

RECONSTRUCTING FORMAL LOGIC:
FURTHER DEVELOPMENTS AND CONSIDERATIONS

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§1. *Introduction.** In [8]¹ I argued for the desirability of founding deduction theory on a system of "pure" first order logic in which:

- (a) all formulas that may appear as lines in derivations, *i.e.* all formulas that are construed as statements or as statement forms, when fully written out in primitive notation contain no individual variables free and do not contain singular terms that are not subject to quantification (proper nouns² or dummy symbols used in their place in statement forms);
- (b) only formulas that are valid in every domain, (including the empty domain) are theorems.

Of a first order quantification system that satisfies condition (a) we say that it is *primitively general*,³ and of one that satisfies condition (b) we say that it is *non-existential*.⁴ 'Primitively general' is a syntactical predicate,

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while 'non-existential' is a semantical predicate. Some systems referred to in §3, §4, §5, and §6 of this paper are non-existential but not primitively general. The completeness of a non-existential system that is not primitively general depends on how validity is conceived for its formulas that contain individual variables free or contain singular terms not subject to quantification, if such formulas are allowed as theorems of the system.

The primitively general non-existential system \mathbf{L} described in detail in [8] can be summarized as follows. The primitive vocabulary of \mathbf{L} consists of a denumerably infinite set of k -place predicate symbols for each $k \geq 1$, a denumerably infinite set of individual variables, the two parentheses, and the truth-functional connectives ' \supset ' and ' \sim '. Other truth-functional connectives and the symbol ' \exists ' are defined as usual. By the formation rules, there may not be in any *formula* (referred to in [8] as wellformed formula) vacuous quantifier occurrences, two or more occurrences of a given quantifier with overlapping scopes, or both free and bound occurrences of a given variable (the closure of a formula being thereby always a formula). In the *closure* (referred to in [8] as the *standard closure*) of a formula ϕ , all variables free in ϕ occur in the prefix to ϕ in alphabetical order from left to right (as in the 1941 edition of Quine's *Mathematical Logic*). Using here a more streamlined syntactical notation than in [8], we prefix ' \vdash ' to a quasi-quotation or to a quotation to assert that the closure of every formula of the form specified by the quasi-quotation or, respectively, the closure of the quoted formula is a theorem, and we follow the notational conventions of [15] also in other respects, indicating any departures from them. The dot cluster punctuation system of [15] is here assumed to be adopted for abbreviating the formulas of \mathbf{L} and is accordingly used also in quasi-quotations. The axioms of \mathbf{L} are the formulas on which theoremhood is conferred by the following meta-axioms:

$$MA1. \vdash \sim \phi \supset \phi \supset \phi^{\neg}$$

$$MA2. \vdash \phi \supset \sim \phi \supset \psi^{\neg}$$

$$MA3. \vdash \phi \supset \psi \supset \psi \supset \chi \supset \phi \supset \chi^{\neg}$$

$$MA4. \vdash (\alpha)(\phi \supset \psi) \supset \phi \supset (\beta) \beta / \alpha \psi^{\neg}$$

$$MA5. \vdash (\alpha)(\phi \supset \psi) \supset (\alpha) \phi \supset \beta / \alpha \psi^{\neg}, \text{ provided } \alpha \text{ occurs in } \psi.$$

Here $\beta / \alpha \psi$ is understood to be the formula that is like ψ except for containing free occurrences of β wherever ψ contains free occurrences of α . *Modus ponens* is the sole primitive rule of derivation of \mathbf{L} .

In the version of \mathbf{L} that we adopt here, we appoint as predicate symbols all capital italic letters in the English alphabet from '*F*' included on (rather than just '*F*', '*G*', and '*H*' as in [8]) with superscripts (which in most contexts may be omitted in abbreviated notation) indicating the intended number of argument places, and with or without numerical subscripts. Apart from changes in syntactical notation, this enlargement of the vocabulary is the only modification that we introduce here into the primitively general, non-existential formulation of logic that was referred to in [8] as the system \mathbf{L} . Thus modified, the system remains virtually the same, and we may hence refer to it as a version of the system \mathbf{L} or simply as the system \mathbf{L} .

The first three meta-axioms of \mathbf{L} may be replaced by a single one decreasing the closure of any tautological formula to be a theorem, in analogy to the systems of [15] (pp. 88, ff.) and [16]. Thereby the number of the meta-axioms of the system is reduced to three and the development of the system is somewhat simplified at the price of the non-independence of its axioms. The system of [16] and its variants described in [9] are other primitively general non-existential systems.

By 'predicates', properly speaking, we mean symbols interpreted, *i.e.* given a meaning, so as to be true or false of specific ordered sets of objects. Rather than predicates in this sense, the language of \mathbf{L} , contains predicate symbols that, in the abstract study of \mathbf{L} , are construed as dummy predicates. The closed formulas of \mathbf{L} are thereby construed as statement forms—we shall occasionally refer to them as such—that become true or false statements when the predicate symbols in them are given specific interpretations as predicates. Besides proofs, we recognize in \mathbf{L} *derivations from assumption forms*, in which closed formulas that neither are axioms of \mathbf{L} nor are derivable from preceding lines in the derivation are admitted as lines under the label of *assumption forms*. Derivations from assumption forms are construed as patterns for arguments, *i.e.* as argument forms; the assumption forms in them represent premises.

Any set of closed formulas of \mathbf{L} that are not theorems of \mathbf{L} may be taken as the set of axioms of a deductive theory formalized within \mathbf{L} , *i.e.* of a deductive theory employing \mathbf{L} as *underlying logic*. To this end, an infinity of primitive predicate symbols is provided in \mathbf{L} , from which to choose any desired (in general finite) number of them, as the primitive predicate symbols specific of a theory.⁵

Derivations from assumption forms in \mathbf{L} , arguments formalized within \mathbf{L} , and proofs in theories formalized within \mathbf{L} (*i.e.* derivations in \mathbf{L} from the axioms of the theory) are three kinds of derivations that are not distinguishable syntactically. Rather the distinction among them is metacontextual and is not very sharp, though useful: a derivation from assumption forms in \mathbf{L} is studied as an abstract form, as part of the study of the system \mathbf{L} as such; an argument formalized within \mathbf{L} , *i.e.* using \mathbf{L} as underlying logic, or, as we may also say, governed by \mathbf{L} ,⁶ is an application of \mathbf{L} in which the predicate symbols of \mathbf{L} that are used are given specific interpretations and whose interest lies in the material content of the premises and the conclusion; in a deductive theory formalized within \mathbf{L} , there is a set of predicate symbols of \mathbf{L} that alone are used as predicate symbols and, subject to this restriction (which may be vacuous, since the whole vocabulary of \mathbf{L} may be adopted for the theory) the deductive consequences by \mathbf{L} of a fixed set of closed formulas of \mathbf{L} that are not theorems of \mathbf{L} (the axioms of the theory) are investigated systematically. The primitive predicate symbols of a theory need not be given specific interpretations, *i.e.* the theory may be abstract. In a theory in which all symbols are interpreted, a proof constitutes an argument.

In [8] the predicate symbols of \mathbf{L} were described as place markers for predicates: the label intended to suggest that in applications of the system

L to an argument those symbols were to be replaced by other symbols functioning as predicates. However, in providing in a general way for the application of the system L to deductive theories and particular arguments, it is conceptually simpler and more elegant to stipulate, as we do here, that all primitive predicate symbols in a deductive theory or particular argument formalized within L be primitive predicate symbols of L, which, in the case of an argument, are for the occasion given specific interpretations as predicates. A predicate symbol may sometimes be adequately interpreted simply by appointing it to be the semantical equivalent of an ordinary discourse predicate. The stipulation that only primitive symbols of L are to count as *primitive* symbols in any application of L is not to be taken as an impediment to the introduction *by definition* of special expressions to be used in place of individual symbols or strings of symbols in the study of L or any of its applications, as may be convenient for reasons of conciseness, suggestiveness, or linguistic habits.

The deductive theory that is formalized within L by retaining for it all predicate symbols of L and taking some one statement form valid exactly in every non-empty domain (as $(\exists x)(Fx \supset Fx)$) as the theory's sole axiom was referred to in [8, p. 152] as existence logic.⁷ We may more appropriately refer to it as the *theory of non-empty domains*. Most deductive theories, as traditionally formulated, contain, as a part, at least a portion of the theory of non-empty domains, obtained by restricting the latter's vocabulary.

In assuming that the universe is not empty, we need not assume that we can name some specific objects in it, as is presupposed in standard, *Principia*-type quantification systems, hereafter referred to collectively as *standard logic*. Standard logic is stronger than the pure theory of non-empty domains, since it contains as additional theorems statements or statement forms that contain proper names or dummy proper names.⁸

§2. *Primitively general, non-existential quantification systems with identity.* The theory of identity is properly regarded as part of pure logic. As such, it is best treated formally as an integral part of a *system of quantification with identity*, which may serve as underlying logic for stronger theories, and which is obtained from a system of quantification that contains an infinity of k -place predicate symbols for every $k \geq 1$ by selecting among the latter one 2-place predicate symbol to be interpreted as the symbol for identity and adjoining to the axioms of quantification an infinity of appropriate new axioms (or a finite number of new axioms subject to a rule of substitution) containing that symbol.

The procedure for extending the system L to include the theory of identity is essentially the same as that for standard logic. To obtain from L a non-existential system of quantification with identity LI, we enlarge its axiom set to include, as additional axioms, the formulas on which theoremhood is conferred by the following two new meta-axioms:

MAIL. $\vdash 'Ix'$,

MAI2. If ϕ is atomic and contains 'x', and if ϕ' is like ϕ except for containing 'y' at exactly one place where ϕ contains 'x', then $\vdash \ulcorner (x)(y)(Ixy \supset \phi \supset \phi') \urcorner$.

In adopting these two meta-axioms for **LI** we interpret ' I^2 ' as the symbol of identity or sameness. Thereby ' I^2 ' functions in **LI** as a predicate proper rather than as a dummy predicate, and some closed formulas of **LI** (e.g. ' $(x)Ixx$ ', an axiom by *MAII*) are statements rather than statement forms. Hereafter, in **LI**, we write $\ulcorner \alpha = \beta \urcorner$ for $\ulcorner I\alpha\beta \urcorner$ and $\ulcorner \alpha \neq \beta \urcorner$ for $\ulcorner \sim I\alpha\beta \urcorner$.

It is important to note that the interpretation of ' I^2 ' as the symbol of identity is not necessary to the satisfaction of the formulas that are axioms of **LI** by *MAII* and *MAI2*. For instance, if the universe is made up of objects no two of which are of the same size, then those formulas are true statements under any interpretation of their predicate symbols under which ' I^2 ' is used for 'are of the same size'. Thus if ' I^2 ' is uninterpreted or is interpreted otherwise than as the predicate that is true of any objects x and y just in case x and y are the same, then the formulas that are axioms of **LI** by *MAII* and *MAI2*, have no special logical status and may serve as premises of an argument or axioms of a non-logical theory formalized within **L**.

All metatheorems proved in [8] for **L** hold in **LI**. The proofs are the same in all cases except that the proof of *MT3* (which asserts that if $\ulcorner (\alpha_1)(\alpha_2) \dots (\alpha_n)\phi \urcorner$ is an axiom then $\vdash \ulcorner (\beta_1)(\beta_2) \dots (\beta_n)\phi \urcorner$), which for **L** is based on the circumstance that no specific variable is mentioned in any meta-axiom of **L**, must be proved for **LI** at first only for the case that ϕ is an *MA1*, *MA2*, *MA3*, *MA4* or *MA5* axiom, this being sufficient for the proof for **LI** of all subsequent metatheorems proved in [8]; *MT3* for the case in which ϕ is an *MAI2* axiom follows from *MT7* immediately and can be easily proved already from *MT6*. *MT3* is trivially true for the case in which ϕ is an *MAII* axiom.

By *MT6*, we have from *MAII* and *MAI2*:

MTII. $\vdash \ulcorner \alpha = \alpha \urcorner$

MTI2. If ϕ is atomic and contains α , and if ϕ' is like ϕ except for containing β at exactly one place at which ϕ contains α , then $\vdash \ulcorner (\alpha)(\beta)(\alpha = \beta \supset \phi \supset \phi') \urcorner$.

As adapted to a system in which only closed formulas count as statement forms or statements and hence only closed formulas may be theorems, the principle of substitutivity of identicals and other familiar metatheorems of the logic of identity are proved without difficulty for **LI**, essentially as for standard systems of quantification with identity, as long as those metatheorems do not assert the theoremhood of formulas not valid in the empty domain—i.e. do not assert the theoremhood of truth-functional compounds of formulas $\ulcorner (\alpha_1)\chi_1 \urcorner$, $\ulcorner (\alpha_2)\chi_2 \urcorner$, ..., $\ulcorner (\alpha_k)\chi_k \urcorner$ that are not associated with the truth value truth under the assignment of truth to all the $\ulcorner (\alpha_i)\chi_i \urcorner$. Closed formulas that are not valid in the empty domain, such as ' $(\exists x)(x = x)$ ' are not theorems of **LI**.

