^{1p}.

Proof: Calculus CF^{1p} has in addition to the schemes embodying the propositional calculus only the axiom schemes **C***305, which is A11, and **C***306, which is readily derived from A9, **R**3.4, A2, and A4 by **R**1.

3.13 Remark. All of the axioms of $\mathfrak{L}_{\mathfrak{F}}$ are theorems of CF^{1p}: Axiom schemes A9 to A13 correspond respectively to **C***330, **C***378, **C***305, **C***382 and an instance of the law of conjunctive tautology combined with **C***301 and **C**34.5. However, no rule equivalent to **R**3, which might be called the rule of exhaustive instantiation, appears in Church's formulation. This rule is only applicable to theorems containing neither individual nor functional constants, hence we compare $\mathcal{S}_{\mathfrak{F}}$ to CF^{1p} rather than to CF^{1}.

3.14 Remark. The condition in **R**3 that all of the free variables of $\exists xP$ be among the y_i is necessitated by A9, and the condition that just one additional variable x be among the y_i suffices because of **R**2.

3.15 Theorem. *The generalized functional calculus* $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$ *contains the infinite-valued predicate calculus axiomatized by Hay* [4].

Proof: The propositional calculus and auxilliary definitions of **H** are equivalent to those of **R** and hence by 3.8 and 3.11 to those of $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$. The axiom schemes **H**A5, **H**A6, **H**A7 and **H**A9 correspond, respectively, to GA13, GA9, GA10 and GA12. Axiom scheme **H**A8 corresponds to the lemma L4, below, which is derivable in $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$ as follows: (The citations within parentheses are to proof schemes, the others to axiom or theorem schemes needed in the proofs)

L1	$.\wedge xF(x) \rightarrow F(y)$	(**H**P-2), GA9, **R**3.4, GA2, GA4, **GR1**
L2	$.\wedge x\ .F(x) \rightarrow Q \rightarrow .\wedge xF(x) \rightarrow Q$	(**C***332), L1, **R**2.1
L3	$.\wedge x\ .F(x) \rightarrow G(x) \rightarrow .\wedge xF(x) \rightarrow \wedge xG(x)$	(**C***333), L2, GA11, **GR2**, GA11, GA2
L4	$.\wedge x\ .F(x) \rightarrow Q \rightarrow .\vee xF(x) \rightarrow Q$	**R**3.5, **GR2**, L3, GA11, **R**3.3, GA14a

where x does not occur free in Q. Rules **H1** and **H2** correspond to **GR1** and **GR2**.

3.16 Remark. Proof that the calculus of **H** is contained in $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$ requires a derivation of GA11 in **H**. Since **H** is complete and GA11 is valid under the truth functions assigned to the connectives in **H**, such a derivation must be possible.

3.17 Definition. Unlike the other logical connectives, equivalence in $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$ and in $\mathcal{S}_{\mathcal{I}}(\mathcal{S}_{\mathcal{I}\mathcal{G}})$ is treated as an abbreviation in the metalanguage:

$$.P \equiv Q \text{ for } :P \supset Q\ \&\ .Q \supset P \text{ and } .P \longleftrightarrow Q \text{ for } :P \rightarrow Q \times .Q \rightarrow P$$

3.18 Theorem. *If P and Q are wff of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{I}}$, P' and Q' are wff of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$, P' is a generalizatum of P and Q' is a generalizatum of Q, then $.P' \longleftrightarrow Q'$ is a generalizatum of $.P \equiv Q$.*

Proof: Follows directly from the definitions 2.15, 2.16 and 3.17.
Thus in a sense the symbol $\longleftrightarrow$ can be said to represent the symbol $\equiv$.

3.19 Theorem. *The conservation of equivalence over a transformation by the replacement of equivalent parts is affirmed by an auxilliary rule of inference:*

R4 (GR4) *If M, N, P and Q are wff of $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$ or $\mathcal{S}_{\mathcal{I}}(\mathcal{S}_{\mathcal{I}\mathcal{G}})$ and if Q is the result of replacing zero or more occurrences of M in P by occurrences of N and if $M \equiv N$ $(M \longleftrightarrow N)$ is a theorem of $\mathfrak{L}(\mathfrak{L}_{\mathfrak{G}})$ or $\mathfrak{L}_{\mathfrak{I}}(\mathfrak{L}_{\mathfrak{I}\mathfrak{G}})$, then $P \equiv Q (P \longleftrightarrow Q)$ is a theorem and, if P is a theorem, Q is also a theorem.*

Proof: By 3.9 the proofs of **C***158 and **C***159 constitute proof of **R4** for $\mathfrak{L}$ and by 3.12 the proofs of **C***341 and **C***342 constitute proof of **R4** for $\mathfrak{L}_{\mathfrak{I}}$. Rule **GR4** may be proved by the methods of **C**34 using L1 of 3.15, **R**3.38, GA1, GA2, **GR1**, **GR2** and the following lemmas of $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$:

L5 $\quad .\wedge x\ .F(x) \longleftrightarrow G(x) \rightarrow .\wedge xF(x) \longleftrightarrow \wedge xG(x)$

(**C***334), **R**3.34, L3, GA2, **R**3.28

L6 $:P \longleftrightarrow Q \longleftrightarrow .\neg P \longleftrightarrow \neg Q$ (PM84.11), R3.5, R3.36, R3.28, R3.10, GA4

L7 $.\wedge x\ .F(x) \longleftrightarrow G(x) \rightarrow .\ \vee x F(x) \longleftrightarrow \vee x G(x)$
L6, GR2, R3.14, L3, L5, L6, GA14ab

L8 $.P \rightarrow :P \rightarrow .Q \longleftrightarrow R \rightarrow :P \rightarrow .S \longleftrightarrow T \rightarrow .P \rightarrow :Q \vee S \longleftrightarrow .R \vee T$
(PM*4.39), R2.19, R3.35, R3.28, R3.30, R3.10; R3.35, R2.8, R3.10

L9 $.P \rightarrow :P \rightarrow .Q \longleftrightarrow R \rightarrow :P \rightarrow .S \longleftrightarrow T \rightarrow .P \rightarrow :Q + S \longleftrightarrow .R + T$
(L8), R3.25 in place of R2.19

L10 $.P \rightarrow :P \rightarrow Q \longleftrightarrow R \rightarrow :P \rightarrow .S \longleftrightarrow T \rightarrow .P \rightarrow :Q \wedge S \longleftrightarrow .R \wedge T$
(L9), R3.28

