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A REDUCTION PROCEDURE FOR SHEFFER STROKE FORMULAS

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In this paper, I shall present a reduction procedure for a propositional calculus employing the Sheffer stroke.[1] I shall show that when this technique is applied to Sheffer stroke formulas (hereafter, 'stroke-formulas'), the result is a finite set of sentences (hereafter, '*FSS*') consisting of sentences which contain fewer strokes than the original. This procedure enables one to test the original formula for tautologousness, self-contradictoriness, or contingency.

1. *Preparatory Considerations* Object languages employing the stroke usually contain sentence variables (for example, 'p', 'q', 'r', 's', and 't' with or without numerical subscripts) and the stroke ('$|$') as primitive signs. Punctuation, where necessary, is achieved by symmetrical groups of dots[2] flanking the appropriate stroke or strokes.

Let *SC* be such a Sheffer calculus. The metalanguage *M* for *SC* consists of (1) the syntactical variables 'P', 'Q', 'R', 'S', and 'T' (with or without numerical subscripts), which range over the wffs of *SC*; (2) the quasi-syntactical[3] variables 'Γ', 'Δ', and 'θ' (with or without numerical subscripts), which range over the *FSSs* of *SC*. The wffs occurring in a non-empty *FSS* are called the *members* of the *FSS*. (3) The negation bar '—', which occurs over wffs; (4) the arrow '$\rightarrow$' (used in the rewrite rules), which permits the expression on its left to be rewritten as the expression on its right; (5) the '#', which indicates a split into two *separate FSSs*; and (6) the variable '$\boldsymbol{R}$', which ranges over reductions, i.e., finite sequences of applications of rewrite rules. An *FSS* Γ is true if and only if at least one of its members is true; otherwise, Γ is false. Γ is a tautology if and only if at least one member of Γ comes out true on every assignment of truth-values to the sentence variables occurring in the members of Γ. Obviously, the order of the members is irrelevant to the truth-value of an *FSS*.

An *FSS* is said to be *basic* if and only if each of its members is a sentence variable or the negation of a sentence variable. A *sequence tautology* is any *FSS* of the form Γ, P, $\overline{P}$, Δ (where Γ and Δ may be empty).

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In the following rewrite rules, Γ and Δ may be empty.

R1: $$\Gamma, P|Q, \Delta \rightarrow \Gamma, \overline{P}, \overline{Q}, \Delta$$

R2: $$\Gamma, \overline{\overline{P}}, \Delta \rightarrow \Gamma, P, \Delta$$

R3: $$\Gamma, \overline{P|Q}, \Delta \rightarrow \Gamma, P, \Delta \# \Gamma, Q, \Delta$$

It is readily verified that the *FSSs* flanking the arrows are truth-functionally equivalent.

The closure rules are as follows:

C1: An *FSS* Γ is said to be *closed* if and only if it is a sequence tautology; otherwise, Γ is *open*.
C2: A reduction ***R*** is said to be closed if and only if each path of ***R*** produces a closed *FSS*; otherwise, ***R*** is open.
C3: A reduction ***R*** is *terminated* if and only if (a) ***R*** is closed, or (b) ***R*** produces a basic *FSS* at the end of each path.

2. *Adequacy of the Method* Every wff of *SC* is assigned a height as follows.
(a) A sentence variable has height 0.
(b) $\overline{P}$ has height n, where P has height $n - 1$.
(c) $P|Q$ has height $m + n + 1$, where P has height m and Q has height n.

Th1: *Each path in the reduction of any stroke-formula terminates in a basic FSS.*

The proof proceeds by induction on the height of the stroke-formula involved.

(i) Let P be a stroke-formula of height 1 (since no stroke-formula has a height less than 1). Then P is $Q|R$, where Q and R are of height 0 (by (c) above). But by **R1** we obtain $\overline{Q}, \overline{R}$ which is basic.

