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A CHARACTERIZATION OF A SPHERICAL m-ARRANGEMENT

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In [1] a simplified definition of an open m-arrangement was presented. The purpose of this paper is to present a simpler characterization of a spherical m-arrangement than that presented in [2], a characterization which because of its similarity to the characterization of an open m-arrangement in [1] leads us to define a new type of structure, an (n,m)-arrangement, of which open m-arrangements and spherical m-arrangements are but special cases. The principal result to be proved in this paper in the following:

Theorem 1: *Let X be a topological space with geometry G of length $m - 1 \geq 0$. Then X and G form a spherical m-arrangement if and only if the following conditions are satisfied:*

i) *Each* 0*-flat consists of precisely two points.*
ii) *If f is a k-1-flat and g is a k-flat with $f \subseteq g$, then f disconnects g into two non-empty convex components which are open in g, $0 \leq k \leq m$.*
iii) *Each* 1*-flat is connected.*
iv) (If f is an m-1-flat, then we call the components of *X-f half-spaces* of X.) *The collection of half-spaces of X forms a subbasis for the topology of X.*

Proof: We note first that i) and ii) are the same as 1) and 5) in the definition of a spherical m-arrangement given in [2]. We now show that i) through iv) also imply 2), 3), and 4) in the definition of a spherical m-arrangement. In the following propositions then we assume that we have a space X with geometry G of length m-1 which satisfies i) through iv).

Proposition 1: *X is* T_1.

Proof: Each m-1-flat is closed and any 0-flat $\{x,y\}$ is the intersection of finitely many m-1-flats, and hence is closed. But by ii) $\{x,y\}$ is disconnected; hence it follows that $\{x\}$ and $\{y\}$ are both closed sets. Since any one point subset of X is contained in some 0-flat, X is T_1.

Proposition 2: *If f is any 1-flat and x is a point of f, then x is a non-cut point of f.*

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Proof: Suppose x is a cut point of f. Since $\{x\}$ is closed, $f - \{x\} = C \cup D$, where C and D are non-empty, disjoint, and open in f. Let x' be the point of f antipodal to x. Then x' is either in C or in D; assume $x' \varepsilon C$. Since $f - \{x, x'\} = A \cup B$, where A and B are convex, non-empty, disjoint, and open in f, we have either $x' \varepsilon \mathrm{Cl} B$, or $x' \varepsilon \mathrm{Cl} A$ (or $\{x'\}$ would be open and f would not be connected), but not both, or $f - \{x\}$ would be connected. Assume $x' \varepsilon \mathrm{Cl} B$. Then $C = \mathrm{Cl} B$ and $A = D$. Since A and B each do not contain any pair of antipodal points, then same is true of C and D.

Choose any point y from C and let y' be its antipodal point in D. Then $f - \{y, y'\} = E \cup F$, where E and F are disjoint, non-empty, convex, and open in f. Assume x' is in E; then x is in F. Now $A \cap E$ is open in f, $C \cap E$ is open in f and non-empty, as is F. Since $E = (D \cap E) \cup (C \cap E)$, $(D \cap E) \cap (C \cap E) = \phi$, and E is connected since it is convex, it follows that $D \cap E = \phi$; therefore $E \subset C$.

Since $E \subset C$, $D - \{y'\} \subset F$. Since F and E are both convex, each cannot contain a pair of antipodal points. But if E is a proper subset of $C - \{y\}$, then F must contain a pair of antipodal points. It follows then that $E = C - \{y\}$ and $F = (D - \{y'\}) \cup \{x\}$. But then we have $f = (F \cup D) \cup C$ and $C \cap (F \cup D) = \phi$ with $F \cup D$ and C both open in f. Therefore f is not connected, a contradiction. Consequently, x is a non-cut point of f.