L11 $.P \rightarrow :P \rightarrow .Q \longleftrightarrow R \rightarrow :P \rightarrow .S \longleftrightarrow T \rightarrow .P \rightarrow :Q \times S \longleftrightarrow .R \times T$
(L9), R3.28

L12 $:P \longleftrightarrow Q \longleftrightarrow .\neg P \longleftrightarrow \neg Q$ (PM*4.11), R3.5, R3.36, R3.28, R3.10

L13 $.P \rightarrow :P \rightarrow .Q \longleftrightarrow R \rightarrow :P \rightarrow .S \longleftrightarrow T \rightarrow .P \rightarrow :Q \dashv S \longleftrightarrow .R \dashv T$
(PM*4.39),
R2.19, R3.35, R3.28, R3.30, R3.10; GA5ab, L10, R3.35, R2.8, R2.7

114 $.P \rightarrow :P \rightarrow .Q \longleftrightarrow R \rightarrow :P \rightarrow .S \longleftrightarrow T \rightarrow .P \rightarrow :Q \rightarrow S \longleftrightarrow .R \rightarrow T$
(L13), R3.25 for R2.19; GA6ab and R3.4 for GA5ab

L15 $:P \rightarrow .Q \longleftrightarrow R \rightarrow :P \rightarrow .\neg Q \longleftrightarrow \neg R$ L10, R2.8, R2.7

No generalizatum of C*340 is obtained since no suitable generalizatum of the tautology employed in Case 1 of Church's proof is available. However a generalized version of C*341 may be derived using the proof scheme of C*340 but with the weaker theorems L8, L9, L10, L11, L13 and L14, expanding Case 1 to the six subcases required by the primitive status of the six binary connectives of $\mathfrak{L}_{\mathfrak{G}}$ and $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$ and expanding case 3 to the two subcases required by the primitive status of the quantification operators of $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$ using GR1 and GR2 in place of C*300 and C*301. The generalization C*342 follows using R3.34 and GR1. GR4 incorporates both generalizations.

The following definitions serve to simplify the diction in some metatheorems and proofs:

3.20 Definition. A wff P is a *transformation of* a wff Q iff $P \equiv Q(P \longleftrightarrow Q)$ is a theorem and the same propositional and predicate variables occur, and the same individual variables occur free, in P as in Q.

3.21 Definition. A wff P is *transformed into* a wff Q iff there exists a proof that Q is a transformation of P.

3.22 Theorem. *If P is a transformation of Q, then P is a theorem iff Q is a theorem.*

Proof: Follows from 3.17, 3.20, R3.3, R3.4 and R1 (GR1) for all four calculi.

4. PROLOGEMINA.

In this section such object language theorem schemes and metalanguage lemmas are presented as are required to prove the generalizability of the propositional and pure first order functional calculi. There is also a metatheorem on degeneration.

4.1 Lemma. *The following are theorem schemes for the calculi as indicated:*

Theorem Schemes for $\mathfrak{L}$ and $\mathfrak{L}_{\mathfrak{I}}$

T1 $:P \supset Q \equiv .\sim P \vee Q$

T2 $.\sim .P \vee Q \equiv .\sim P \,\&\, \sim Q$

T3 $.\sim .P \,\&\, Q \equiv .\sim P \vee \sim Q$

T4 $.\sim \sim P \equiv P$

T5 $:P \vee Q \equiv .Q \vee P$

T6 $.:P \vee Q \,\&\, .P \vee R \equiv .P \vee .Q \,\&\, R$

T7 $.\because .P \vee Q \vee R \equiv .P \vee .Q \vee R$

T8 $:P \,\&\, Q \equiv .Q \,\&\, P$

T9 $.\because .P \,\&\, Q \,\&\, R = .P \,\&\, .Q \,\&\, R$

T10 $.P \vee \sim P$

T11 $.P \supset .Q \supset .P \,\&\, Q$

T12 $:P \,\&\, Q \supset P$

T13 $.P \supset .P \vee Q$

T14 $:P \,\&\, Q \supset .P \,\&\, Q$

Theorem Schemes for $\mathfrak{L}_{\mathfrak{G}}$ and $\mathfrak{L}_{\mathfrak{IG}}$

GT1a $:P \rightarrow Q \longleftrightarrow .\neg P + Q$ GA6ab, R3.4

GT1b $:P \rightarrowtail Q \longleftrightarrow .\neg P \vee Q$ GA5ab, R3.4

GT2a $.\neg .\ P + Q \rightarrow .\neg P \times \neg Q$ GA6ab, GA8ab, R3.4

GT2b $.\neg .P \vee Q \rightarrow .\neg P \wedge \neg Q$ GA7a, R3.4

GT3a $.\neg .P \times Q \rightarrow .\neg P + \neg Q$ GA2a, R3.4

GT3b $.\neg .P \wedge Q \rightarrow .\neg P \vee \neg Q$ GA2b, R3.4

GT4 $.\neg\neg P \longleftrightarrow P$ R3.4

GT5 $:P + Q \longleftrightarrow .Q + P$ R3.11

GT6 $.\because .P + Q \wedge .P + R \longleftrightarrow .P + .Q \wedge R$ R3.44

GT7 $.\because .P + Q + R \longleftrightarrow .P + .Q + R$ R3.29

GT8 $:P \wedge Q \longleftrightarrow .Q \wedge P$ R3.12

GT9 $.\because .P \wedge Q \wedge R \longleftrightarrow .P \wedge .Q \wedge R$ R3.20

GT10 $.P + \neg P$ R2.10, GA6b

GT11a $.P \rightarrow .Q \rightarrow .P \wedge Q$ R3.22

GT11b $.P \rightarrow .Q \rightarrow .P \times Q$ R3.36

GT12a $:P \wedge Q \rightarrow P$ R3.14

GT12b $:P \times Q \rightarrow P$ R3.34

GT13 $.P \rightarrow .P \vee Q$ R2.4

GT14 $:P \times Q \rightarrow .P \wedge Q$ R3.52, R3.5, GA8a, GA7b

Theorem Schemes for $\mathfrak{L}_{\mathfrak{I}}$

T15 $.\sim \exists x F(x) \equiv \forall x \sim F(x)$

T16 $.\sim \forall x F(x) \equiv \exists x \sim F(x)$

T17 $.\forall x F(x) \rightarrow F(y)$

T18 $:P \vee \forall x F(x) \equiv \forall x\ .P \vee F(x)$

Theorem Schemes for $\mathfrak{L}_{\mathfrak{IG}}$

GT15 $.\neg \bigvee x F(x) \longleftrightarrow \bigwedge x \neg F(x)$ GA14ab, R3.4