(ii) Let Th1 hold for all stroke-formulas of height $k - 1$ or less. Let P be a stroke-formula of height $k > 1$. Then P is either (a) $\overline{Q}$ or (b) $Q|R$, where Q and R are of height $k - 1$ or less. (a1) Suppose Q is not a stroke-formula. Then Q must be $\overline{S}$, and P is $\overline{\overline{S}}$. By **R2**, we obtain S, whose height is less than k. Hence, Th1 by the hypothesis of the induction. (a2) Suppose Q is a stroke-formula. Then P is $\overline{R|S}$, where R and S are of height $k - 1$ or less. By **R3** we get $R \# S$, each of which reduces to a basic *FSS* by the hypothesis of the induction. (b) Let P be $Q|R$, where Q and R are of height $k - 1$ or less. By **R1** we have $\overline{Q}, \overline{R}$. Hence, Th1 by (a1), (a2), and the hypothesis of the induction.

Th2: *Each of* **R1-R3** *is sound.*

Proof by truth conditions for *FSSs*.

Th3: *If Γ is obtained from Δ by* **R1** *or* **R2**, *then if Γ is a tautology then Δ is a tautology.*

(i) If Γ comes by **R1**, then Γ is θ, $\overline{P}$, $\overline{Q}$, Δ_1 and Δ is θ, $P|Q$, Δ_1. Since Γ is a tautology, each truth-value assignment to the sentence variables of the members of Γ will either (a) make something in θ true, which will also be true in Δ; (b) make $\overline{P}$ true, which makes $P|Q$ true in Δ; (c) make $\overline{Q}$ true, which will make $P|Q$ true in Δ; or (d) make something in Δ_1 true, which will also be true in Δ. So, Δ is made true by every assignment that makes Γ true, namely, all possible assignments. Hence, Δ is a tautology.

(ii) If Γ comes by **R2**, then Γ is θ, P, Δ_1 and Δ is θ, $\overline{\overline{P}}$, Δ_1. Proof similar to (i).

Th4: *If* Γ # θ *comes from* Δ *by* **R3**, *then if* Γ *and* θ *are tautologies,* Δ *is a tautology.*

Γ # θ will be Δ_1, P, Δ_2 # Δ_1, Q, Δ_2 and Δ will be Δ_1, $\overline{P|Q}$, Δ_2. Since Γ and θ are tautologies, each truth-value assignment to the sentence variables of the members of Γ and θ will either (a) make something in Δ_1 or Δ_2 true, which will also be true in Δ; or (b) make P true in Γ and Q true in θ. But this renders $P|Q$ false and $\overline{P|Q}$ true, which renders Δ true. So, Δ is made true by every assignment that makes each of Γ and θ true, namely, all possible assignments. Hence, Δ is a tautology.

Th5: *Let P be a stroke-formula. If the reduction of P closes, then P is a tautology.*

If *P*'s reduction closes, then each path produces a closed *FSS*. That is, each path has as its last *FSS* a sequence tautology (and, hence, a tautology). By repeated uses of Th3 and Th4, *P* is a tautology.

Th6: *If P is a tautology, then the reduction of P closes.*

By Th1, each path in the reduction of *P* terminates in a basic *FSS*. Since *P* is a tautology, then by Th2, tautologousness is hereditary. Hence, each path in the reduction of *P* terminates in a basic *FSS* which is a tautology. But such a tautology is a sequence tautology, hence, the reduction of *P* closes.

Th7: *P is a tautology if and only if its reduction closes.*

By Th5 and Th6.

3. *Application of the Procedure* I shall now give an example of a reduction. A sentence to which one of the rewrite rules is applied will be called the *active* sentence. In the example, I shall enclose the active sentence in square brackets for the reader's convenience. Such brackets, however, are neither necessary nor are they part of *SC* or *M*. In the reduction Γ is used as representing part of a given *FSS* which is inactive for an application of a rule.