Proposition **3**: *If $\{x,y\}$ is any two point subset of X and $\{x,y\} \subset f$, a* 1*-flat, then $f = S \cup T$, where S and T are both subsets of f irreducibly connected between x and y. Moreover, if $\{x,y\}$ is linearly independent, then either S or T is the convex hull $\overline{xy}$ of $\{x,y\}$.*

Proof: It is easily shown that any 1-flat satisfies Wilder's definition of a *quasi-closed curve* (11.18, [4]), hence applying Lemma 11.19 of [3], we obtain that $f = S \cup T$, where S and T are both subsets of f irreducibly connected between x and y. Suppose $\{x,y\}$ is linearly independent, and x' is antipodal to x; that is, $\{x, x'\}$ form a 0-flat. Also assume x' is in T. Now $\{x, x'\}$ disconnects f into two convex components A and B, which are each open in f; assume y is in A. From Wilder [4], 11.4, we have that $B \cup \{x, x'\}$ is irreducibly connected between x and x' and is a subset of T; hence $B \subset T$. But then $S \subset A$; hence S contains no 0-flat. This proves then that S is a convex set ([3], Proposition **2.2**). If W is any convex set which contains $\{x,y\}$, then W must contain either S or T, or it could be shown that $f \cap W$ is not connected. But W cannot contain T since T contains antipodal points. Therefore $S \subset W$. Thus $S = \overline{xy}$.

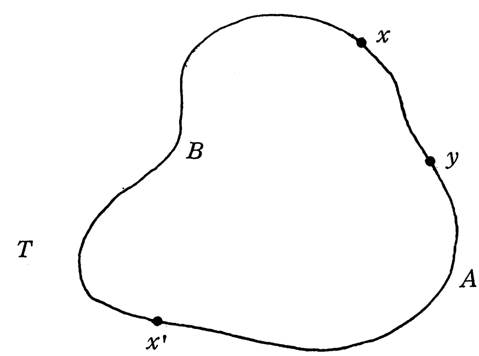

Figure 1

Corollary 1: *A subset W of X is convex if and only if given any linearly independent subset $\{x,y\}$ of W, $\overline{xy} \subset W$, and W contains no antipodal points.*

Proof: Suppose $\overline{xy} \subset W$ for any linearly independent subset $\{x,y\}$ of W, W contains no antipodal points, and f is any 1-flat of X. If $f \cap W$ is empty or consists of a single point, then $f \cap W$ is connected. Suppose x and y are in $f \cap W$. Then $\{x,y\}$ is linearly independent since $f \cap W$ can contain no antipodal points. Thus $xy \subset f \cap W$. But then x and y are both in the same component of $f \cap W$, hence $f \cap W$ is connected. Therefore W is convex.

Suppose W is convex and $\{x,y\}$ is a linearly independent subset of W. Then $\overline{xy} \subset W$ by Proposition 3. Moreover, since W is convex, W can contain no antipodal points.

Corollary 2: *G is a topological geometry.*

This corollary follows from Corollary 1 which can be used to show that the intersection of any family of convex sets is convex, and from the fact that each k-flat is closed (since any k-flat is the intersection of finitely many m-1-flats which are each closed).

Using the simplified characterization of an m-arrangement found in [1], it is now easy to prove

Proposition 4: *If W is any convex subspace of X, then W with geometry G_W is an open $(\delta(W) + 1)$-arrangement.* [This is 4) in the definition of a spherical m-arrangement.]

Proposition 5: *If f is a k-flat, $k \geq 1$, then no flat of dimension less than $k - 1$ disconnects f.*

Proof: Suppose f' is a k-2-flat which is contained in some k-flat f, $k \geq 1$. Let g be any k-1-flat which contains f' and is contained in f. Then g disconnects f into convex open components A and B. Also f' disconnects g into convex components C and D. Let x and y be any two points of $f - f'$. If x and y are both in A, B, C, or D, then $\overline{xy} \subset f - f'$. If x and y are in A and B, respectively, choose z in $g - f'$. Then $\overline{xy} \cup \overline{zy} \subset f - f'$ and is connected. If x and y are in C and D, respectively, choose z in A. Again, $\overline{zx} \cup \overline{zy} \subset f - f'$. Any two points of $f - f'$ are therefore in the same component of $f - f'$; hence $f - f'$ is connected.

Proposition 6: *G is semi-projective* [2) in the definition of a spherical m-arrangement].

Proof: Suppose f and f' are k-1-flats contained in some k-flat g and $f \neq f'$. We must prove that $f \cap f'$ is a k-2-flat, $1 \leq k \leq m$. The proposition is trivial for $k = 1$. Suppose $k = 2$. Then if $\dim(f \cap f') \neq 0$, $f \cap f' = \phi$. Now f' disconnects g into convex components A and B. Since $f - f' = f$ is connected, $f \subset A$, or $f \subset B$. If $f \subset A$, then A contains two points from some 0-flat, hence is not convex; therefore $f \not\subset A$. Similarly, $f \not\subset B$. It follows then that $f \cap f' \neq \phi$; hence $f \cap f'$ is a 0-flat.