GT16 $.\neg \bigwedge x F(x) \longleftrightarrow \bigvee x \neg F(x)$ GA14ab, R3.4, GR4

GT17 $.\bigwedge x F(x) \rightarrow F(y)$ L1

GT18a $:P + \bigwedge x F(x) \longleftrightarrow \bigwedge x\ .P + F(x)$ (C*335), L1, R3.23, GR2, L3, R3.36, GT1a, GT4

	GT18b $:P \vee \bigwedge xF(x) \longleftrightarrow \bigwedge x\ .P \vee F(x)$ L1, R2.18, GR2, GA11, L3, GR18a, GT1a, GT3ab, R3.36
T19 $:P \vee \exists xF(x) \equiv \forall x\ .P \vee F(x)$	GT19a $:P + \bigvee xF(x) \longleftrightarrow \bigvee x\ .P + F(x)$ GA9, R3.23, GR2, L4, GA12, GT1a, R3.36
	GT19b $:P \vee \bigvee xF(x) \longleftrightarrow \bigvee x\ .P \vee F(x)$ GT20a, GT2b, GT15, L6
T20 $:P\ \&\ \forall xF(x) \equiv \forall x\ .P\ \&\ F(x)$	GT20a $:P \wedge \bigwedge xF(x)$ $\bigwedge x\ .P \wedge F(x)$ L1, R3.14.13, GR2, GA11, R3.19; L1, R3.17, GR2, GA11; R3.36
	GT20b $:P \times \bigwedge xF(x) \longleftrightarrow \bigwedge x\ .P \times F(x)$ GT19a, GT3a, GT15, L6
T21 $:P\ \&\ \exists xF(x) \equiv \exists x\ .P\ \&\ F(x)$	GT21a $:P \wedge \bigvee xF(x) \longleftrightarrow \bigvee x\ .P \wedge F(x)$ GT18b, GT15, GT3b, L6
	GT21b $:P \times \bigvee xF(x) \longleftrightarrow \bigvee x\ .P \times F(x)$ GT18a, GT15, GT3a, L6

Note: By 4.3, below, any wff of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{J}}$ is a theorem of $\mathfrak{L}$ or $\mathfrak{L}_{\mathfrak{J}}$ if its generalizatum is a theorem of $\mathfrak{L}_{\mathfrak{G}}$ or $\mathfrak{L}_{\mathfrak{JG}}$; hence proof citations are indicated only for the theorems of $\mathfrak{L}_{\mathfrak{G}}$ and $\mathfrak{L}_{\mathfrak{JG}}$.

4.2 Lemma. *The junction of $P_1, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_n$ is a transformation of the junction of $P_i, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots P_n$ and also of the junction of the junction of $P_1, \ldots, P_i$ with the junction of $P_{i+1}, \ldots, P_n$.*

Proof: Follows from the definitions 2.4 and 2.5 together with T5(GT5) and T7(GT7), T8(GT8) and T9(GT9), R2.2 and R2.21, and R3.10 and R3.30.

4.3 Lemma. *Every theorem of $\mathfrak{L}_{\mathfrak{G}}$ or of $\mathfrak{L}_{\mathfrak{JG}}$ is a generalizatum of one and only one theorem of $\mathfrak{L}$ or $\mathfrak{L}_{\mathfrak{J}}$.*

Proof: (1) Every axiom of $\mathfrak{L}_{\mathfrak{G}}$ or $\mathfrak{L}_{\mathfrak{JG}}$ is a generalizatum of some axiom of $\mathfrak{L}$ or $\mathfrak{L}_{\mathfrak{J}}$ as is evident from the structure of the axiom schemes and the definitions 2.15 and 2.16. (2) From (1), isomorphism of GR1 with R1 and GR2 with R2, and the many-one character of the relations *represents* and *generalizatum of* it follows that every theorem of $\mathfrak{L}_{\mathfrak{G}}$ or $\mathfrak{L}_{\mathfrak{JG}}$ is a generalizatum of some theorem of $\mathfrak{L}$ or $\mathfrak{L}_{\mathfrak{G}}$. (3) A given wff of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{JG}}$ can be a generalizatum of only one wff of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{J}}$ by virtue of 2.15 and 2.16.

4.4 Definition. A wff of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{J}}$ ($\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{JG}}$) is in *positive form* iff

Implicator does not (Weak Implicator and Strong Implicator do not) occur in it and Negator, if it occurs in it, occurs only immediately preceding occurrences of elementary wff.

4.5 Definition. A wff of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$ is in *weak positive form* iff it is in positive form and Strong Disjunctor and Strong Conjunctor do not occur in it.

4.6 Lemma. *Each wff of $\mathcal{S}$($\mathcal{S}_{\mathcal{G}}$) or $\mathcal{S}_{\mathcal{I}}$($\mathcal{S}_{\mathcal{I}\mathcal{G}}$) can be transformed into a unique wff in positive form in which the same elementary wffs occur with the same frequency, order and grouping.*

Proof: By zero or more applications of **R4** with T1 (**GR4** with GT1a or GT1b) a sequence of wffs is obtained, beginning with the given wff, P, and ending with a wff, Q, such that each of the wffs is a transformation of the preceeding one with one less (weak or strong) implicator and such that Q is free of (strong or weak) implicators. By means of zero or more applications of **R4** with T2, T3, T4, T15 or T16 (**GR4** with GT2a, GT2b, GT3a, GT4, GT15 or GT16) a sequence of one or more wffs is obtained, beginning with Q and ending with a wff, R, such that each of the wffs except Q is a transformation of its predecessor in which a pair negators has been eliminated, a negator preceeding a quantifier has been replaced by a negator following a quantifier, or a negator preceeding a dot has been replaced by two negators following the dot, and such that in R no negator immediately preceeds another negator, a quantifier or a dot. R is thus a transformation of P in positive form and no permutation, duplication, elimination or regrouping of elementary wffs has occurred in the process.