$$[P|P.|.P|P\therefore|\therefore P|P.|.Q|Q:|:P|P.|.Q|Q]$$

$$\overline{P|P.|.P|P},\ [\overline{P|P.|.Q|Q:|:P|P.|.Q|Q}] \qquad \textbf{R1}$$

$$[P|P.|.Q|Q],\ \overline{P|P.|.P|P} \qquad \# \qquad [P|P.|.Q|Q],\ \overline{P|P.|.P|P} \qquad \textbf{R3}$$

$$\overline{P|P},\ \overline{Q|Q},\ [\overline{P|P.|.P|P}] \qquad \textbf{R1} \qquad \overline{P|P},\ \overline{Q|Q},\ [\overline{P|P.|.P|P}] \qquad \textbf{R1}$$

$$P|P,\ \overline{P|P},\ \Gamma\ \#\ P|P,\ \overline{P|P},\ \Gamma \qquad \textbf{R3} \qquad P|P,\ \overline{P|P},\ \Gamma\ \#\ P|P,\ \overline{P|P},\ \Gamma \qquad \textbf{R3}$$

closed *closed* *closed* *closed*

A somewhat neater job can be done by adding the following set of equivalences to **R1-R3**.

E1: $P|Q \longleftrightarrow Q|P$

E2: $P.|.P|Q \longleftrightarrow P|\overline{Q}$

E3: $P.|.Q|Q \longleftrightarrow P|\overline{Q}$

Below is an application of **R1-R3** and **E1-E3**.

$$[P.|.Q|R::|::P.|.R|Q\therefore|\therefore S|Q:|:P|S.|.P|S]$$

$$\overline{P.|.Q|R},\ [\overline{P.|.R|Q\therefore|\therefore S|Q:|:P|S.|.P|S}] \qquad \textbf{R1}$$

$$P.|.[R|Q],\ \overline{P.|.Q|R} \qquad \# \qquad [S|Q:|:P|S.|.P|S],\ \overline{P.|.Q|R} \qquad \textbf{R3}$$

$$P.|.Q|R,\ \overline{P.|.Q|R} \qquad \textbf{E1} \qquad [S|Q.|.\overline{P|S}],\ \overline{P.|.Q|R} \qquad \textbf{E3}$$

closed

$$\overline{S|Q},\ [\overline{\overline{P|S}}],\ \overline{P.|.Q|R} \qquad \textbf{R1}$$

$$\overline{S|Q},\ [P|S],\ \overline{P.|.Q|R} \qquad \textbf{R2}$$

$$[\overline{S|Q}],\ \overline{P},\ \overline{S},\ \overline{P.|.Q|R} \qquad \textbf{R1}$$

$$S,\ \overline{S},\ \Gamma \quad \#\ Q,\ \overline{P},\ \overline{S},\ [\overline{P.|.Q|R}] \qquad \textbf{R3}$$

closed

$$P,\ \overline{P},\ \Gamma \quad \#\ [Q|R],\ Q,\ \overline{P},\ \overline{S} \qquad \textbf{R3}$$

closed

$$\overline{Q},\ \overline{R},\ Q,\ \Gamma \qquad \textbf{R1}$$

closed

NOTES

1. For an axiomatic approach to propositional calculi using single operators, see T. W. Scharle, "Single Axiom Schemata for **D** and **S**," *Notre Dame Journal of Formal Logic*, vol. 7 (1966), pp. 344-348, and J. Riser, "A Gentzen-type Calculus of Sequents for Single-operator Propositional Logic," *The Journal of Symbolic Logic*, vol. 32 (1967), pp. 75-80. As the title of the present paper indicates, the present procedure differs from that of Riser and Scharle in being a reduction procedure rather than a deductive one.

2. Punctuation could be eliminated by adapting the Sheffer stroke to a Polish notation, e.g., '$|p|qr$' instead of '$p\cdot|\cdot q|r$'. But such adaptations are less frequent for the Sheffer notation than for propositional calculi using more standard operators,

hence, I have retained the dot punctuation. The technique presented here, however, is in no way dependent on the type of notation (i.e., Polish or non-Polish) used.

3. I say *quasi*-syntactical variables because, strictly speaking, syntactical variables range over expressions of an object language, not over *sets* of such expressions.

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