The notion of validity (i.e. of validity in every domain) as made precise

in [8] for the closed formulas of L (pp. 151, 152), is extended for those of LI containing ' I^2 ' by extending for them the notion of validity in every non-empty domain in the usual manner (see e.g. [1, p. 282]). In view of the completeness of L (*MT 20* [8, p. 152]), it is then easily established that the system LI is complete with respect to validity as the term is defined for its formulas.

The procedure for obtaining a primitively general non-existential quantification system with identity from any primitively general non-existential quantification system other than L is essentially the same as that described above for L .

§3. *Existential and non-existential systems as underlying logics for deductive theories.* In this section we will use some simple illustration to bring home the import of a non-existential system as underlying logic for deductive theories (*cf.* [8, p. 148]). We could not have fruitfully done so before L was extended to become the system LI , for most deductive theories use the theory of identity.

When the axioms of a deductive theory impose special conditions upon the sound interpretations of some predicate symbols occurring in them (unlike the closed formula that may serve as the sole axiom for the theory of non-empty domains), we may refer to those predicate symbols as predicates, even though they be not considered under any specific interpretation (in other words, even though the theory be abstract), as has been customary in the logico-mathematical literature.

The abstract theory of simple ordering—to take an example not involving yet operator symbols or proper noun symbols—may be typically formulated in a non-formalized language as follows:

The elements of a non-empty class C are said to be *simply ordered* by a dyadic relation $<$ iff the following assumptions hold:

- a. if x and y are in C and if $x \neq y$, then $x < y$ or $y < x$.
- b. if x and y are in C and if $x < y$, then $x \neq y$.
- c. if x, y, z are in C , if $x < y$, and if $y < z$, then $x < z$.

Here 'is in C ' and ' $<$ ' should be viewed as the primitive predicates of the theory. Moreover, as long as the theory is not formalized and hence no clear distinction exists between a formal part and an informal context, the non-emptiness of the class C may be viewed as assumed axiomatically, though the statement expressing it be not listed among the enumerated assumptions or axioms. Identity is treated in such a formulation as a relation governed by the rules of an informal underlying logic, which becomes a formal quantification system with identity when the theory is formalized.⁹

In formalizing a theory such as the above, it is customary to dispense with the primitive predicate 'is in C ' by the device of restricting the "universe of discourse" to the domain of objects that are in C , *i.e.* by the device of regarding the domain C as exhausting the universe. Furthermore, in standard formalizations, the non-emptiness of the universe thus restricted is presupposed in the underlying logic used. The axioms of the

theory then become (with variations that depend upon the exact syntactical rules of the language used):

- $a'. (x)(y)(x \neq y \supset \blacksquare Kxy \vee Kyx)$
- $b'. (x)(y)(Kxy \supset \blacksquare x \neq y)$
- $c'. (x)(y)(z)(Kxy \blacksquare Kyz \supset \blacksquare Kxz)$

Hereafter, we will write $\lceil \alpha < \beta \rceil$ for $\lceil K\alpha\beta \rceil$.

If instead of standard logic, we use **LI** as underlying logic for the theory of simple ordering and wish the universe of discourse to be non-empty, then the latter condition, which is not a logical truth, must be postulated explicitly by means of an additional axiom such as $\lceil (\exists x)(x=x) \rceil$, or $\lceil (\exists x)(y)(x < y \supset \blacksquare x < y) \rceil$. Let it be the first and let us refer to it as d' .

Let us note that *an existence assumption like d' hardly adds anything of interest to our theory*, since in the study of abstract simple ordering all interesting theorems are universal. Contrary to what is perhaps a prevailing impression, the same can be said of all deductive theories in which one does not postulate the existence of objects of a specific kind within the universe of discourse. On the other hand, in deductive theories in which the existence of objects of a specific kind within the universe of discourse is postulated, as a geometry in which it is axiomatically asserted that there are at least three points or a group theory in which it is axiomatically asserted that there is an identity element, no additional general assumption of existence of elements in the universe of discourse as a whole is required if **LI** is used as the underlying logic. By the same token, *in theories of the later kind an underlying existential system of logic is stronger than it needs to be for the derivation of any theorem*.

Once more, consider the use of **LI** for derivations from the axioms $a'-c'$ or $a'-d'$ as written above. It is best to view the resulting calculus as an abbreviated formalism, in which every expression ϕ that ostensibly constitutes the scope of a quantifier $\lceil \alpha \rceil$ is an abbreviation of $\lceil (\Lambda \alpha \supset \phi) \rceil$, wherein Λ is some fixed one-place predicate symbol—let it be F^1 —used for 'is in C '. Thereby, for instance, anything written as $\lceil \sim(\alpha) \sim \phi \rceil$ or as $\lceil (\exists \alpha) \phi \rceil$ is short for the formula $\lceil \sim(\alpha)(F\alpha \supset \sim \phi) \rceil$, which is equivalent to $\lceil \sim(\alpha) \sim \sim(F\alpha \supset \sim \phi) \rceil$, which may be abbreviated as $\lceil (\exists \alpha)(F\alpha \blacksquare \phi) \rceil$. Thereby, not even the restriction to a special universe of discourse need be regarded as presupposed, *i.e.* as assumed extrasystemically in the informal context of the formal system. Thus understood, the above writing of d' is short for $\lceil \sim(x)(Fx \supset \blacksquare x \neq x) \rceil$, equivalent to $\lceil (\exists x)(Fx \blacksquare x = x) \rceil$. If it is then desired to assume that the set of objects that are in C is not empty, instead of the proposed d' we may use as fourth axiom the simpler $\lceil (\exists x)Fx \rceil$, with the understanding that in the abbreviated procedure the latter's occurrence as a line in a derivation may be omitted.

Let standard logic, which admits open formulas as lines in derivations, be applied to a set of axioms assumed to hold in a restricted domain C . If we wish to view the procedure as shorthand for the application of the same system of logic to appropriate axioms in an unrestricted universe, then an expression ϕ containing a variable α free has to be interpreted as short for

$\lceil \Lambda \alpha \supset \phi \rceil$, where ' Λ ' is a one-place predicate chosen to stand for 'is in C ', not only when ϕ ostensibly exhausts the scope of an occurrence of $\lceil (\alpha) \rceil$, but also when it ostensibly exhausts the largest expression of which it is a part. Apart from this detail, when standard logic is applied directly to the axioms $a' - c'$ as originally written above, no further axiom is needed to insure the non-emptiness of the domain C *only as long as the procedure is not viewed as an abbreviated one in the manner indicated*. In fact, by standard logic, for given ϕ and α , the formula $\lceil (\exists \alpha)(F\alpha \supset \phi) \rceil$, for instance, can be derived from the formula $\lceil (\alpha)(F\alpha \supset \phi) \rceil$, but from the latter it is not possible to derive the formula $\lceil \sim(\alpha)(F\alpha \supset \sim \phi) \rceil$ equivalent to $\lceil (\exists \alpha)(F\alpha \bullet \phi) \rceil$. Hence, in a standard formalization of simple ordering, for given α and ϕ , one cannot interpret the derivation of $\lceil (\exists \alpha)\phi \rceil$ from a theorem $\lceil (\alpha)\phi \rceil$ as shorthand for the derivation of $\lceil (\exists \alpha)(F\alpha \bullet \phi) \rceil$ from $\lceil (\alpha)(F\alpha \supset \phi) \rceil$ where ' F^1 ' stands for 'is in C '.

We see thus that, if a formalization of simple ordering in a restricted universe of discourse C is to be regarded as an abbreviated procedure for a formalization in an unrestricted universe and if it is desired to base the theory on the assumption that C is not empty, then either the set of axioms must contain an item like d' which as written above is viewed as abbreviated, or the axiom ' $(\exists x)Fx$ ', (where ' F^1 ' stands for 'is in C ') must be regarded as omitted in the abbreviated procedure, *whether standard logic or a non-existential system is used*, and hence standard logic is superfluously strong.

Needless to say, what has been just remarked about simple ordering holds for any deductive theory that in standard formalizations does not have axioms $\lceil (\exists \alpha)\phi \rceil$ or their equivalents and whose axioms do not contain proper noun symbols. We already saw earlier that standard logic is superfluously strong as underlying logic for theories with specific existence axioms even if the procedure is not regarded as abbreviated and we shall take up later the case of theories in which proper noun symbols are used primitively.

Nothing that has been said in this section should be interpreted as an objection to, or even as a frowning upon customary theory formalizations that can be viewed as abbreviated procedures. The plea here is for an awareness of the justifiability of customary formal procedures in terms of a non-existential logic and explicit formal axioms. That is why we are speaking of reconstructing formal logic rather than of destroying anything. The attitude that we are taking is similar to that of the rigorous mathematical analyst toward the customary terminology of actual infinitesimals used by the applied mathematician and the physicist. The analyst is not interested in depriving the practitioner of the differential and integral calculus of his convenient and suggestive ways of expression, provided these be recognized as a shorthand for a clear, rigorous treatment of the subject in terms of limits in the ϵ - δ -sense. The errors that an unbridled use of the shorthand language of infinitesimals may lead to are as a rule avoided thanks to the "mathematical sense" of the user. No doubt, iron-clad guarantees against the occurrence of such errors could also be built into the language of infinitesimals by subjecting it to precise syntactical rules

determining the procedures permitted. Even so, in the name of conceptual clarity, the analyst may be expected to demand that such language be viewed as a shorthand, just as we like here to view standard formalizations of deductive theories as shorthand for more elaborate ones in which the underlying logic is non-existential, *i.e.* pure logic, and all non-logical assumptions are explicit assumptions in the formal language.

§4. *Open formulas with valid closures as theorems of non-existential systems.* Free occurrences of variables in lines of derivations was proscribed in setting up the system L in order to avoid the surreptitious introduction of existence assumptions into the derivations. As was noted in [8, pp. 131-132], such theorems of standard logic as ' $(x)Fx \supset Fy$ ' or ' $Fy \supset (\exists x)Fx$ ', of which at least one is usually an axiom, are valid in *every* domain if they are construed as statement forms semantically equivalent to their closures, *i.e.* as forms of statements expressing universal propositions by means of variables that occur free. Yet formulas not valid in the empty domain are derivable from those formulas and other axioms, because the conclusion of a derivation by *modus ponens* from a conditional and its antecedent containing variables free whose closures are true statements or valid statement forms is not necessarily a true statement or, respectively, a valid statement form, unless it contains some variable free (in its role of expressing universality) or the universe of discourse is not empty. It is this legalized slight of hand in proving as theorems formulas not valid in the empty domain that was mainly objected to in [8].

Some logicians have proposed non-existential systems of quantification in which open formulas are admitted as lines of derivations harmlessly, thanks to a suitable restriction on the application of *modus ponens* to them (Mostowski [14], Hintikka [3], Schneider [20], and recently Kearns [7]). In these non-primitively general systems, as in standard logic, an open formula is considered valid and is provable as a theorem just in case its closure is valid.¹⁰ We shall refer to such systems as *restricted ponens systems*.

The exponents of restricted *ponens* systems differ somewhat in their account of the semantics of open formulas, but essentially the latter formulas may be viewed in the context of these non-existential systems either (a) as statements in which variables are used free to express universality or forms of such statements, or (b) as forms of statements that in place of the variables occurring free in the forms contain nouns presupposed to name individual objects. In the latter case, open formulas are regarded as vacuously valid in the empty domain: all statements of their form are true, but if the universe is empty there are no statements of their form nor *a fortiori* false statements of their form, since there are then no objects to be named. The construal put upon the open formulas of a restricted *ponens* system materially affects the manner in which it may be applied. Proper nouns may occur as primitive symbols in an argument using a restricted *ponens* system as underlying logic only if construal (b) is put upon the latter's open formulas.

It is important to note that under construal (b) of its open formulas, a

restricted *ponens* system is like standard logic in the following respect: in a discourse governed by it, the only primitive proper nouns that are admitted as significant, *i.e.* as conferring significance as statements to the expressions in which they occur, are those that name individual objects, *i.e.*, those that are proper names. In fact, in a restricted *ponens* system, if ϕ is a formula containing a variable α free and if ϕ' is like ϕ except for containing a free occurrence of a variable β in place of a free occurrence of α , then $\lceil \phi \supset (\exists \beta)\phi' \rceil$ is a theorem, and hence all statements of its form must be valued as true, as cannot be done if expressions containing proper nouns that do not name in place of the free variables of a form are countenanced as statements.

To restricted *ponens* systems, there are no philosophical objections of the kind raised in [8] against standard logic. As we shall presently see, there are none the less reasons for preferring a primitively general system like **L** to a restricted *ponens* system.