4.7 Lemma. *For every wff, P, of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{I}}$ in positive form there is one and only one wff, P', of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$ such that P' is a generalizatum of P and is in weak positive form.*

Proof: A direct consequence of definitions 2.15, 2.16, 4.4 and 4.5.

4.8 Definition. *The positive form of* a wff, P, of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{I}}$($\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$) is that wff which is a transformation of P, which is in positive form and in which the elementary wffs occur with the same frequency, order and grouping as in P.

4.9 Lemma. *A wff of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$ which is in weak positive form and which is a generalizatum of the positive form of some wff P of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{I}}$ can be transformed into a unique generalizatum of P of which it is the positive form.*

Proof: Let $A_0, A_1, \ldots, A_n$ be a sequence of wffs of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{I}}$ such that A_n is the positive form of A_0 and for each i, $1 \leq i \leq n$, A_i is a transformation of A_{i-1} in that the first wff occurring in which an implicator occurs is transformed by applying **R4** with T1 or, if no implicator occurs in A_{i-1} in that the first wff in A_{i-1} beginning with a negator followed by a dot or a quantifier is transformed by applying **R4** with T2, T3, T4, T15 or T16. Let $B_0, B_1, \ldots, B_n$ be a sequence of wffs of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$ such that B_0 is a generalizatum of A_n in weak positive form and for each i, $1 \leqq i \leqq n$, B_i is a transformation of B_{i-1} by GT16, GT15, GT4, GT3a, GT3b, GT2a, GT2b, GT1b, or GT1a so chosen that for each i, $0 \leqq i \leqq n$, B_i is a generalizatum of A_{n-1}. This is always possible since each GT theorem is a generalizatum of one of the T theorems. Although there are two generalizata of some T theorems among the GT theorems only one is applicable to a given B_{i-1}, hence the B sequence is completely determined by B_0 and the A sequence. The A sequence is completely determined by A_0 and the instructions given for its construction. B_0 is determined by A_n in accordance with 4.7. Also by 4.7, if a different A sequence had been specified the same generalizatum, B_n, of A_0 would have been obtained.

4.10 Definition. A wff Q of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{I}}$ is the *weakest generalizatum* of a wff P of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$ iff the positive form of Q is in weak positive form and Q is a generalizatum of P.

4.11 Definition. A wff of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{I}}$ is in *conjunctive normal form* iff it is in positive form and if no wff occurring in it is a disjunction of wff any of which is a conjunction.

4.12 Definition. A wff of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$ is in *weak conjunctive normal form* iff it is in weak positive form and no wff occurring in it is a weak conjunction.

4.13 Definition. A wff of $\mathcal{S}_{\mathcal{G}}$ or $\mathcal{S}_{\mathcal{I}\mathcal{G}}$ is in *strong positive form* iff it is in positive form and neither Weak Disjunctor nor Weak Conjunctor occur in it.

4.14 Lemma. *A wff of $\mathcal{S}_{\mathcal{G}}$ or of $\mathcal{S}_{\mathcal{I}\mathcal{G}}$ which is in strong positive form and a generalizatum of the positive form of some wff P of $\mathcal{S}$ or $\mathcal{S}_{\mathcal{I}}$ can be transformed into a unique generalizatum of P of which it is the positive form.*

Proof: The proof is exactly the same as that of 4.9 excepting only that B_0 is to be in strong positive form.

4.15 Theorem. *The calculi $\mathfrak{L}_{\mathfrak{G}}$ and $\mathfrak{L}_{\mathfrak{I}\mathfrak{G}}$ degenerate respectively into the calculi $\mathfrak{L}$ and $\mathfrak{L}_{\mathfrak{I}}$.*

Proof: By virtue of 4.3 and 2.17 the relations defined in 2.15 and 2.16 satisfy the requirements of 1.18.

5. GENERALIZABILITY OF THE PROPOSITIONAL CALCULUS.

5.1 Postulate. *If P is a disjunction (weak disjunction) of wffs of $\mathcal{S}_{\mathcal{G}}$ ($\mathcal{S}_{\mathcal{G}}$) each of which is either an elementary wff or the negation of an elementary wff, then P is a theorem of $\mathfrak{L}$($\mathfrak{L}_{\mathfrak{G}}$) only if there occurs in P the negation of at least one elementary wff which also occurs in P as one of the terms of the disjunction (weak disjunction).*

Justification: Although this follows from the truth table for disjunction (truth function for weak disjunction) and the mutual independence of the truth values assigned to the elementary wffs it cannot be derived from the formal axiom schemes and rules and is hence introduced as a postulate.

5.2 Postulate. *If P is a strong disjunction of wffs of* $\mathcal{S}_{\mathcal{G}}$ *each of which is either an elementary wff or the negation of an elementary wff, then P is not a theorem of* $\mathfrak{L}_{\mathfrak{G}}$.

Justification: The postulate also follows from the truth function, in this case for strong disjunction, and the independence of the elementary wffs.

5.3 Lemma. *If P is a theorem of* $\mathfrak{L}$ *and is in positive form and Q is a generalizatum of P and is in weak positive form, then Q is a theorem of* $\mathfrak{L}_{\mathfrak{G}}$.