Assume Proposition 6 is true for $k - 1 \geq 2$, but $\dim(f \cap f') < k - 2$. By Proposition 5, f does not disconnect f. Again, however, f' disconnects g

into convex components A and B. Therefore $f - f' \subset A$, or $f - f' \subset B$. But then either A or B must contain some 0-flat, and hence could not be convex. Therefore $f \cap f'$ is a k-2-flat and the proposition is proved.

Proposition **7**: *Let f be a k-1-flat contained in a k-flat g; then f disconnects g into convex components A and B which are open in g. Then f = $\mathrm{Fr} A$ in g = $\mathrm{Fr} B$ in g.*

Proof: If $f \neq \mathrm{Fr} B$ in g, there is a point x of f and a neighborhood U in g of x such that $U \cap A = \phi$, or $U \cap B = \phi$. Choose y in A. Then $f_1(x,y) \cap f = \{x,x'\}$, where x' is antipodal to x. If $\{x,x'\} \subset U$, then $f_1(x,y)$ is not connected. If only x is in U, then x' disconnects $f_1(x,y)$, a contradiction of Proposition **2**.

Corollary **3**: *In the situation of Proposition* **7** *if $W \subset f$, then $A \cup W$ and $B \cup W$ are connected. Moreover, if W is convex, then $A \cup W$ and $B \cup W$ are also convex.*

This corollary follows from Proposition **7** and Corollary **1** of Proposition **3**, together with the well-known fact that if A is connected and $A \supset B \supset \mathrm{Cl} A$, then B is connected.

Proposition **8**: *If $S = \{x \;, x \;, \ldots, x_k\}$ is a linearly independent set, then S has a convex hull.*

Proof: Because of Corollary **2** of Proposition **3**, it suffices to show that S is contained in one convex set. We know the proposition is true for $k = 1$. Assume it is true for $k - 1 \geq 1$. Then $S_k = S - \{x_k\}$ has a convex hull in $f_{k-1}(S_k)$. Now $f_{k-1}(S_k)$ disconnects $f(S)$ into convex components A and B with x_k in A. Then by the corollary to Proposition 7, $A \cup C(S_k)$ is convex and contains S.

Proposition **8**, which is 3) in the definition of a spherical m-arrangement, completes the proof that if i)-iv) of Theorem 1 are assumed, then we have a spherical m-arrangement. We now show that if we have a spherical m-arrangement that i)-iv) hold. Assume therefore that X and G form a spherical m-arrangement. 1) is identical to i) and 5) is identical to ii). It remains to prove iii) and iv). iii), however, follows at once from Lemma **2** of [2], hence we direct our efforts to proving iv).

Suppose x is a point of X and U is any neighborhood of x. Let f be any m-1-flat which does not contain x. Then f disconnects X into convex open components A and B; assume $x \varepsilon A$. Then $A \cap U \subset U$ is a neighborhood of x. Now by 4) of the definition of a spherical m-arrangement and the results of [1], the half-spaces of X intersected with A form a subbasis for the topology of A, hence $A \cap U$ contains a finite intersection W of half-spaces such that $x \varepsilon W$. Therefore iv) is proved and the proof of Theorem 1 is complete.

From the results of [1] and this paper, we are led to make the following definition:

Definition: Let a space X have a geometry G of length $m - 1 \geq 0$. Then and G form an *open* (n,m)-*arrangement* if:

i) Each 0-flat consists of precisely n points.
ii) If f is a k-1-flat and g is a k-flat with $f \subset g$, then f disconnects g into $\max(2,n)$ convex components which are open in g, $0 \leq k \leq m$.
iii) Each 1-flat is connected.
iv) If f is an m-1-flat, then we call the components of X-f *half-spaces* of X. The collection of half-spaces of X forms a subbasis for the topology of X.

Thus, an open m-arrangement is but an open $(1,m)$-arrangement and a spherical m-arrangement is an open $(2,m)$-arrangement.

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