Especially under construal (b) of its open formulas, a restricted *ponens* system acquires some features that seem awkward or outright objectionable to this writer. To begin with, under that construal of its open formulas, statements of the form of an open formula that is a theorem (of which statements there need not be any), though they are all true, are not logical truths in the sense in which non-existential systems are intended to do justice to this concept. For, on the one hand, under a rule of significance that recognizes primitive proper nouns as significant only if they are proper names, any statement containing a primitive proper noun has, semantically speaking, any statement $\lceil (\exists \alpha)(\phi \supset \phi) \rceil$ as a consequence, whether or not a formula of the form of such a statement is derivable from an open formula by the syntactical rules of the system of logic governing the discourse; and, on the other hand, in any discourse using primitive proper nouns that is governed by a restricted *ponens* system of logic, some statements containing proper nouns are of the form of theorems of that system. Furthermore, if construal (b) is put on its open formulas, a restricted *ponens* system is not deductively complete, in the sense that, in a discourse governed by it, a statement that semantically is a consequence of another statement is not always derivable from it: a statement $\lceil (\exists \beta)\Lambda\beta \rceil$ wherein Λ is a one-place predicate, is semantically a consequence of, but is not in the system formally derivable from the corresponding statement $\lceil \Lambda\alpha \rceil$ wherein α is a proper noun and hence, by the rules of significance of the system, a proper name.

If construal (a) is put upon the formulas of a restricted *ponens* system, the above criticism does not apply. On aesthetic or quasiaesthetic grounds, however, it seems preferable to this writer to sanction the use of *modus ponens* primitively only where the truth and validity preserving character of this rule is transparent on the strength of purely truth-functional considerations. Which it is not when the rule is applied, under restrictions, on open sentences that are construed as synonyms of their closures, *i.e.* as elliptical statements.

Also if construal (b) is put upon the open formulas of a restricted

ponens system, it is not on purely truth-functional considerations that one justifies a restricted use of *modus ponens* on the ground that, to an application of *modus ponens* to open formulas that is unauthorized under the restriction, there need not correspond any derivation of a statement from statements of their form.

Nothing that has been said above about restricted ponens systems should be construed as constituting an objection to the use of open formulas in a primitively general non-existential system in the capacity of syntactically introduced abbreviations of their own closures. A restricted rule of *modus ponens*, applicable to such elliptical formulas directly, may be adopted as a derived rule of derivation in a primitively general non-existential system, upon proof of the appropriate metatheorem, as *Cor2-MT2* in \mathcal{L} [8, p. 140]. In [8] the possibility of adopting such convenient abbreviated notation in \mathcal{L} was intended to be tacitly understood.

§5. *“Intended names” and their elimination in formal discourse.* In ordinary discourse, not all expressions that grammatically are proper nouns name individual objects. When they do not, as in the case of ‘Pegasus’, they are not proper names as we wish to use the term here. For short, hereafter, we shall refer to proper nouns and to those among them that are proper names respectively as nouns and names.

Nouns occurring in the statements of a formal language under a rule of significance requiring that they be names may be referred to as *presupposed* names. The primitive nouns in a discourse governed by standard logic or a restricted *ponens* system are presupposed names in that discourse. The rule of significance that makes them so is necessary to the soundness of those systems. In a discourse governed by standard logic or a restricted *ponens* system, a statement-like expression containing, as a predicate argument, a primitive symbol to which is assigned the meaning that ‘Santa’ has in ordinary language could not be regarded as not true on account of the fact that there is no such thing as Santa; rather, on that account, it would have to be regarded as ill-formed or ill-interpreted (depending on whether or not the use of that symbol is syntactically allowed), and hence as an expression that is not a statement.¹¹ Thus all presupposed names in the statements of a formal language are names. But not all names that may occur in a language are presupposed names in that language, for nouns occurring in a language admitting them as significant even if they do not name may nonetheless happen to be names.

It is clear that presupposed names cannot occur as primitive symbols in theorems of a non-existential system. Dummy presupposed names or place markers for presupposed names may, we have seen, be countenanced in the theorems of a non-existential system at the cost of not requiring that every formula that may occur as a line in a derivation be of the form of some statement, *i.e.* have some interpretation as a statement; on the strength of the considerations of the last section, it does not seem desirable to this writer thus to structure a non-existential system.

There remains however an important question, to the consideration of which this section is devoted. In setting up a non-existential system that is

to be adequate as underlying logic for all deductive theories and particular arguments, is it necessary, and if not is it desirable to provide for the occurrence of primitive noun symbols (symbols occupying the syntactical position of nouns) in assumption forms, letting them endow the formulas in which they occur, when the latter are interpreted, with existential import in the same manner as in standard logic?

From **LI**, a non-existential system **LI***, endowed with the feature just described, is easily obtained as follows:

1. The primitive vocabulary of **LI** is enriched with a denumerable infinity of noun symbols.
2. The formation rules of **LI** are modified to allow noun symbols to occur in formulas as unquantified arguments of predicate symbols. The new formulas count as closed *i.e.* as statement forms, if they do not contain any variable free and hence are legitimate lines in derivations, but the syntactical symbol '⊢' must now be understood to refer only to formulas that do not contain noun symbols.
3. To *modus ponens* is adjoined a second primitive rule of derivation, authorizing the immediate derivation of a closed formula ϕ containing a noun symbol α from two formulas of which one is any closed formula χ containing α and the other is any formula $\ulcorner(\beta) \beta/\alpha \phi \urcorner$ (where β is foreign to ϕ , as is necessary to its wellformedness).

Note that noun symbols never occur in the theorems of **LI***, which are exactly those of **LI**. If $\ulcorner(\beta)\psi \urcorner$ is a theorem of **LI*** and if α is a noun symbol, $\alpha/\beta\psi$ is not a theorem of **LI***. In **LI***, $\ulcorner(\beta)((\gamma)\phi \supset \beta/\gamma \phi) \urcorner$, and hence from any assumption form containing a noun symbol α , one can derive a formula $\ulcorner(\gamma)\phi \supset \alpha/\gamma \phi \urcorner$. Thus it is clear that **LI***, though it lacks some of the theorems of standard logic with identity (those that are not valid in the empty domain) has the same force as the latter with respect to what can be derived from assumption forms containing noun symbols and has hence the same force as the latter when used as underlying logic for theories containing noun symbols (so-called constants) primitively.¹² The usual meta-theorem of deduction clearly does not hold in **LI***, though a weakened form of it does.

When a primitive noun symbol α is given meaning as a noun (in a way yet to be made clear) in an argument employing **LI*** as underlying logic, it need not be presupposed, *i.e.* extrasystemically assumed that it names, as in standard logic. Rather the assumption that α names may be viewed as introduced into the argument formally, if only implicitly, by regarding any premise (interpreted assumption form) in which α occurs as a statement that is not true (*i.e.*, is false, or, to accommodate Strawson's point of view, valueless) unless α does name. Interpreted noun symbols functioning in this fashion in a formal language may be referred to as *asserted names*. By construing the noun symbols in interpreted assumption forms of **LI*** as asserted names, we avoid making the significance of an expression depend upon an extralinguistic fact, as when presupposed names are used.

Let us note that a noun symbol introduced by definition as an

abbreviation of a definite description with maximum scope (the definite description being itself an item of abbreviated notation, in Russell's sense) is an asserted name when all the symbols in its definiens are interpreted, whether it is used in standard logic, in LI or in LI*. When such a defined asserted name α is used in a formal language, then within the same formal language it is possible to express a proposition to the effect that α does not name, as cannot be done either in the case of a presupposed name or in the case of a primitive asserted name. In the case of a statement containing an asserted name α , neither it nor the result of prefixing the negation symbol to it are true if α does not name—the two statements are inconsistent, but they are contrary, not contradictory.

With respect to their consequences, premises with primitive asserted names have the same strength as those with presupposed names. In many contexts, therefore, we need not distinguish between the two kinds of nouns, important as the distinction is conceptually. We will refer collectively to presupposed and asserted names as *intended names*. The label is conveniently applied also to nouns that occur in ordinary language so as to imply that there are objects named by them, though it be not determined by any rule of grammar or composition whether they are asserted or presupposed names.¹³

In discoursing about a formal deductive theory, it is most of the time terminologically convenient to speak as if the theory were soundly interpreted, even if it is not being considered under any specific interpretation. In this spirit, in §3 we agreed that, in the case of abstract deductive theories whose axioms impose conditions upon the sound interpretations of their predicate symbols, we would refer to the latter as predicates. If such a theory is formalized within either standard logic or LI*, the primitive noun symbols occurring in its axioms are names under any of its *sound* interpretations. We will most of the time refer to them as names, speak of objects named in the theory, and use similar locutions, though we may have no specific sound interpretation in mind.

In [8, note 17, p. 156], I stated that, in abstract deductive theories, in whose context there could be no question of exhibiting any of the objects named, there was no need of names (or constants, as I called them) in their primitive vocabulary.¹⁴ As I briefly remarked there, any object named in such a theory is identified by axiomatically asserting or proving of it that it uniquely satisfies certain conditions, and hence any symbol intended to function as its name can and properly ought to be introduced into the theory by definition as a definite description, *i.e.*, as a device for abbreviating formulas containing a clause asserting the object's existence and uniqueness. We shall now proceed to elaborate on and to elucidate the above contention.

The use of names in formalized theories resemble, in certain respects, that of compound terms, *i.e.* of terms¹⁵ constructed out of other terms with the help of operators (or operator symbols, as symbols playing the syntactical role of operators may be referred to when they are primitively supplied in infinite number in a system intended to serve as underlying logic

for any deductive theory¹⁶) and punctuation marks.¹⁷ The use of such compound terms as arguments of the predicate of identity with a suitable logic of identity amounts to the implicit assumption (actually a presupposition, as we have been using this word) of the existence and uniqueness of the result of an operation on any objects. It is well known how those assumptions can be made explicit by regarding the compound terms as defined within a formal language in which operators are not primitive. To this end, a term consisting, besides punctuation, of a k -place operator and of its arguments a_1, a_2, \dots, a_k , which are terms that may in their turn be compound, is understood to be short for a definite description $\ulcorner (\exists \beta)(\Lambda a_1 a_2 \dots a_k \beta) \urcorner$ (where Λ is a $(k+1)$ -place predicate), occurring with appropriate scope as part of an abbreviated formula in Russell's sense. If the formulas containing compound terms are to be read thus, it is better, as a rule, to choose a different set of axioms for a given theory than is done when the operators are primitive, to keep the assertions of existence and uniqueness from occurring in an axiom in conjunction with other assertions and from recurring superfluously in more than one axiom.

Let us illustrate the use of operators and of names as primitive symbols in the case of group theory. In a non-formalized language, group theory may be typically formulated as follows:

- A "system" consisting of a non-empty class C and the two operations $^{-1}$ and \circ is a group iff the following axioms hold for it:
- a. If x is in C , then x^{-1} is a uniquely determined member of C .
 - b. If x and y are in C , then $x \circ y$ is a uniquely determined member of C .
 - c. If x, y, z are in C , then $(x \circ y) \circ z = x \circ (y \circ z)$.
 - d. e is in C , and if x is in C , $x \circ e = x$
 - e. If x is in C , then $x \circ x^{-1} = e$.¹⁸

In formalizing such a theory, often not only the explicit reference to a class C , *i.e.* the explicit use of a primitive predicate corresponding to 'is in C ' is suppressed, as we have seen in the case of simple ordering, but through the use of operators as primitive symbols with the proper underlying logic, the explicit postulation of the closedness of the system, the all important group characteristics expressed by axioms a and b above, is avoided. Thus, typically, in a formalized group theory, the axioms may be (see [22, p. 105]).

- c'. $(x)(y)(z)(x \circ (y \circ z) = (x \circ y) \circ z)$
- d'. $(x)(x \circ e = x)$
- e'. $(x)(x \circ x^{-1} = e)$

Plainly, to say of any x, y and z that $x \circ y = z$ is to say that they stand in a certain triadic relation H^3 , and the use of the symbol of identity in the expression merely serves to insure implicitly, by the use of the appropriate logic, the existence and uniqueness of the third term¹⁹ of the relation for any given pair of objects. By the recognized way, previously referred to, of dispensing with operators as primitive symbols, upon restriction of the universe of the discourse to the class C , a set of axioms for group theory is the following:

- a'' . $(x)(\exists y)(z)(Gxz \equiv \bullet z = y)$
- b'' . $(x)(y)(\exists z)(u)(Hxyu \equiv \bullet u = z)$
- c'' . $(x)(y)(z)(u)(v)(w)(Hxyu \bullet Hyzv \bullet Huzw \bullet \supset Hxvw)$
- d'' . $(x)Hxex$
- e'' . $(x)(y)(Gxy \supset Hxye)$

If $\ulcorner \alpha^{-1} \urcorner$ is short for $\ulcorner (\gamma)G\alpha\gamma \urcorner$ and $\ulcorner \alpha \circ \beta \urcorner$ is short for $\ulcorner (\gamma)H\alpha\beta\gamma \urcorner$ (where α and β are understood as syntactical variables ranging over all variables, all definite descriptions, and 'e'), each with a scope coinciding with the common part of the scopes of the quantifiers binding the variables free in them, then c' , e' , and d' are provable from the above axioms by standard logic with identity or by LI*. Indeed, by either logic, c' , d' and e' in conjunction are deductively equivalent to the conjunction of the axioms a'' - e'' (while severally, they are stronger than c'' , d'' and e'' respectively).