Proof: (1) Let $A_0, A_1, \ldots, A_n$ be a sequence of one or more wffs of $\mathcal{S}$ such that A_0 is P and such that each member A_i which is not in conjunctive normal form has a successor A_{i+1} which is the result of replacing the first wff occurring in A_i and having the form of a disjunction of some wff with a conjunction by its transformation, by way of T6, into a conjunction of disjunctions or, in case no wff occurs in A_i in the form of a disjunction of some wff with a conjunction, is the result of replacing the first wff occurring in A_i and having the form of a disjunction of a conjunction with some wff by its transformation, by way of T5, into a disjunction of the wff with the conjunction. The sequence terminates in a wff A_n which is in conjunctive normal form and, being a transformation of P, is by **R4** and the hypothesis a theorem of $\mathfrak{L}$. (2) The theorem A_n is, or by one or more applications of T9 may be transformed into, a wff of the form R_0 or of the form ${}^{r}\!\cdot R_0 \,\&\, R_1 \,\&\, \ldots \,\&\, R_r$ where, in either case, each R_i is of the form ${}^{s}\!\cdot S_{i0} \vee S_{i1} \vee \ldots \vee S_{is}$ such that each S_{ij} is either an elementary wff of $\mathcal{S}$ or the negation of an elementary wff. The wff A_n is, by T12 and 4.2, a theorem of $\mathfrak{L}$ only if each R_i is a theorem of $\mathfrak{L}$. However, by 5.1, each R_i is a theorem only if there is an S_{ij} and an S_{ik} such that S_{ik} is the negation of S_{ij}. (3) Let B_0 be the generalizatum of A_n which is in weak positive form, then, by 4.5 and 4.12, B_0 is in weak conjunctive form, that is of the form ${}^{r}\!\cdot R_0' \wedge R_1' \wedge \ldots \wedge R_r'$ where each R_i' is of the form ${}^{s}\!\cdot S_{i0}' + S_{i1}' + \ldots + S_{is}'$ in which, for each i and j, S_{ij}' is a generalizatum of S_{ij} and hence is either an elementary wff of $\mathcal{S}_{\mathcal{G}}$ or the negation of an elementary wff. Wff B_0 is a theorem of $\mathfrak{L}_{\mathfrak{G}}$ by GT12 if each R_i' is a theorem of $\mathfrak{L}_{\mathfrak{G}}$. However, by GT10 and 4.5, each R_i' is a theorem if there is an S_{ij}' and an S_{ik}' such that S_{ik}' is the negation of S_{ij}'. (4) Consider the sequence $B_0, B_1, \ldots, B_n$ of one or more wff of $\mathfrak{L}_{\mathfrak{G}}$ such that for each i, $1 \leq i \leq n$, B_i is that transformation of B_{i-1} by either GT5 or GT6 which makes it a generalizatum of A_{n-1}. This is always possible since B_0 is a generalizatum of A_n and GT5 and GT6 are generalizata of T5 and T6. Thus B_n is a generalizatum of A_0, i.e. P, and, is a theorem of $\mathfrak{L}_{\mathfrak{G}}$ since by (4) it is a transformation of B_0 which by (2) and (3) is a theorem of $\mathfrak{L}_{\mathfrak{G}}$ if A_n is a theorem of $\mathfrak{L}$ which by (1) is the case. Also, since the same connectives occur after transformation by either GT5 or GT6 and the negators remain fixed relative to the elementary wff, the weak positive form of B_0 gives rise to a weak positive form for B_n. Hence B_n is the required Q.

5.4 Lemma. *The weakest generalizatum of every theorem of* $\mathfrak{L}$ *is a theorem of* $\mathfrak{L}_{\mathfrak{G}}$.

Proof: (1) Let P be a theorem of $\mathfrak{L}$, then, by 4.6, P_1, the positive form of P, is a theorem of $\mathfrak{L}$. (2) Since P_1 is in positive form and a theorem of $\mathfrak{L}$, then, by 5.3, Q_1, that generalizatum of P_1 which is in weak positive form, is a theorem of $\mathfrak{L}_{\mathfrak{G}}$. (3) By 4.9 and 4.19, Q, the weakest generalizatum of P is a transformation of Q_1 and hence, by (2), a theorem of $\mathfrak{L}_{\mathfrak{G}}$.

5.5 Theorem. *The calculus $\mathfrak{L}$ generalizes into the calculus $\mathfrak{L}_{\mathfrak{G}}$.*

Proof: By 2.17 the language $\mathcal{S}_{\mathcal{G}}$ is a generalization of the language $\mathcal{S}$. The relations *represents* defined in 2.15 and *generalizatum of* defined in 2.16 constitute a mapping such that by 5.4 for every theorem P of $\mathfrak{L}$ there is a theorem Q of $\mathfrak{L}_{\mathfrak{G}}$ such that each symbol which occurs in Q represents, i.e. is an image of, a symbol occurring in a corresponding position in P. Thus the definition of generalizes in 1.18 is satisfied.

5.6 Remark. In general, the weakest generalizatum of any theorem P of $\mathfrak{L}$ will contain either or both representatives of each binary connective which occurs in P even though it is constructed by means of an intermediate weak positive form in which only two binary connectives occur.

5.7 Lemma. *Disjunctor occurs in the positive form of every theorem of $\mathfrak{L}$.*

Proof: The axioms and **R1** are such that a binary connective occurs in every theorem. By T12 and 4.2 a conjunction is a theorem only if all of the conjoined wff are theorems. Hence there is at least one disjunctor in every theorem which is in positive form and therefore, by 4.6, 4.8 and 3.22 in the positive form of every theorem of $\mathfrak{L}$.

5.8 Lemma. *Weak Disjunctor occurs in the positive form of every theorem of $\mathfrak{L}_{\mathfrak{G}}$.*

Proof: The axioms of $\mathfrak{L}_{\mathfrak{G}}$ and **GR1** are such that a binary connective occurs in every theorem of $\mathfrak{L}_{\mathfrak{G}}$. By GT12a and 4.2 a weak conjunction and by GR12b and 4.2 a strong conjunction is a theorem only if all of the conjoined wff are theorems. By 5.2 a wff of $\mathcal{S}_{\mathcal{G}}$ in which Strong Disjunctor is the sole binary connective is not a theorem of $\mathfrak{L}_{\mathfrak{G}}$ and hence, by GT12a, GT12b, **R**2.17 and **R**3.23 neither is a wff containing as connectives only Strong Disjunctor together with either or both of the representatives of Conjunctor.

5.9 Theorem. *Among the generalizata of every theorem of $\mathfrak{L}$ there is at least one wff of $\mathcal{S}_{\mathcal{G}}$ which is not a theorem of $\mathfrak{L}_{\mathfrak{G}}$.*

Proof: Let P be a theorem of $\mathfrak{L}$, P_1 the positive form of P and Q_1 a generalizatum of P_1 in strong positive form. Since, by 4.13, Weak Disjunctor does not occur in Q_1 it is, by 5.8, not a theorem of $\mathfrak{L}_{\mathfrak{G}}$. However, by 4.14, Q_1 is the positive form of a wff Q of $\mathcal{S}_{\mathcal{G}}$ which is a generalizatum of P and, being a transformation of Q_1, is also not a theorem of $\mathfrak{L}_{\mathfrak{G}}$.

5.10 Lemma. *If P and Q are generalizata of the same wff of $\mathcal{S}$ and are in positive form and if Q is the result of replacing zero or more strong conjunctors in P by weak conjunctors, then the weak implication of Q by P is a theorem of $\mathfrak{L}_{\mathfrak{G}}$.*

Proof: Since both P and Q are in positive form, repeated application of GT14 together with R2.17.18, R3.16.17.23.24.26.27 can be used to construct proofs for all possible cases.