Since a name occurs in the axioms a'' - e'' , no additional axiom is needed if the non-existential system LI* is directly applied to the axioms in the form in which they are written above. By the same token, an existential system is stronger than necessary as underlying logic for the axioms as written above. Whether an existential or a non-existential system is used, the restriction of the universe of discourse to C may be viewed as an abbreviated procedure by regarding all formulas as abbreviated in the manner we indicated when considering simple ordering and further assuming 'Fe' (where ' F^1 ' is used for 'is in C ') as an additional axiom, omitted in the abbreviation. Hereafter, the possibility of viewing the formalization of group theory within a universe of discourse C as an abbreviated procedure will be assumed as understood.

In the formulation of an argument or deductive theory, *it is not necessary ever to use primitive noun symbols as intended names or dummy intended names*, nor is it ever necessary therefore to use a system like LI*, rather than one like LI, as underlying logic for a theory or argument. For intended names can be dispensed with as primitive symbols by a device like that proposed by Quine [15, pp. 149-50]; or [17, pp. 7-8], similar to that whereby compound terms are primitively avoided. That is, instead of using a primitive intended name α , we can use a primitive one-place predicate Λ of which we assert axiomatically that it is true of exactly one object, and make α , if used at all, synonymous with $\ulcorner (\gamma)\beta \urcorner$ used with maximum scope. Thereby we make explicit the assumption, implicit in the use of α as a primitive intended name, that there is an object which uniquely bears the characteristics whereby we identify it as the object named by α . However, my remarks in [8, note 14] were intended to make a stronger claim than that of the possibility of dispensing with primitive intended names by the method just indicated. What I wished there to maintain and will now explain in detail is the *possibility and desirability of primitively dispensing with names in any of the abstract deductive theories actually studied in mathematics, by suitably modifying the theory's axioms, without introducing any new predicates into its vocabulary.*

For some theories formulated with primitive names, alternative formulations are known, which are free of primitive intended names, but

otherwise have the same primitive vocabulary. Such are for instance the formulation of group theory given by Suppes in [22, p. 113], and that of Boolean algebra given by Rosenbloom in [18, p. 9], in which latter neither the universal nor the empty class are primitively named. But, to this writer's knowledge, the general conditions under which primitive names can be dispensed with in a theory without the introduction of fresh primitive predicates have never been spelled out, though they are contained implicitly in Beth's definability theorem.

Speaking informally, when an object is primitively named in a mathematical theory, as in the various formulations of group theory that we have considered so far, it is on the tacit ground that what is said of it in the axioms collectively is, within the theory, provably false of every other object. For, if all that is asserted in the axioms of the object named could, consistently with the rest of the theory, be assumed to be true also of other objects, then there would hardly be any point in referring by name to a particular object, which had such-and-such characteristics but could not be identified on the basis of anything said of it in the theory, rather than assuming the existence of objects having those characteristics and speaking about all of them.²⁰ But is it proper to name an object primitively in a theory on the tacit understanding that a proof of its uniqueness with respect to what is said of it in the axioms is forthcoming? It seems more in line with the ideal of a systematic deductive development of a theory—which should remain free of any contamination, as it were, by knowledge gained through some prior process of discovery—to assert in the axioms only the existence of an object having such-and-such characteristics, and only subsequently to introduce an asserted name for it by an abbreviative definition in use, upon proof of its uniqueness with respect to those characteristics.²¹

To accomplish this for group theory, for instance, we can replace d'' and e'' , in the last set of axioms we considered for that theory, with the single axiom ' $(\exists x)((y)Hyxy \cdot (y)(z)(Gyz \supset Hyzx))$ ' or, equivalently, with the following two:

$$\begin{aligned} d''' &. (\exists x)(y)Hyxy \\ e''' &. (x)((y)Hyxy \supset (y)(z)(Gyz \supset Hyzx)). \end{aligned}$$

This is noted without prejudice to the possibility of alternative, simpler formulations of group theory that make no use of primitive names, as that of [22, p. 113] or the corresponding one in which primitive predicates are used in place of operators. The method we have here illustrated for eliminating a primitive name is applicable just as well, of course, to formulations of an abstract theory that are not formalized or are formalized with the help of primitive operators, as the first two formulations of group theory considered above.

Speaking now in more precise syntactical terms, let S be a formulation of a deductive theory in which a name α is used primitively, and which consists of a set of axioms used with a system of standard logic with identity or with LI^* (or with either system modified to allow operators as primitive symbols) as underlying logic. Let $\chi_1, \chi_2, \dots, \chi_k$ be all axioms of S that

contain α , and let $\chi_1', \chi_2', \dots, \chi_k'$ be the corresponding formulas in which a variable γ occurs free, wherever α occurs in the χ_i' 's. Then the following points hold, as the interested reader can easily verify:

1. The provability in S of a formula $\ulcorner(\beta)(\phi \equiv \beta = \alpha)\urcorner$ where β occurs free in ϕ is sufficient and necessary for the existence of an alternative formulation S' of the same theory, with the same underlying logic and the same primitive vocabulary, except that α is not used as a primitive symbol in S' . (Note that if there are no primitive names in S' , the use of LI^* for it reduces to that of LI .) The systems S and S' , which use the same primitive vocabulary except that α is not primitive in S' , are here to be understood to be formulations of the same theory in the following sense:

(a) the formulas provable in S' are exactly those provable in S that do not contain α ;

(b) for some fixed formula ψ in which a variable γ occurs free and no other variable is free, upon (i) introduction of the expression $\ulcorner(\gamma)\psi\urcorner$ to abbreviate formulas in accordance with the customary conventions and (ii) the further agreement to use α in S' as short for $\ulcorner(\gamma)\psi\urcorner$ occurring with maximum scope, the formulas provable in S that contain α are exactly the abbreviations containing α of formulas provable in S' (which are also provable in S).²²

2. If a formula $\ulcorner(\beta)(\phi \equiv \beta = \alpha)\urcorner$, where β is free in ϕ , is provable in S , then $\ulcorner(\gamma)(\chi_1' \cdot \chi_2' \cdot \dots \cdot \chi_k' \equiv \gamma = \alpha)\urcorner$ is provable in S .

3. If $\ulcorner(\gamma)(\chi_1' \cdot \chi_2' \cdot \dots \cdot \chi_k' \equiv \gamma = \alpha)\urcorner$ is provable in S , then a formulation S' of the same theory (in the sense explained under 1) is obtained from S by replacing in its axiom set all the χ_i' 's with the single formula $\ulcorner(\exists\gamma)(\chi_1' \cdot \chi_2' \cdot \dots \cdot \chi_k')\urcorner$.

4. If $\ulcorner(\gamma)(\chi_1' \cdot \chi_2' \cdot \dots \cdot \chi_k' \equiv \gamma = \alpha)\urcorner$ is provable in S , we obtain another formulation S'' of the same theory (in the sense explained under 1) as follows: for an arbitrary χ_n among the χ_i' 's, we replace χ_n with $\ulcorner(\exists\gamma)\chi_n\urcorner$ and every $\chi_i \neq \chi_n$ with the corresponding $\ulcorner(\gamma)(\chi_n \supset \chi_i)\urcorner$.

For reasons explained a few paragraphs back, it is a tacit rule in mathematical practice to use a primitive name in a theory only when by a suitable reformulation of the theory that name can be eliminated from its primitive vocabulary without otherwise modifying the latter. However, contrary to what may have been inadvertently suggested by my remarks in [8, note 14], the adherence to this rule is not dictated by any logical necessity. Clearly, it is possible to set up an abstract deductive theory, employing primitive names, in which all that is said in the theory, when it is interpreted, about some object primitively named, can be true of more than one object consistently with the theorems of the theory. However, no useful purpose could be conceivably served by setting up a theory in this form; it would amount to prescribing that, for any given interpretation of the predicates of the theory, one object among more than one that would in general qualify be selected for special attention in being made the subject, by name, of certain theorems—as if anticipating that among the objects that might qualify for the role, there would always be one that would be of special in-

terest for reasons other than anything provable in the theory, an object whose identifying characteristics need not even be expressible in terms of the theory.

The situation is quite different if, instead of abstract, self-contained theories, we consider particular arguments, in which all words used are taken to be well understood, as when we discourse about the Rev. Ralph H. Abernathy or the Suez Canal. An intended name used primitively in such an argument is assumed to be defined in a larger context. But even in the case of such discourse, for each participant in it, that is for the maker of an argument and his listeners or readers—I should maintain—*every intended name functions as a descriptive phrase*. Certainly, one learns the meaning of some names ostensively, but in discoursing about an object, one must refer to it in his mind, if only in an inarticulate fashion, as the such-and-such, though he may refer to it in his mind as the object experienced on such-and-such an occasion, *i.e.* experienced in juxtaposition with such-and-such other experiences. And this—let it be noted for the moment only parenthetically—holds also for nouns that are not used as intended names, except that then the descriptive phrase that may replace such a noun synonymously in a statement does not endow it with existential import.

It is often said that the intension of a name is more likely to be subjective, *i.e.* to vary from user to user, than that of a predicate and that all that counts for communication is that the denotation (to use the traditional terminology) of a name be understood, not its intension. To be sure, one participant in a discourse may think of Caesar as the author of the *Gallic Wars*, while another may think of Caesar as the ancient Roman dictator who was stabbed to death on the Ides of March of some year or other, about the middle of the first century B.C. But unless at least one of the two participants knows that the two descriptions denote the same man or unless some third description is known by each participant to have the same denotation, *i.e.*, to name the same object, as does the description he most readily associates with Caesar, clearly there would be a failure of communication.

In using a noun as an intended name, one assumes something about reality. Ideally, to make all non-logical premises of an argument explicit and make their denials expressible severally in the same language, one should use primitively only predicates, variables, and logical words and assume explicitly the existence and uniqueness of any object he wishes to refer to by name. This should be definitely the practice in mathematical theories. In individual arguments, the characterization of a well-known object by means of predicates that are understood in a wider context would often be excessively pedantic, but one can always have recourse to Quine's device, already referred to earlier, of using an asserted name defined in terms of a single one-place primitive predicate instead of a primitive intended name. The main advantage of the latter procedure over the use of primitive intended names, is that the use of a special one-place predicate does not carry with it the necessity of assuming that it is true of exactly one object. Thus one can consistently both deny that Santa Claus will come and deny that he will not come to the house on Christmas Eve, or assert that there is more than one Santa Claus.²³

Still, one may often find it convenient to use intended names in a particular argument as a shortcut. Then, if there is to be any pretense of formal reasoning, all existence assumptions should be viewed as provided by the premises, through the use of asserted names or otherwise, rather than by the underlying logic, which should be thought of as a non-existential system like $L1^*$

§6. *Nouns that are not intended names—“free logic” and the “logic of uni-predicates”*. In some quarters, there have been attempts in recent years to give new philosophical life to what had been regarded as old quibbles about existence and non-existence. To this writer, these efforts seem, for the most part, to be raising a lot of dust by reintroducing obscurities of language that were successfully cleared away over half a century ago.

As was noted long ago, such a statement as ‘Pegasus does not exist’ does not attribute some mysterious property of non-existence to an object named ‘Pegasus’. Unless it is construed as a statement about the word ‘Pegasus’, synonymous with ‘‘Pegasus’ does not name anything’ (in which the noun ‘‘Pegasus’’ *does* name something), all that the statement ‘Pegasus does not exist’ can possibly mean is that there is no unique such-and-such. Note that since ‘Pegasus’ in fact does not name anything, it cannot possibly have been defined ostensively and hence cannot have any meaning for any participant in a discourse otherwise than as a synonym of a descriptive phrase. This simple fact is too often forgotten.

In §5, p. 54 above, I expressed the view that a noun always functions as a descriptive phrase in a meaningful discourse. Most certainly this must be so at least in the case of a non-naming noun or of one naming at most a historical object that no participant in the discourse could have experienced directly. And, at any rate, in all cases we can replace a noun with a descriptive phrase by having recourse to a specially introduced one-place predicate, as proposed by Quine for the case of intended names. If ‘Pegasus’ is understood, however it be understood, ‘that which pegasizes’ can be understood in the same sense.

However, to say of a noun that it means the same as some phrase such as might be referred to by ‘the such-and-such’ hardly fixes the noun’s meaning in discourse. For even apart from any inherent vagueness that may be present in the predicates appearing in the descriptive phrase constituting the noun’s translation, the full syncategorematic meaning of such a phrase in ordinary discourse is not determined by the form of the statement in which it occurs and can be inferred, if at all, only from the material content of that statement and/or from a larger context in which the statement is embedded.