5.11 Theorem. *If a theorem of $\mathfrak{L}$ is such that its positive form is in conjunctive normal form, then every generalizatum of said theorem in which Strong Disjunctor does not occur is a theorem of $\mathfrak{L}_{\mathfrak{G}}$.*

Proof: Let P_1 be a theorem of $\mathfrak{L}$ in conjunctive normal form and the positive form of a theorem P, and let Q be a generalizatum of P such that in its positive form Q_1 the only binary connectives occurring are Strong Conjunctor and Weak Disjunctor. By definitions 2.15, 2.16 and 4.4 Q_1 is a generalizatum of P_1. Since P_1 and Q_1 are in conjunctive normal form the argument used in (2) and (3) of the proof of 5.3 can be used with strong conjunctors in place of weak conjunctors and GT12b in place of GT12a to prove that Q_1 must be a theorem of $\mathfrak{L}_{\mathfrak{G}}$ if P_1 is a theorem of $\mathfrak{L}$, and hence by 4.6 that Q is a theorem of $\mathfrak{L}_{\mathfrak{G}}$ since P is a theorem of $\mathfrak{L}$. Consider now another generalizatum of P, the wff R, such that in its positive form R_1 Weak Conjunctor occurs. By 5.10 if Q_1 is a theorem so is R_1 and hence by 4.4 also R. Thus any generalizatum of P such that Strong Disjunctor does not occur in its positive form is a theorem of $\mathfrak{L}_{\mathfrak{G}}$.

5.12 Remark. Other conditions can be enunciated which are either sufficient or necessary in order that a generalizatum of a theorem of $\mathfrak{L}$ be a theorem of $\mathfrak{L}_{\mathfrak{G}}$; however there is no simple set of conditions which is both sufficient and necessary.

The following lemma establishes the lemmas and theorems of this section as lemmas for Section 7 on the pure predicate calculus of the first order.

5.13 Lemma. *Every general statement which holds for wff of $\mathcal{S}$ ($\mathcal{S}_{\mathcal{G}}$) and/or theorems of $\mathfrak{L}$($\mathfrak{L}_{\mathfrak{G}}$) also holds for quantifier-free wff of $\mathcal{S}_{\mathcal{F}}$ ($\mathcal{S}_{\mathcal{F}\mathcal{G}}$) and/or quantifier-free theorems of $\mathfrak{L}_{\mathfrak{F}}$($\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$).*

Proof: This is a direct result of the common formation rules of $\mathcal{S}$($\mathcal{S}_{\mathcal{G}}$) and $\mathcal{S}_{\mathcal{F}}$($\mathcal{S}_{\mathcal{F}\mathcal{G}}$) and the common axioms and rules of derivation for $\mathfrak{L}$($\mathfrak{L}_{\mathfrak{G}}$) and $\mathfrak{L}_{\mathfrak{F}}$($\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$) applicable to quantifier-free formulae.

6. NONGENERALIZABILITY OF CALCULI WITH EQUIVALENCE.

6.1 Definition. Let $\mathcal{S}_{\mathcal{E}}$ be a language differing from $\mathcal{S}$ only in that its logical symbols include Equivalor, prototype $\equiv$, and its formation rules include provision for the formation of wffs called equivalences of two wffs in which Equivalor occurs as a binary connective.

6.2 Definition. Let $\mathcal{S}_{\mathcal{G}\mathcal{E}}$ be a language differing from $\mathcal{S}_{\mathcal{E}}$ only in that its logical symbols include representatives of Equivalor and its formation rules include provisions for the formation of wff which are generalizata of equivalences.

6.3 Definition. Let $\mathfrak{L}_{\mathfrak{E}}$ be a calculus differing from $\mathfrak{L}$ only in that its language is $\mathcal{S}_{\mathcal{E}}$ and its axioms include a definition of Equivalor equivalent to 3.14. Similarly let $\mathfrak{L}_{\mathfrak{G}\mathfrak{E}}$ differ from $\mathfrak{L}_{\mathfrak{E}}$ only in that its language is $\mathcal{S}_{\mathcal{G}\mathcal{E}}$ and its axioms include generalizata of the axioms embodying Equivalor effectively defining the added symbols.

6.4 Definition. A wff of $\mathcal{S}_{\mathcal{E}}(\mathcal{S}_{\mathcal{G}\mathcal{E}})$ is in *positive form* iff the only binary connectives occurring in it are Disjunctor and Conjunctor and Negator occurs only immediately before elementary wff.

6.5 Definition. A wff of $\mathcal{S}_{\mathcal{G}\mathcal{E}}$ is in *weak positive form* iff the only binary connectives occurring in it are Weak Disjunctor and Weak Conjunctor and Negator occurs only immediately before elementary wff.

6.6 Lemma. *Lemma* 4.6 *is not valid for* $\mathcal{S}_{\mathcal{E}}(\mathcal{S}_{\mathcal{G}\mathcal{E}})$ *in place of* $\mathcal{S}(\mathcal{S}_{\mathcal{G}})$.

Proof: When an equivalence such as $P \equiv Q$ occurs in a wff which is to be transformed to its positive form it must be replaced by a conjunction of disjunctions, such as $:\sim P \vee Q \;\&\; .\sim Q \vee P$, or a disjunction of conjunctions, such as $:P \;\&\; Q \vee .\sim P \;\&\; \sim Q$. The requirement of 4.6 that the elementary wffs occur with the same frequency thus cannot be maintained.

6.7 Lemma. *Lemma* 4.9 *is not valid with* $\mathcal{S}_{\mathcal{G}\mathcal{E}}$ *and* $\mathcal{S}_{\mathcal{E}}$ *replacing* $\mathcal{S}_{\mathcal{G}}$ *and* $\mathcal{S}$.

Proof: Consider a wff of $\mathcal{S}_{\mathcal{E}}$ in which an equivalence occurs between wffs at least one of which is compound. Say $.p \equiv .q \vee r$ in which p, q and r are supposed to be elementary wffs. Transformation to positive form can be accomplished in three steps: (1) removal of Equivalor to yield $:p \supset q \vee r \;\&\; :q \vee r \supset p$; (2) removal of Implicator to yield $:\sim p \vee q \vee r \;\&\; .\sim .q \vee r \vee p$; (3) shifting of Negator when it precedes a dot to yield $:p \vee q \vee r \;\&\; :\sim q \;\&\; \sim r \vee p$.