Let ϕ_α be a statement containing a noun α of which we will assume that it is not *presupposed* (in the sense explained earlier) that it names, and let ϕ_α be other than an explicit statement of the existence or of the non-existence of an object named by α , such as, typically, are statements ‘ α exists’ or ‘ α does not exist’. Let also ϕ_α be other than a statement containing α in what Frege calls an *oblique* and Quine calls an *opaque* context. Then, one possible construal of ϕ_α is that it means the same as a statement ‘There

exists exactly one such-and-such, and is so-and-so⁷. Thereby, α is construed as a definite description in Russell's sense, with the widest possible scope within ϕ_α , and is what we have referred to as an asserted name. (If, with Strawson, we wish to say that ϕ_α has existential import, but is neither true nor false if α does not name, then we also construe α as an asserted name). If ϕ_α is construed in the manner just indicated, we will also say of α that, as it occurs in it, it is an *asserted* (proper) *noun*, on account of the circumstances that under such construal ϕ_α is not true if there are more than one object that fit the description such-and-such associated with α . (We are using 'such-and-such' and 'so-and-so' as syntactical variables, with obvious ranges). But it is also possible to construe ϕ_α as semantically equivalent to a statement \lceil if there exists exactly one such-and-such, then it is so-and-so⁷, thereby making ϕ_α true if there is no object fitting the description such-and-such or if there are more than one object that fit it. We will say of a noun used in the latter fashion that it is a *hypothetical noun* and a *hypothetical name*. By another construal yet, ϕ_α has the same meaning as a statement \lceil There is at most one such-and-such and if there is a such-and-such, then it is so-and-so⁷. Of a noun α so understood we will say that it is an asserted noun but a hypothetical name.

Further construals of a statement ϕ_α of the kind that we have considered are obtained by construing every atomic component of ϕ_α in which α occurs in one of the three ways that we have considered above for ϕ_α as a whole, *i.e.*, by restricting the scope of the noun α , at each of its occurrences in an atomic component statement, to that atomic component statement. So understood in one of three ways, the noun α will be said to be respectively (a) a *restricted asserted noun* and a *restricted asserted name*, (b) a *restricted hypothetical noun* and a *restricted hypothetical name*, or (c) a *restricted asserted noun* and a *restricted hypothetical name*. Furthermore, we may construe ϕ_α so that α in it is an asserted noun but a restricted hypothetical name.

The labels that we have introduced classify nouns (*nonexhaustively*) on the basis of a correct rendering, in a precise formal language, of statements containing them, on the assumption that in the use of a noun there is no presupposition of the existence or uniqueness of an object of a certain kind in virtue of some rule of significance that would deprive an expression of significance rather than of truth if the presupposed conditions did not obtain. In the ordinary use of nouns, it is however possible to see a presupposition rather than an assertion that certain conditions of *uniqueness* obtain, *i.e.*, it is possible to see a violation of grammar rather than the assertion of a falsehood in the syntactical use of a word or sequence of words as a noun when certain conditions of uniqueness are not satisfied. We may then speak of *presupposed nouns* rather than of asserted nouns, or in-so-far as the distinction between the two is not determined or is irrelevant to the matter at hand, of *intended nouns*. A formal system may also be set up with presupposed nouns that are asserted names or that are hypothetical names, as we will see later. At any rate, an asserted name, as we use the term here, is always an intended (asserted or presupposed) noun.

As far as ordinary discourse is concerned, to this writer it seems rather far-fetched to suppose that a statement be intended to be construed so that the nouns in it have a restricted scope. It seems more likely that in ordinary discourse, in most cases, if ϕ_α is an ostensibly atomic statement, then ϕ_α , the statement \lceil It is not the case that ϕ_α \rceil and any statement $\lceil\psi \vee \phi_\alpha\rceil$, for instance, alike possess or lack existential import, or, more specifically, alike do or do not presuppose or implicitly assert, or—to accommodate Strawson's point of view—alike imply or do not imply the existence and/or the uniqueness of any object of such-and-such description. However, this writer holds rather strongly that *any attempt to decide what the real meaning of a statement containing a proper noun or phrase* \lceil the such-and-such \rceil *in ordinary discourse is or, God forbid, ought to be, easily degenerates into logomachy.* 'Pegasus does not exist,' as ordinarily understood, has the force of a statement \lceil There is no unique such-and-such \rceil and hence has no existential import, 'Pegasus exists', as ordinarily understood has the force of a statement \lceil There is a unique such-and-such \rceil and hence has existential import. But there is no simple answer to the question of whether such statements as 'Pegasus is sometimes hungry', 'The king of France in 1968 is bald', or 'Santa will come or he will not come tonight' have existential import. Any disagreement on whether they do might conceivably be resolved by an empirical investigation on how people use and understand such statements, but most likely is a verbal dispute. For, to all appearance, the meaning of most statements containing a phrase \lceil the such-and-such \rceil or a noun is not fully determined in ordinary language, there being no general rule on whether they should be regarded as true, false, or neither, in case the singular term in question does not name. At any rate, *it seems to this writer to be of no philosophical moment whether in ordinary discourse 'The King of France in 1968 is bald' has the meaning of 'There is one and only one king of France in 1968 and he is bald' or that of 'If there is one and only one king of France in 1968, then he is bald', or would be ordinarily regarded as neither true nor false if there is no king of France in 1968. What is important is to be able, in a formal language, severally to express each of the possible meanings which such an ordinary language statement may have.*

For any of the construals that we have considered here under which a statement regarded as presupposition free and containing a possibly non-naming noun always has a truth value, the statement can be unambiguously translated into the standard symbolic language of quantification, provided nouns that need not be names are not taken into that language as primitive symbols, as, for this very reason, in my opinion they ought not to be.²⁴

Some logicians have thought differently. Hintikka in [4] and Lambert in [10] have proposed non-existential systems of quantification in which, unlike in L1, variables may occur free in the lines of derivations, but in which, unlike in restricted *ponens* systems, not all formulas in which some variable is free and whose closures are theorems are themselves theorems. Thus in both systems (except for differences in notation in Hintikka's system), ' $Fx \supset Fx$ ' and ' $x = x$ ', whose closures are theorems, are themselves

theorems, while ' $Fx \supset (\exists y)Fy$ ' and ' $(\exists y)(y = x)$ ' are not theorems, though their closures are. In these systems, which, following Lambert, are referred to as systems of *free logic*, the rationale for not allowing some open formulas whose closures are theorems to be theorems is that the free variables in them are construed as place-markers for nouns that need not be names, such as occur in ordinary language. In particular, in the intent of the proponents of free logic, statements of the form ' $(\exists x)(x = y)$ ' and of the form ' $\sim(\exists x)(x = y)$ ' are to be regarded as contingent and synonymous with ordinary language statements of the form 'y exists' and of the form 'y does not exist' respectively.

Let ' a ' be a noun in a discourse governed by free logic, with the ordinary meaning of, say, 'Pegasus'. If the statements ' $(\exists x)(x = a)$ ' and ' $\sim(\exists x)(x = a)$ ' are read, respectively as 'There is something identical with the object a ' and 'There is nothing identical with the object a ', then the first appears to be a truism, while the second seems to be inconsistent with the very notion of identity or sameness, and more specifically seems to contradict the theorem of free logic ' $(y)(\exists x)(x = y)$ '. Hence we must understand ' $=a$ ' in those two statements as a complex predicate, which, by virtue of the syntax imposed upon '=' by the axioms of identity, cannot without contradiction be asserted of more than one object, *i.e.* as a complex phrase 'is the only such-and-such'. In other words, ' $x = a$ ' in the above statement has the same syncategorematic meaning as some expression ' $\lceil \chi_1 x \cdot \chi_2 x \cdot \dots \cdot \chi_n x \cdot (y)(\chi_1 y \cdot \chi_2 y \cdot \dots \cdot \chi_n y \cdot \supset x = y) \rceil$ ', wherein the χ_i 's are one-place predicates. If we use ' P ' for 'pegasizes', in its turn understood as having the same meaning as the complex predicate consisting of the conjunction of the χ_i 's, then ' $(\exists x)(x = a)$ ' says the same as ' $(\exists x)(Px \cdot (y)(Py \supset y = x))$ ', equivalent to the shorter ' $(\exists x)(y)(Py \equiv y = x)$ ', which translates informally as 'One and only one thing pegasizes'. Using ' P ', we can say that there is exactly one pegasizer, as we have just done, or that there is at most one of them, or that there is at least one of them, or that there are more than one of them, or that there are no pegasizers, or that there are any specific (finite) number of pegasizers, while we cannot do all of this if we use ' a ' as a primitive noun instead.

Russell never intended his theory of descriptions to be a literally true report of how people always use and understand phrases of a certain form in ordinary conversation, which is what Strawson appears to be claiming for his own rival theory.²⁵ The philosophic significance of Russell's theory lay in his having shown by it that our use of non-naming descriptive phrases and nouns as singular terms does not commit us to maintain the existence of non-existent objects or similar absurdities. By his axioms and syntactical rules, Lambert prevents the formal derivation of such absurdities in his system, but he makes it a point to retain in the latter some of the misleading grammatical forms of ordinary language, which may lend themselves to informal misinterpretations, as by reading ' $=a$ ' as 'identical with the object a '.

The difficulties that I find with free logic are however of a more general nature than that connected with possible misconstruals of statements

containing expressions $\ulcorner \alpha \urcorner$ where α is a noun. The analysis we have made of such expressions does not throw any light on the meaning of statements, in a discourse governed by free logic, that contain a noun otherwise than as the second argument of the predicate of identity, and in the original papers proposing free logic, the authors give us no indication of the conditions under which such a statement is to be regarded as true if a noun in it does not name. In [10, p. 290] Lambert does suggest though that 'Pegasus flies' need not have existential import. This remark of his seems to indicate that he regards 'Pegasus flies' as true, even though Pegasus does not exist, by viewing it as analytic. That is, it seems from his remark that he does associate a descriptive phrase with 'Pegasus', but instead of construing the statement 'Pegasus flies' as synonymous with a statement \ulcorner There is one and only one thing that flies and has such-and-such other properties and which moreover flies \urcorner , in accordance with the Russellian analysis, he construes it as synonymous with a statement \ulcorner If there is one and only one thing that flies and has such-and-such other characteristics, then it flies \urcorner .²⁶ Under such construal as a conditional, however, every statement containing 'Pegasus' is true, though in general it is not analytic. In [8, p. 138], I remarked with reference to free logic:

The merits of such an approach and more specifically the completeness of these systems can be determined only in the context of a semantics for non-referential names [in this paper's terminology, non-naming nouns] stipulating which statements containing non-referential names are to be regarded as true and which as false. Though they discuss the use of non-referential names in ordinary language at some length, both authors fail to furnish their respective calculi with such a semantics. If 'Pegasus = Pegasus' is to be regarded as true by being construed as synonymous with some such statement as 'if there is such a thing as Pegasus, then it is identical with Pegasus', one may just as well regard 'Pegasus eats daisies $\supset (\exists x) x$ eats daisies' as true by analogously construing it as synonymous with 'if there is such a thing as Pegasus, then if it eats daisies, there is something that eats daisies'. And the same remark applies to statements of the form of their paradigm non-theorem $(\exists y)(y = x)$.

Since I wrote the above lines, Bas C. van Fraassen has come out with a semantics for a formal language containing nouns that need not name, with respect to which he proves the completeness of free logic [23]. Before considering van Fraassen's proposed semantics, which is made to measure to insure the completeness and soundness of free logic, let us examine whether any of the construals of statements containing nouns that we considered a while ago is suitable for a discourse governed by free logic.

In a discourse governed by free logic, we cannot construe nouns either as asserted names or as asserted nouns but hypothetical names. For, under either of these construals, statements $\ulcorner \phi_\alpha \supset \phi_\alpha \urcorner$ and $\ulcorner \alpha = \alpha \urcorner$ need not be true while $\ulcorner Fx \supset Fx \urcorner$ and $\ulcorner x = x \urcorner$ are theorems of free logic. Nor can we, in such a discourse, construe a noun α as a hypothetical noun and a hypothetical name. In fact, under the latter construal, all statements $\ulcorner (\exists \beta)(\beta = \alpha) \urcorner$ are true and have no existential import, since each means the same as a statement \ulcorner If there is exactly one such-and-such, then there is something iden-

tical with it⁷, while $(\exists x)(x = y)$ is not a theorem of free logic and statements of its form are intended, in a discourse governed by free logic, to be understood as contingent statements of existence.