The weak generalizatum of this in weak positive form is $:\urcorner p + q + r \wedge : q \wedge \urcorner r + p$. Transformation by the generalized reversal of step (3) yields $:\urcorner p + q + r \wedge .\urcorner .q \vee r + p$; reversal of step (2) yields $:p \rightarrow q + r \wedge :q \vee r \rightarrow p$ which however is not transformable to a generalizatum of any wff that can be formed by joining p and $q \vee r$ with a binary connective. Since the dual of each weak disjunction or conjunction is a strong conjunction or disjunction this result is perfectly general.

6.8 Theorem. *The calculus* $\mathfrak{L}_{\mathfrak{E}}$ *does not generalize into the calculus* $\mathfrak{L}_{\mathfrak{G}\mathfrak{E}}$.

Proof: The failure of 4.9 and hence the inadmissibility of definition 4.10 for $\mathcal{S}_{\mathcal{G}}$ and $\mathcal{S}_{\mathcal{G}\mathcal{E}}$ invalidates the use of the proof given in 5.5 for the extension of theorem 5.5 to $\mathfrak{L}_{\mathfrak{E}}$ and $\mathfrak{L}_{\mathfrak{G}\mathfrak{E}}$ but does not directly disprove the theorem or prove 6.8. The easiest proof of 6.8 is by counterexample. Thus, **PM*5.32** is a theorem of $\mathfrak{L}_{\mathfrak{E}}$ but none of its generalizata are theorems of $\mathfrak{L}_{\mathfrak{G}\mathfrak{E}}$. This is most easily demonstrated by the use of truth functions for the seven possible representatives of Equivalor which can be characterized in terms of the positive forms of equivalences given in 6.6, together with the truth functions for the other connectives as set forth in **R1**. The truth functions for the seven generalizata of equivalence are as follows:

$\max(\min(1-p,\ 1-q),\ \min(p,q))$; $\max(\min(1-p,\ 1-q),\ p+q-1))$; $\min(1-p+q,\ 1-q+p)$; $\max(1-p-q,\ \min(p,q))$; $\max(1-p-q,\ p+q-1)$; $\min(1-q+p,\ \max(1-p,q))$; $\min(1-p+q,\ \max(1-q,p))$. There are 392 different generalizata in $\mathcal{S}_{\mathcal{G}\mathcal{E}}$ of the theorem.

$$\text{PM*5.32}\ :P \supset .Q \equiv R \equiv :P \,\&\, Q \equiv .P \,\&\, R$$

none of which are theorems of $\mathfrak{L}_{\mathfrak{G}\mathfrak{E}}$ as each can be shown to be contravalid by assigning to P, Q and R one or more of the following sets of truth values: $(\frac{1}{2},0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2},0)$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2},1)$, $(\frac{1}{2},1,1)$.

6.9 Remark. It is often convenient to consider the theorems asserted in **PM** as theorems of $\mathfrak{L}_{\mathfrak{E}}$ and to seek among their generalizata in $\mathcal{S}_{\mathcal{G}\mathcal{E}}$ theorems of $\mathfrak{L}_{\mathfrak{G}\mathfrak{E}}$. However, by 6.8, such a search may be fruitless. In such cases, if the principal connective is Equivalor, the theorem is often best decomposed into two implications, or, if a complex subsidiary equivalence occurs in it, this may be transformed into a conjunction of implications or a disjunction of conjunctions. As an example it may be noted that there are several generalizata of the implications arising from the decomposition of **PM*5.32** which are theorems of $\mathfrak{L}_{\mathfrak{G}\mathfrak{E}}$ and retain the subsidiary equivalences,

(1) $:P \dashv .Q \longleftrightarrow R \rightarrow :P \wedge Q \longleftrightarrow .P \wedge R$
(2) $:.P \wedge Q \longleftrightarrow .P \wedge R \rightarrow .P \rightarrow .Q \longleftrightarrow R$

The following theorem of $\mathfrak{L}_{\mathfrak{G}\mathfrak{E}}$ is a generalizatum of a theorem formed from **PM*5.32** by transforming the complex subsidiary equivalence to a conjunction of implications but retaining the principal equivalence:

(3) $:P \rightarrow .Q \longleftrightarrow R \longleftrightarrow .\because P \times Q \rightarrow .P \wedge R \longleftrightarrow :P \times R \rightarrow .P \wedge Q$

Transforming the simple subsidiary equivalence would produce no new independent generalizata.

7. GENERALIZABILITY OF THE PURE PREDICATE CALCULIS.

7.1 Definition. A wff of $\mathcal{S}_{\mathcal{J}}$ ($\mathcal{S}_{\mathcal{J}\mathcal{G}}$) is in *prenex form* iff no dot or negator appears in it preceding any occurrence of a quantifier.

7.2 Definition. The *matrix* of a wff in prenex form is the longest quantifier-free formula occurring therein. The *prefix* of a wff in prenex form is the longest formula occurring before any dot or negator.

7.3 Theorem. *The matrix of a wff in prenex form is wf. A quantifier-free wff is in prenex form, has no prefix and is identical with its matrix.*

Proof: Follows directly from the definitions.

7.4 Lemma. *Every wff of $\mathcal{S}_{\mathcal{J}}$ in positive (of $\mathcal{S}_{\mathcal{J}\mathcal{G}}$ in postive or weak positive) form can be transformed into a wff in prenex form which is also in positve (positive or weak positive) form.*

Proof: By the use of T18, T19, T20, T21 and A10 (GT18ab, GT19ab, GT20, GT21ab and GA10) a sequence of wff may be constructed which begins with the given wff and ends in a wff in prenex form and is such that each member of the sequence is a transformation of its predecessor resulting from the replacement of the first quantifier occurring after a dot by one occurring before the dot. The connectives are not changed or displaced in such transformations and hence the property of being in positive (positive or weak positive) form is conserved.

7.5 Definition. A wff is in prenex positive (positive or weak positive) form iff it is in positive (positive or weak positive) form and also in prenex form.

7.6 Lemma. *If a theorem* $\mathfrak{L}_{\mathfrak{F}}$ *is in prenex positive form its generalizatum in weak positive form is a theorem of* $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$.