Free logic does not fare any better if we construe the nouns in a discourse governed by it as having restricted scopes in any of the three ways that we have considered. Indeed, for statements $\ulcorner \alpha = \alpha \urcorner$, which are atomic, the construal of the noun α in them as a restricted asserted noun amounts to its construal as an asserted noun, and, as we have noted, under these construals the statements in questions need not be all true, though they are of the form of a theorem of free logic. If, on the other hand the noun α is construed in a statement $\ulcorner (\exists \beta)(\beta = \alpha) \urcorner$ as a restricted hypothetical noun and hypothetical name, the statement has the same meaning as a statement $\ulcorner \text{There is a } \beta \text{ such that, if there is exactly one such-and-such, then } \beta \text{ is identical with it} \urcorner$, which, contrary to the intended import of the original statement in free logic, is true even if there is nothing or there are more than one thing satisfying the description such-and-such, as long as the universe is not empty; it follows that under such construal, statements $\ulcorner (\exists \gamma)(\gamma = \gamma) \supset (\exists \beta)(\beta = \alpha) \urcorner$ are all true, though they are not of the form of any theorem of free logic.

It appears that the soundness and completeness of free logic can be insured only by (1) viewing the nouns in a discourse governed by it as intended nouns but otherwise restricting their scopes to the atomic component statements in which they occur, and (2) construing an atomic component of a statement, with respect to the import of any nouns in it, in a different way if it contains the predicate of identity than if it contains some other predicate. This is what van Fraassen has in fact done [23] in establishing, by the rules of his semantics, the permissible assignments of a truth value to the formulas of free logic, so as to make the latter sound and complete.

In his system of free logic LF, equivalent to Lambert's, van Fraassen adopts Quine's axioms of quantification, modified to allow the use of special noun symbols as dummy nouns (which he calls 'constants'), thus dispensing with a rule of generalization and with Lambert's use of free variables as place markers for nouns. His semantics for the language of LF allows for alternative construals of statements containing nouns, by permitting that, under certain restrictions, such statements be severally regarded as true, false, or neither true nor false, if any nouns in them do not name. In essence his semantics can be described as follows:

1. (i) All statements $\ulcorner \alpha = \alpha \urcorner$ are true.
- (ii) If just one of nouns α, β is a name, then $\ulcorner \alpha = \beta \urcorner$ is false.
- (iii) If $\ulcorner \alpha = \beta \urcorner$ is a true statement, if χ is a statement that contains the noun α , and if χ' is like χ except for containing the noun β wherever χ contains α , then χ and χ' are both true, both false, or both neither true nor false.
2. Subject to the restrictions (i), (ii), (iii), every atomic statement containing some non-naming noun may be arbitrarily regarded as true, false, or neither true nor false.
3. An interpretation of the language of LF in a domain D consists of

(a) the assignment of an extension in D to each predicate symbol,
 (b) the assignment of an object in D , as the object named by it, to each of zero or more noun symbols, and
 (c) the assignment of a truth value to some or all atomic statement forms (formulas not containing any variable free) that contain some noun symbol to which no object is assigned under (b), this assignment of truth values being subject to the restrictions (i), (ii), and (iii).

4. Under an interpretation I in a domain D , every statement form of the language of LF is regarded as a statement with or without a truth value in accordance with the following rules:

I. If all atomic statement forms have a truth value under I (by direct assignment or through assignments (a) and (b)) and hence every atomic formula has a truth value under I for each set of values in D of its variables, then every compound statement form has that truth value under I that accrues to it by the usual rules.

II. If some atomic statement forms do not have a truth value under I , then a compound statement form is true (false) under I iff it is true (false) under all those interpretations in D of the language of LF that differ from I only by the assignment of a permissible truth value to each atomic statement form that has no truth value under I .

Van Fraassen shows that, with respect to this semantics, every valid statement form of the language of LF—*i.e.* every statement form of the language of LF that is true (becomes a true statement) under any interpretation in any domain—is a theorem of LF.

Van Fraassen's position is that disputes about the truth value of statements containing non-naming nouns are philosophical in nature and that logic should be neutral with regard to them. Since the same statement forms of free logic are valid under different concepts of validity, reflecting different points of view concerning the truth value, if any, of statements containing non-naming nouns, he believes to have shown with his proof of completeness that, as a formal tool, free logic may serve different philosophies of language equally well.

To this writer, it seems that, in restricting the scope of nouns to the atomic constituents of statements and in further imposing conditions on the truth value that some atomic statements containing non-naming nouns may have, van Fraassen has tailored his semantics to fit free logic rather than to accommodate different construals that are likely to be put on ordinary statements containing nouns of which, unlike in standard logic, it is not presupposed that they name. If somebody regards 'Santa likes whisky' as false on the ground that there is no such thing as Santa, he is likely to regard 'Santa does not like whisky' also as false on the same ground. But such a construal of these statements is not permissible under any of the "philosophical" views of language that van Fraassen's semantics accommodates and deprives of validity the theorems of free logic $\lceil \phi \vee \sim \phi \rceil$.

Further van Fraassen claims that if any statement is logically true or logically false, it is assigned the truth value true or the truth value false respectively in his semantics. But his thinking here appears to be

circular, for, whether a statement is logically true (or logically false), depends on how it is construed. Thus all that he is really saying in making such claim is that if a statement, *by his semantics*, is logically true (*i.e.* of a form that is true under all interpretations permitted in his semantics), then by his semantics it is true. With the suggestion that a statement may be recognized as logically true independently of any semantics, *i.e.* independently of how it is construed (implicit in his claim that his semantics make true all statements that are logically true), van Fraassen takes a position that is more philosophically loaded than any of those with respect to which he says that free logic is neutral.

The semantic fiat by van Fraassen that all statements $\lceil \alpha = \alpha \rceil$ are true appears to be particularly artful in view of his non-commitment with regard to other atomic statements containing nouns. It is not only "reasonable," as he says [24, p. 489], but necessary to regard both or neither of 'Cicero = Cicero' and 'Pegasus = Pegasus' as logically true, by the very meaning of 'logically true' (as long as both statements are regarded as written out in primitive symbolism and hence not analyzable into different finer structures). But whether either statement is logically true and hence both are, depends on how they are construed. Ordinary usage does not help us here by giving any guidelines (and we would need not follow them if it did), since statements $\lceil \alpha$ is identical with $\alpha \rceil$ hardly ever occur in ordinary conversation. It is quite conceivable that someone, in the proper context and the proper tone of voice, may use 'Pegasus is Pegasus' rhetorically, as a way of asserting the existence of Pegasus—or, if this is unlikely because nobody in fact believes in Pegasus, we may well conceive of 'The monster of Loch Ness is the monster of Loch Ness' being used to assert the monster's existence, which use would make it a false statement.

In introducing undergraduates to logic, it is customary to call their attention to the circumstance that an ordinary language statement $\lceil A$'s are B 's \rceil may mean, depending upon its content and context, either the same as a statement \lceil All A 's are B 's \rceil or the same as a statement \lceil Some A 's are B 's \rceil , and should be rendered in a symbolic language accordingly. To cite a more complicated example, we caution our students with regard to such statements as 'He took off his clothes and went to bed', which should not be rendered as a simple conjunction in which the conjuncts can be commuted *salva veritate*; that statement, we point out, means, with possible further connotations, the same as 'There is a time interval Δ_1 and there is a time interval Δ_2 , such that Δ_2 is subsequent to Δ_1 , and such that he took off his clothes during Δ_1 , and such that he went to bed during Δ_2 '. All this serves to show that the grammatical form of an ordinary language statement is not by itself a reliable guide to its meaning. Any attempt therefore to render all ordinary language statements of a given grammatical form into the same syntactical form in a precise formal language, as the proponents of free logic seek to do for statements containing nouns, is misguided.

Citing Leonard [12] van Fraassen notes in [23] that 'The ancient Greeks worshipped Zeus' must be regarded as true on historical grounds. This is a good illustration of the inadvisability of uncritically rendering ordinary

language statements containing nouns into a symbolic language, using their grammatical form alone as a guide. Just as 'Zeus exists' or 'Zeus does not exist', 'The Greeks worshipped Zeus' does not refer to any object called 'Zeus' nor does it assert that if there is such-and-such an object, then this-and-this is true of it. Rather that statement asserts that the ancient Greeks engaged in rituals of a certain kind, during which the word 'Zeus' was uttered, or which, at least, were referred to by a descriptive phrase containing the word 'Zeus'. That statement is vague enough to be construable either as implying or as not implying that the Greeks believed in Zeus. If it is construed with that implication, then 'Zeus' occurs in it opaquely (Quine) or obliquely (Frege), and the statement cannot be fully translated into the language of ordinary quantification logic, with or without primitive nouns.

Lambert introduced the label 'free logic' for systems of logic like his, on the alleged ground that they are presupposition free, since free logic does not presuppose, as standard logic does, that the universe is not empty and that every noun names. However, in the use of a noun in a discourse governed by standard logic, it is presupposed not only that there is an object that fits the description understood in the use of that noun, but also that there is only one such object. I maintain that *a discourse governed by free logic is not presupposition free, since, when a noun is used in it, it is presupposed that no more than one object fits the description understood in the use of that noun.* If it were not for these presuppositions, one could not soundly make use of those axioms of free logic that contain the predicate of identity with nouns as arguments.

Every possible meaning of a statement regarded as presupposition free and containing a noun non-opaquely can be expressed in the symbolism of standard quantification logic or of **LI** without using a noun as a primitive symbol, by expressing in terms of a one-place predicate what is expressed by means of the noun in the original statement, as long as the latter is understood as having a truth value under any factual conditions. The one-place predicate appointed for this purpose in the case of any given noun will be understood, if the noun is understood, without need that the former be defined as a complex predicate. We saw already how this is done if the noun in the original statement is understood as an asserted name. Let us now illustrate the method more in general.

Assuming the vocabulary and formation rules of **LI**, let us appoint '*P*' to stand for 'pegasizes'. Then 'Pegasus exists' is rendered symbolically and precisely as ' $(\exists x)(y)(Py \equiv \bullet y = x)$ '. If we further appoint '*H*' to stand for 'eats daisies', then 'Pegasus eats daisies' construed as containing 'Pegasus' as an asserted name is rendered as ' $(\exists x)((y)(Py \equiv \bullet y = x) \bullet Hx)$ ', while 'Pegasus eats daisies' construed as containing 'Pegasus' as a hypothetical noun and hypothetical name is rendered as ' $(\exists x)(y)(Py \equiv \bullet y = x) \supset (\exists x)((y)(Py \equiv \bullet y = x) \bullet Hx)$ ' or equivalently and more shortly as ' $(x)((y)(Py \equiv \bullet y = x) \supset Hx)$ '. Under the same interpretation of the predicate symbols, ' $(x)(y)(Px \bullet Py \supset \bullet x = y \bullet Hx)$ ' is the symbolic translation of 'Pegasus eats daisies', if 'Pegasus' is understood therein as an asserted noun but hypo-

tical name. For convenience, a special symbol may be introduced into the symbolic language by an appropriate definition in use, to stand for 'Pegasus' or any other noun as an asserted name, as a hypothetical noun and name, or as an asserted noun but hypothetical name. Or, for any given noun of ordinary discourse, three different symbols (whose difference may consist in the absence or presence of asterisks or other distinguishing marks) may be introduced, by appropriate definitions in use, to stand for the nouns in the three mentioned senses respectively.

Some may wish to set up a formal system to which the rule be applicable that a statement (interpreted formula that may occur as a line in derivations) has no truth value if any definite description or noun occurring in it does not uniquely denote, on the alleged ground that such is the ordinary usage of those terms. To cater to this desire, we still need not make use of primitive noun symbols, but rather, doing justice to the sense in which nouns are meaningful, we may modify **L1** to obtain a system **L1**** as follows:

1. We adopt ' γ ' as a new primitive symbol.
2. We revise the formation rules so that, if χ is a formula in which the variable α is free and no other variable is free, then the definite description $\ulcorner(\gamma\alpha)\chi\urcorner$ is a term, *i.e.* is allowed to occur in formulas as argument of predicates²⁷; the variable α is regarded as bound in a definite description and hence formulas containing definite descriptions may occur as lines in derivations, but the reading of ' \vdash ' is revised so that no axiom contains definite descriptions.
3. To *modus ponens*, we adjoin the following new primitive rules of derivation: (a) if ϕ is any closed formula containing the definite description $\ulcorner(\gamma\alpha)\chi\urcorner$, then the formula $\ulcorner(\exists\beta)(\alpha)(\chi_{\alpha} \equiv \alpha = \beta)\urcorner$ is derivable from it; (b) if $\ulcorner(\gamma\alpha)\chi\urcorner$ is a term, then $\frac{\ulcorner(\gamma\alpha)\chi\urcorner}{\alpha}\chi$ is derivable from $\ulcorner(\exists\beta)(\alpha)(\chi_{\alpha} \equiv \alpha = \beta)\urcorner$; (c) if ϕ and χ are formulas wherein the variable α and only the variable α is free, then $\frac{\ulcorner(\gamma\alpha)\chi\urcorner}{\alpha}\phi$ is derivable from the formula $(\alpha)\phi$ and any closed formula containing the definite description $\ulcorner(\gamma\alpha)\chi\urcorner$.