Proof: (1) Consider a theorem P of $\mathfrak{L}_{\mathfrak{F}}$ which is in prenex positive form with a prefix consisting of n quantifier occurrences. In $n > 0$ then P is either in the form $\forall xQ$ or $\exists xQ$. (2) If P is in the form $\forall xQ$ then there is, by T17, a wff Q which is a theorem; if P is in the form $\exists xQ$ then there is, by **R3**, a set of wf, Q_0, Q_1, . . . , Q_m, at least one of which is a theorem. (3) Hence by (1) and (2) if $n > 0$ there is a set α_1 of one or more wffs in positive prenex form at least one of which is a theorem and each of which has the same prefix consisting of n - 1 quantifier occurrences. (4) if $n > 1$ there is, corresponding to each member Q_i of α_1, a set Q_{i1}, Q_{i2}, . . . , Q_{ip} of wffs in prenex positive form at least one of which is a theorem in the case that Q_i is a theorem. (5) Since by (3) some member Q_k of α_1 is a theorem then by (4) some member of α_2, the union of the Q_{ij}, is a theorem. Also each member of α_2 has the same prefix of n - 2 quantifier occurrences. (6) In general, each α_k for $0 < k < n$ determines a set of sets of wffs in prenex positive form the union α_{k+1} of which is such that at least one of its members is a theorem and each has the same prefix consisting of $n + k - 1$ quantifier occurrences. (7) The sequence terminates in a set α_n of quantifier-free wffs each in positive form and at least one a theorem. (8) Let β_n be the set of weakest generalizata of α_n. The members of β_n will be wff of $\mathcal{S}_{\mathfrak{F}\mathfrak{G}}$ in quantifier-free weak positive form. (9) Since, by (7), at least one member of α_n is a theorem of $\mathfrak{L}_{\mathfrak{F}}$ it follows by 5.3 and 5.13 that at least one member of β_n is a theorem of $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$. (10) The derivation of α_n from α_{n-1} involved the use of T17 or **R3**, hence, by the use of **GR2** or GA9, resp., a set β_{n-1} of wffs of $\mathcal{S}_{\mathfrak{F}\mathfrak{G}}$ can be derived from β_n such that β_{n-1} constitutes the generalizata of α_{n-1} in prenex weak positive form and such that each member of β_{n-1} which is the generalizatum of a theorem of $\mathfrak{L}_{\mathfrak{F}}$ is a theorem of $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$. The same single quantifier will occur as the prefix of each member of β_{n-1}. (11) In general, to each α_k for $0 < k < n$ there corresponds a β_k which is derived from β_{k+1} by **GR2** or GA9, is made up of the weakest generalizata of the α_k each with a prefix consisting of the same $n - k$ quantifiers and is such that every member of β_k which is the generalizatum of a theorem of $\mathfrak{L}_{\mathfrak{F}}$ will be a theorem of $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$. (12) From β_1 which, being derived stepwise from β_n includes among its members at least one theorem of $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$, a wff is obtained by **GR2** or GA9 which has a prefix consisting of n quantifiers is a theorem of $\mathfrak{L}_{\mathfrak{F}\mathfrak{G}}$ and is the weakest generalizatum of P.

7.7 Lemma. *The weakest generalizatum of every theorem of* $\mathfrak{L}_{\mathfrak{J}}$ *is a theorem of* $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$.

Proof: Let P be a theorem of $\mathfrak{L}_{\mathfrak{J}}$. By 4.6, 5.13 and 3.22 the positive form P_1 of P is a theorem of $\mathfrak{L}_{\mathfrak{J}}$; by 7.4 and 3.22 the prenex form P_2 of P_1 is a theorem of $\mathfrak{L}_{\mathfrak{J}}$; by 7.6 the weakest generalizatum Q_2 of P_2 is a theorem of $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$; hence, by 4.6 and GR4, the generalizatum Q of P of which Q_2 is the positive form is a theorem of $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$ and by 4.10 is the weakest generalizatum of P.

7.8 Theorem. *The calculus* $\mathfrak{L}_{\mathfrak{J}}$ *generalizes into the calculus* $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$.

Proof: By 2.17 the language $\mathcal{S}_{\mathcal{J}\mathcal{G}}$ is a generalization of the language $\mathcal{S}_{\mathcal{J}}$. The relation, *generalizatum of,* of 2.16 constitutes a mapping such that by 7.7 for every theorem P of $\mathfrak{L}_{\mathfrak{J}}$ there is a theorem Q of $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$ such that each symbol which occurs in Q represents and hence is an image of a symbol occurring in a corresponding position in P, thus satisfying the definition 1.18 of *generalizes.*

7.9 Lemma. *Weak Disjunctor occurs in the positive form of every theorem of* $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$.

Proof: (1) Let Q_1 be the positive form of a theorem Q of $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$ and let Q_2 be the prenex form of Q_1. (2) By (1) and 7.4 Q_2 is a theorem of $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$ and the same connectives occur in it as occur in Q_1. (3) From Q_2, by **GR3** and GT17 in a series of steps analagous to steps (1)—(7) in the proof of 7.6, a set of quantifier-free wffs is derivable in each of which the same connectives occur as occur in Q_2 and hence by (2) as in Q_1 and at least one of which, say Q_3 is a theorem of $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$. (4) Since Q_3 is a theorem then by 5.8 and 5.13 Weak Disjunctor occurs in it, and hence by (3) in Q_2 and hence by (2) in Q_1.

7.10 Theorem. *Among the generalizata of every theorem of* $\mathfrak{L}_{\mathfrak{J}}$ *there is at least one wff of* $\mathcal{S}_{\mathcal{J}\mathcal{G}}$ *which is not a theorem of* $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$.

Proof: Let P be a theorem of $\mathfrak{L}_{\mathfrak{J}}$, P_1 the positive form of P and Q_1 a generalizatum of P_1 in strong positive form. Since, by 4.13, Weak Disjunctor does not occur in Q_1, Q_1 is, by 7.9, not a theorem of $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$. However, by 4.14, Q_1 is the positive form of a wff Q of $\mathcal{S}_{\mathcal{J}\mathcal{G}}$ which is a generalizatum of P and, being a transformation of Q_1 is also not a theorem of $\mathfrak{L}_{\mathfrak{J}\mathfrak{G}}$.

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