Noun symbols can be introduced in **L1**** by definition, as short for specific definite descriptions. For any noun occurring in ordinary discourse or some technical subject, we may appoint a one-place predicate symbol Λ to be interpreted so that, for any given variable α , $\ulcorner(\gamma\alpha)\Lambda\alpha\urcorner$ has the same meaning as that noun.

The following semantical rule may be adopted for **L1****: under any interpretation of its predicate symbols, a closed formula containing the term $(\gamma\alpha)\chi$ is a statement that has no truth value (rather than being false), if the formula $(\exists\alpha)(\beta)(\chi_{\alpha} \equiv \alpha = \beta)$ happens to be a false statement under the same interpretation. We may also modify standard logic so as to make the same semantical rule applicable to the resulting system. It is difficult to see what is gained by such modifications either in **L1** or in standard logic—the import of any formula that occurs as a line of derivation remains the same.²⁸

At the end of §5, we noted that in formalizing particular arguments with premises that ordinarily would contain nouns, it may be convenient to employ asserted names primitively, using **LI*** as underlying logic, rather than have recourse to the more pedantic formulations governed by **LI**. However, it may be called into question whether in ordinary discourse proper nouns are in general intended names, even if we use this label broadly enough to allow for the occurrence of intended names in a fictional universe of discourse, as in mythology or in children stories beginning with ‘there was upon a time’. On the other hand, there can hardly be any doubt that a singular term is ordinarily used with the presupposition or the implication that it names at most one object. More specifically, words or sequences of words that are ordinarily employed as nouns (as distinct from descriptive phrases ‘the such-and-such’ or similar ones, in that the meanings of nouns are independent of their structure) are so used with the *presupposition* that they name at most one object—which circumstance does not prevent them from being ambiguous, as in the case of personal given names, or to be used in other contexts as common nouns. This suggests a way of retaining in a formal discourse some of the syntactical simplicity of the ordinary use of nouns, while avoiding its amphibolies, by introducing in it as primitive symbols a special class of recognizable one-place predicates, each presupposed to be true of at most one object. We refer to these predicates, which, in virtue of a rule of significance, cannot be true of more than one object, as *unipredicates*.

A system **LIU** of the *logic of unipredicates*, to serve as underlying logic for formal arguments that as ordinarily formulated would contain nouns, is obtained from **LI** as follows:

1. The primitive vocabulary of **LI** is enlarged to include a denumerable infinity of *unipredicate symbols*. As unipredicate symbols, we appoint the small letters of the English alphabet from ‘f’ included on, with or without numerical subscripts.
2. The formation rules of **LI** are modified for **LIU** to allow unipredicate symbols to occur in formulas in the same way as other one-place predicate symbols.
3. The meta-axioms of **LI** are retained for **LIU**, modified by the understanding of the statement of each as referring to all formulas of **LIU** (rather than just to those of **LI**). Further axioms of **LIU** are the formulas on which theoremhood is conferred by the following new meta-axiom, wherein ‘ ω ’ is a syntactical variable ranging over the unipredicate symbols.

MAU. $\vdash \ulcorner \omega\alpha \ulcorner \omega\beta \ulcorner \supset \ulcorner \alpha = \beta \urcorner \urcorner$

In any argument formalized within **LIU**, the formulas referred to in *MAU* are viewed as axioms of the underlying logic rather than as premises which need not be true. This means that the only legitimate interpretations of the unipredicate symbols is as unipredicates, *i.e.* as predicates that are true of at most one object.²⁹

In a discourse governed by **LIU**, we may appoint the unipredicate symbol ‘*p*’ to stand for ‘pegasizes’ understood as a unipredicate. Under the

same interpretation, we may prefer to read '*p*' as 'is Pegasus', with the understanding that it is synonymous with a predicate 'uniquely possesses such-and-such characteristics' (compare with the predicates ' $\ulcorner = \urcorner$ ' of free logic, discussed earlier). The symbolic rendering of 'Pegasus exists' is then ' $(\exists x)px$ ' (which may be read as 'there is an *x* such that *x* is Pegasus', or more simply as 'there is an *x* that is Pegasus'). If we, further, interpret the predicate symbol '*H*' as standing for 'eats daisies', then the statement, 'Pegasus eats daisies' understood as non-implying the existence of Pegasus is translated into the language of LIU as ' $(x)(px \supset Hx)$ '; while the same statement construed as containing 'Pegasus' as an intended name is translated into our formal language (making 'Pegasus' an asserted name) as ' $(\exists x)(px \blacksquare Hx)$ '. To the theorems of free logic ' $\ulcorner \alpha = \urcorner$ ', correspond in LIU the theorems ' $\ulcorner (\beta)(\omega\beta \supset \blacksquare \beta = \beta) \urcorner$ ', which fully express what must be understood by the former to make them valid. Thus 'Pegasus is identical with Pegasus' is rendered in LIU as ' $(x)(px \supset \blacksquare x = x)$ ' ('for every *x*: if it is Pegasus, then it is self-identical').

By appropriate definitions in use, we may easily introduce presupposed nouns that are hypothetical names or such that are asserted names in the applications of LIU, or dummies for such in the uninterpreted calculus. We may namely stipulate as follows:

1. If ω is a unipredicate symbol and ϕ contains the variable α free, then we may write $\frac{\ulcorner \omega \urcorner}{\alpha} \phi$ for ' $\ulcorner (\alpha)(\omega\alpha \supset \phi) \urcorner$ ' whenever the latter formula does not occur as part of another formula.
2. Let ω be a unipredicate symbol and ϕ a formula containing the variable α free. Then, using ' \ulcorner ' as a new object language symbol to appear in expressions ' $\ulcorner \ulcorner \omega \urcorner$ ', we may write $\frac{\ulcorner \ulcorner \omega \urcorner \urcorner}{\alpha} \phi$ for ' $\ulcorner (\exists \alpha)(\omega\alpha \cdot \phi) \urcorner$ ' whenever the latter formula does not occur as part of another formula.

By 1, we agree to use unipredicate symbols in abbreviated notation as presupposed nouns and hypothetical names or dummy symbols for such. By 2, if ω is a unipredicate symbol, we agree to use ' $\ulcorner \ulcorner \omega \urcorner$ ' as a presupposed noun and asserted name or as a dummy symbol for such. Thus 'Pegasus eats daisies' understood without existential import may now be written in abbreviated symbolic notation simply as ' Hp '. The same statement understood with existential import may be written in abbreviated symbolic notation as ' $H\ulcorner p \urcorner$ '.

In adopting LIU as underlying logic for argumentation, we retain most of the syntactical simplicity that accrues to ordinary discourse from the use of nouns, without paying as heavy a price for it in presuppositions about reality and in obliteration of distinctions between the possible meanings of nouns in ordinary usage as we do with standard or free logic. But we do sacrifice, with LIU, some of the power that we have with LI to express all possible states of affairs. We presuppose namely, that certain predicates are true of at most one object—as long as LIU is the logic that underlies our discourse and we use '*p*' (for 'pegasizes') as a unipredicate, we cannot

consistently say that there are two pegasizers. In ordinary discourse, we may for the occasion use 'Pegasus' as a common noun (as we noted earlier for 'Santa'²³), but ordinary language pays heavily in ambiguity for this flexibility.

§7. *Some larger philosophical questions.* The considerations and results presented in this paper are part of a continued study. In this last section, I shall broach—just broach—some of the larger philosophical questions they point to.

As is well known, Hume distinguished two kinds of objects of inquiry: relations of ideas and matters of fact [5, p. 25]. Hume's choice of labels was somewhat unfortunate, since 'ideas' has psychological connotations, and as psychological entities ideas may bear relations to each other, such as that of being in succession in free association, that are definitely matters of fact as Hume understood the latter term.

The origin of Hume's dichotomy goes back at least to Locke, who, in a somewhat confused account had maintained [13, Bk IV, chap. 4] that knowledge is only "the perception of the agreement or disagreement of our ideas", but is "real" knowledge and not just knowledge of the "agreements of a man's own imaginations" only if "there is conformity between our ideas and the reality of things." Locke is then forced to qualify this view somewhat in order to grant the status of real knowledge to geometric knowledge, which is not about any existing things, and thus avoid putting the knowledge that a harpy is not a centaur into the same class as the demonstrations of Euclid and the knowledge that a square is not a circle (his own examples). But throughout his account there runs a conviction—which he regretfully qualifies in various ways, making artificial distinctions and exercising considerable verbal gymnastics—that "real" knowledge is knowledge about "real" things, *i.e.* knowledge of propositions with existential import. This notion runs through the history of the distinction between knowledge of matter of fact and another kind of knowledge, even when, as in Hume, the latter knowledge is not disparaged.

Not only statements with existential import are synthetic (as we would say today). For instance, a universal material conditional in general denies that certain possibilities are the case without asserting the existence of anything. But undoubtedly statements with existential import are, with qualifications to be made shortly, about matters of fact *par excellence*. It appears that Hume's *intended* distinction between relations of ideas and matters of fact can be made precise in terms of formal logic as follows: the theorems of a non-existential first order system of quantification with identity, such as LI, when interpreted, express what Hume intended to refer to by 'relations of ideas', while the interpreted closed formulas of LI that are not theorems of LI, if true, are statements of matters of fact. We shall see shortly that the second part of this statement needs some important qualifications.

It would not constitute an objection to the above tentative explication of Hume's intended distinction between relations of ideas and matters of fact to note that analytic statements in the original sense of the term, such as

'All brothers are male', belong among the statements that Hume regarded as being about relations of ideas. For, in the formal language of LI, the symbolic equivalent of the statement 'All brothers are male' viewed as analytic would contain a symbol for 'is a brother' only when written in abbreviated notation, in accordance with a definition by which the symbolic equivalent of 'x is a brother' is short for the symbolic equivalent of 'x is male and x's parents have had other children than x'. Hence, as symbolically rendered in the language of LI, the statement in question is a theorem of LI. The difficulties found in recent years, notably by Quine, with the analytic-synthetic dichotomy and the attendant difficulties with synonymy, does not seem to this writer to concern the descriptions of the world that are carried out in a formal language such as that of LI.

But do the theorems of LI represent the symbolic renderings of *all* statements about "relations of ideas" or, as we shall say for lack of a better term, of all analytic statements? We may well suppose that Hume, after becoming acquainted with LI in a one-semester course in modern elementary logic, would enthusiastically subscribe to an affirmative answer. But he would not know then anything about higher order logic, set theory, or the foundations of mathematics and would not suspect the problems about existence of abstract entities that lurk there.

The qualifications we referred to a short while ago to the assertion that statements with existential import are all about matters of fact and that the interpreted closed formulas of LI that are not theorems of LI are all about matters of fact concern statements of existence of abstract entities, as properties or classes, which can be technically formulated in the language of a first order system by allowing these entities to be values of individual variables, as when set theory is formalized within first order logic. Whether there are or not triangular objects, one may feel that, if 'P' is interpreted as meaning the same as 'is a property', 'K' as denoting the relation of being an instance of, and 'T' as meaning the same as 'is triangular', then ' $(\exists x)(Px \cdot (\exists y)(Kyx \equiv Ty))$ ' is a true statement, though doubtfully one about matters of fact.

If we face the *traditional* problem of the "existence of universals" with the habits of thought that comes from an acquaintance with modern *first level* class theory—as by having studied some Boolean algebra in its most usual interpretation—we find a *prima facie* very strong argument against saying that properties or classes exist in any sense like that in which we say that individual objects exist. For, as long as classes are treated as collections of objects and the same objects are not talked about both as individuals and as classes in the same context, the term 'null class' having been introduced into our vocabulary, to say that there is such-and-such a class is to say something that is always true and not to say anything about the world. And the same holds if the discourse is in terms of properties rather than classes. In ordinary conversation, one may in an optimistic mood say 'There *is* honesty in this world'. But all that he would thereby mean is that there are honest persons, or perhaps that there are honest actions (bits of behavior), and hence nothing that could not be said without

seeming to assert the existence of a property or “universal”. On the other hand, if using a more technical language, we “admit” empty properties into our universe of discourse, to say that there exists honesty is to say a truism, not anything like an ordinary existential statement, which is typically about matters of fact—and the same goes for saying that there exists triangularity as we did a while ago. The universe of discourse can thus be made to encompass, it would seem, more than the actual universe and cannot be empty,³⁰ but only as a result of a linguistic convention, a convenient device to allow certain propositions to hold with complete generality, just as when points at infinity are introduced into a projective geometry that is embedded in Euclidean or hyperbolic geometry.

The real problems, of which the participants in the traditional nominalism-realism controversy were not aware, make themselves felt when we go into higher order logic or use a set of axioms for a theory of sets with our first order logic,³¹ as is necessary to the formalization of mathematics—the problems are there of course, and crop out when we probe into the basic concepts of mathematics, whether we formalize it or not. In set theory, the sets (as we refer to them preferably than as classes when we start considering sets of sets) acquire a substantiality of their own, as it were. When we allow ourselves to speak of sets of sets, and distinguish, say, between a set of sets and the union of its elements, we can no longer dispose of all talk of sets as just a way of speaking, translatable into a language in which variables do not range over sets, nor can we amputate from our discourse all that is not so translatable without losing a great deal that is beautiful and valuable, and clearly not all nonsense. Furthermore we cannot harmlessly, for any given description, assert the existence of the set of all objects fitting that description, and the restrictions that we must impose on free “set constructions” to avoid the antinomies seem entirely pragmatic. Set theory and the mathematical edifice that can be built on it hardly deals with matters of fact as the term would be understood in natural science (which uses that mathematics), yet it is difficult to view the entire subject as a free logical construction—the subject seems to constrain us as if we encountered a reality to be discovered. But against the Platonic view, militates the circumstance that in set theory there is not one reality to be discovered, but alternative ones, depending upon the choice of axioms.

Set theory can of course be treated as an abstract theory, without assigning any meaning to the predicates ‘is a set’ and ‘belongs’. Then there is no occasion to be surprised at the circumstance that an unrestricted comprehension axiom leads to contradictions anymore than there is when any other set of uninterpreted axioms turns out to be inconsistent. The surprise or rather the *Unbehagen* arises from the circumstance that we do believe that we understand the meaning of ‘set’ and ‘belongs’. But do we really? The concepts of set and of membership in a set turns out to be the most basic on which mathematics can be formally founded. Perhaps because of the *formal* primacy of these concepts, we have been deluding ourselves that they are also the simplest psychologically and the easiest to understand. But the man in the street does not understand them—not as we

are using them in set theory. I suspect that, as we use them in set theory, they are about as nebulous as the concepts of infinitesimals and infinite sums were in the eighteenth century. They need a clarification in terms of simpler concepts so that a consistent and adequate axiom set for set theory may arise naturally out of those basic concepts. It is in set theory that the important reconstruction is needed.

NOTES

1. This study is a sequel to [8] for which the following corrigenda are in order.
 - a. Of the several misprints in [8], the following should be noted here lest it affect the intelligence of the text: on page 144, line 22, instead of 'MT1-4' there should occur 'Cor1-MT4'.
 - b. Throughout [8], expressions such as $\left\langle \frac{\alpha_1}{\beta_1} \frac{\alpha_2}{\beta_2} \dots \frac{\alpha_n}{\beta_n} \phi \right\rangle$ are used to denote the result of several simultaneous substitutions of variables in a formula. Lest an expression of this form be misunderstood as denoting the result of consecutive substitutions in a formula, it is better, for the intended purpose, to use in its stead an expression such as $\left\langle \frac{\alpha_1, \alpha_2, \dots, \alpha_n}{\beta_1, \beta_2, \dots, \beta_n} \phi \right\rangle$. This improved notation was adopted by the author in [9] at the suggestion of the referee.
 - c. On page 136 of [8], the formulation of '*101*' is not quite correct. A correct formulation of '*101*' is given in [9], page 39.
2. In the logical literature, these are variously called 'individual constants', 'proper names', or simple 'names'. In [8] I called them 'individual constants', but I find this term to be ambiguous. 'Individual constant' is some times employed to designate symbols that may be significantly used in statements as arguments of predicates, while at other times the same term is used to designate dummy symbols that play the same syntactical role in statement forms. Moreover in physics 'constant' has a different sense than it has acquired in mathematics, as when we speak of the physical constants, which are not linguistic entities at all.

In this paper I employ 'proper noun' to designate any *simple* symbol (*i.e.* excluding definite descriptions or compound terms not replaced by definition with special symbols), whether primitive or defined, that may be used significantly in statements as arguments of predicates and are not subject to quantification, reserving 'proper names' to designate those among them that name individual objects, as, *e.g.* 'Pegasus' does not. 'Singular term' or simply 'term', as we shall later say for short, refers in this paper to any expression that may be used as argument of a predicate, including variables, open or closed compound terms, *etc.*
3. 'General' is here modified by 'primitively' to allow for the introduction into the system *by definition* of symbols functioning as unquantified singular terms in abbreviated formulas, as by introducing into it a definite description, in the manner of *Principia*, and abbreviating the latter in its turn by a single symbol functioning as a proper noun that is assumed to name.

4. Occasionally in the past [8 and 9] I have followed Quine [16] in referring to non-existential systems as inclusive systems, but I find the term not felicitous, because 'inclusive' suggests greater strength, *i.e.* more theorems, as when we say of a system that it is inclusive of or that it includes the theory of identity. The description of a system of *logic* as non-existential would be superfluous from the standpoint taken in [8] and in this paper, were it not for the tradition of regarding existential systems (*i.e.* systems that are not non-existential) as systems of pure logic.
5. Thus the notion of underlying logic adopted here differs from that of Church [1, §55], in which the vocabulary of a theory includes that of its underlying logic. The use that we are making here of the notion of dummy predicates, as distinct from that of predicate variables that serve to express generality with respect to properties and relations, accounts for the difference.
6. We shall also have occasion to speak of a *discourse* governed by **L** or by some other logic system, referring thereby to any theory in a general way or to assertions and arguments that may be only loosely related employing that system as underlying logic. Such a discourse is formalized, as distinct from *ordinary* discourse.
7. As was shown in [8] (see there *MT19*, p. 151 and remarks on p. 152), all closed formulas valid in every non-empty domain (existentially valid, as they are called there) are derivable within **L** from a single such object language assumption form, though no rule of substitution is applicable to such assumption form primitively.
8. This is so even in so-called pure systems of standard logic, wherein variables are the only symbols that may occur as unquantified arguments of predicate symbols, since the free individual variables function therein as place markers for proper names, though not only so. (*Cf.* below §4, and [8, pp. 131-32].)
9. Let us note in passing the well known fact that since the theory and hence the underlying logic use only a finite number of primitive predicates, the needed axioms of identity are finite in number (*Cf. e.g. MA11* and *MA12* above) and may be treated, in any formalization of the theory, as axioms of the theory proper rather than as part of the underlying logic. Here we will assume instead that the underlying logic includes the logic of identity.
10. Hintikka uses different symbols as place-markers for nouns and as variables subject to quantification, but refers to the symbols of both kinds as variables.
11. This use of 'presupposed names' is unrelated to the sense in which, according to Strawson [21], the existence of a certain object is presupposed in the use of a noun or definite description, since in Strawson's account, a statement-like expression does not lack significance if a noun or definite description in it does not name anything.
12. Also, of course, as already **L1**, **L1*** has the same force as standard logic with respect to what can be derived from a set of assumption forms containing an assumption form $\ulcorner (\exists \beta)\phi \urcorner$.
13. If we construe an ordinary language statement containing a noun as neither true nor false yet significant, if the noun does not name, as Strawson wants us to [21], then that noun is an asserted name rather than a presupposed name in our terminology, and at any rate it is an intended name. *Cf.* above, footnote 11.

14. In [8], the reference really was to abstract axiomatic theories and "such that deal only with abstract entities, as sets or numbers." But the distinction between the two kinds of theories is nebulous, and any attempt at its clarification here would take us too far from the matter at hand. We may harmlessly renounce making the distinction here since we are thinking of theories that at any rate can be treated abstractly and involve no ostensive definitions.
15. In this context we are using 'term' in the sense of 'singular term' or any object language expression that may occur as argument of a predicate, not, as often up to now, to refer to an item of terminology, usually of the metalanguage.
16. Some authors refer also to truth-functional connectives and/or to quantifiers as operators. This usage is correct only if statements are regarded as names (of truth values, propositions, or whatever), *i.e.* as terms.
17. No set of formation rules for compound terms or conventions for omitting punctuation marks from them will be spelled out here, since we will not refer to any specific formal system in which such terms are admitted and no ambiguity will arise in our use of them.
18. *Cf.* the typical formulation of group theory in [6]. Kayser does not introduce there any explicit assertion of the uniqueness in C of the result of an operation. This omission may be justified on account of the tacit assumption of an informal logic of identity, except that the assertion of the existence of the result of an operation in C , which Kayser explicitly makes, could also be regarded as provable on the tacit assumption of the appropriate underlying logic, as in the standard formalization that we give next in the text above. At any rate, Kayser is redundant in axiomatically asserting the commutativity of the product of an element of C by its reciprocal or by i (as he calls the identity element), which is provable from the other assumptions. The opening phrase "there is in C an element i " of Kayser's axiom (c), may make it appear that ' i ' is used as a variable, but the use of ' i ' as an unquantified argument of identity in the next axiom leaves no doubt that it is intended as a name. *Cf.* the formulation of Boolean algebra in [18, p. 9], where Rosenbloom explicitly asserts the existence and uniqueness in C of the result of any operation; he assumes there a logic of identity weaker than the usual, for which the principle of substitutivity of identity does not hold [*ibid.*, p. 1], and thus is necessitated to spell out his axioms $A6$ and $A7$ as specific axioms of Boolean algebra rather than assume them as part of the logic of identity.
19. Here 'term' refers to an object in the universe of discourse, not to a linguistic item.
20. In deductive procedures, often, of course, we reason about an arbitrary object of a certain kind, but we do so only as a deductive technique, to prove something about all objects of that kind. As a formal technique in derivation this procedure is authorized primitively in many systems, but in this writer's opinion, the authority for it should be derivative (see [8, p. 130]).
21. In the formulation of a theory, even if the uniqueness of an object with respect to some characteristics of it should be explicitly asserted already in the axioms (as is not the case in any formulation of a theory that comes to this writer's mind), there would be a redundancy of a sort in referring to that object by a primitive name.

22. In other words, there is a redundancy in *S*'s vocabulary, consisting in the circumstance that, by taking a suitable set of *S*'s theorems not containing α , all of its other theorems can be proved by introducing α by definition.
23. Note that in the English statement 'There is more than one Santa Claus', 'Santa Claus' is not used as a proper but as a common noun, *i.e.* as part of a one-place predicate 'is a Santa Claus'.
24. As we will see later, there is no difficulty in modifying **L1** or standard logic so that a statement containing a noun can be rendered symbolically in the language of those systems without need of assigning any truth-value to it if the noun in it does not name.
25. As Russell makes it clear in his witty reply to Strawson's attack on his theory of descriptions, it is not difficult to imagine situations in ordinary life in which one could not claim that he did not consider a statement to be false on account of its containing a non-naming definite description [19, p. 339].
26. In a recent paper [11], Lambert throws more light on his views concerning the truth values of statements containing non-naming nouns. The paper has come to the attention of this writer too late for him to include any comments on it here, except for remarking that it seems to support the opinion expressed above that for Lambert a statement containing a non-naming noun is true only if it is analytic.
27. Thus, we do not admit in the language of **L1**** open definite descriptions. To do it would only be occasion for raising fresh verbal issues concerning the conditions under which a statement containing open definite descriptions is true, false, or neither.
28. As is well known to mathematical logicians, long before Strawson raised his objections to Russell's theory of descriptions, Hilbert and Bernays noted in their *Grundlagen der Mathematik*, without making a philosophical issue of it, that in ordinary usage and especially in mathematical practice, an expression 'the such-and-such' is employed only when it is already established that there is one and only one object fitting the description such-and-such [2, p. 384]. To conform to this usage, they proposed an alternative formalism for definite descriptions to that of *Principia* [*ibid.*]. In their system, ' γ ' is primitive, and, for a given formula ϕ an expression ' $\lceil(\gamma\alpha)\phi\rceil$ ' counts as a term that can be substituted for a free variable in a theorem of quantification or of a formalized theory only after the formulas ' $\lceil(\exists\alpha)\phi\rceil$ ' and ' $\lceil(\alpha)(\beta)(\phi \cdot \frac{\beta}{\alpha} \phi \supset \alpha = \beta)\rceil$ ' have been derived from the axioms. A supplementary rule of derivation is then adjoined to those of standard logic, to the effect that for a given ϕ , $\frac{\lceil(\gamma\alpha)\phi\rceil}{\alpha} \phi$ is derivable from ' $\lceil(\exists\alpha)\phi\rceil$ ' and ' $\lceil(\alpha)(\beta)(\phi \cdot \frac{\beta}{\alpha} \phi \supset \alpha = \beta)\rceil$ '. Analogously, we may modify **L1** by allowing definite descriptions in derivations from assumption forms only after the appropriate existence and uniqueness formulas have been derived and adjoining to *modus ponens* the rules (b) and (c) given above in the text.
29. Alternatively, we could retain **L1** as underlying logic, and assert in the premises of particular arguments or axioms of a theory that certain predicates are true of at most one object. Thereby, however, we should lose the advantage of having a special class of symbols recognizable as unipredicates. We are accustomed to recognize nouns in ordinary discourse by the syntactical position they take.

30. This circumstance constitutes no reason for using an existential system as underlying logic. Since in set theory, or a theory of properties, the existence axioms of the theory must be spelled out anyway, a non-existential system is indicated as underlying logic.
31. The problems would be the same if we introduced a set of axioms for a theory of properties.

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