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ON A MODAL SYSTEM OF D. C. MAKINSON AND B. SOBOCIŃSKI

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It is well-known that if *N*, *K* and *M* are taken as primitive, with *C* and *L* defined in the usual manner, the theses of Prior's Diodorean system D = S4.3 + *CLCLCpLppCMLpp*[1] can be characterized as those, and only those, formulas verified by the matrix $\mathfrak{P} = < V, d, -, \cap, P >$, where:

1. V is the set of all ω sequences $(x_0, x_1, \ldots)$ of 0's and 1's.
2. d is the designated element $(1, 1, 1, \ldots)$ of V.
3. $-$ and $\cap$ are operations in V defined in pointwise fashion from the familiar Boolean operations $-$ and $\cap$ in $\{0, 1\}$.
4. P is the operation in V such that if $(x_0, x_1, \ldots) \varepsilon V$, then $P(x_0, x_1, \ldots) = (y_0, y_1, \ldots)$ where, for each i, $y_i = 1$ iff $x_j = 1$ for some $j \geq i$.

In [2], Makinson observes that if 4. is replaced by

5. P^* is the operation in V such that if $(x_0, x_1, \ldots) \varepsilon V$, then $P^*(x_0, x_1, \ldots) = (y_0, y_1, \ldots)$ where, for each i, $y_i = 1$ iff $x_j = 1$ for some $j \leq i$.

then D*, defined as the system for which the resulting matrix $\mathfrak{P}^* = < V, d, -, \cap, P^* >$ is characteristic, is a proper extension of D and, like D, admits of a very natural tense-logical interpretation.

We here show that D* can be axiomatized and is equivalent to the system K3.1 = S4.3 + *CLCLCpLppp* discussed by Sobociński in [4]. To this end, let S = D + *CLMpMLp*. It is readily established that S $\subseteq$ K3.1 $\subseteq$ D*—use the known fact that *CLMpMLp* is a thesis of K3.1 and note that *CLCLCpLppp* and *CpCMLpp* yield *CLCLCpLppCMLpp*—and so it will suffice to show that D* $\subseteq$ S.

Suppose $\gamma_1, \ldots, \gamma_m$ are the subformulas of α. Then for each γ_i, we put $\beta_i = MKC\gamma_i L\gamma_i CN\gamma_i LN\gamma_i$ and let β be the conjunction of all β_i's. Where μ is any assignment into $\mathfrak{P}$ or $\mathfrak{P}^*$ and $\mu(\delta) = (x_0, x_1, \ldots)$, we let $\mu_j(\delta) = x_j$.

Lemma 1. *If* $\vdash_D C\beta\alpha$, *then* $\vdash_S \alpha$.

Proof. Using the matrix $\mathfrak{P}$ it is easily checked that *CCLMpMLpMKCpLp CNpLNp* is a thesis of D and therefore of S. Then since $\vdash_S$ *CLMpMLp*, we

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have $\vdash_S MKCpLpCNpLNp$, from which it follows that $\vdash_S \beta$. And thus, if $\vdash_D C\beta\alpha$, $\vdash_S C\beta\alpha$ and hence $\vdash_S \alpha$.

Lemma 2. *If α is verified by $\mathfrak{M}^*$, then $C\beta\alpha$ is verified by $\mathfrak{M}$.*

Proof. Suppose $C\beta\alpha$ is falsified by $\mathfrak{M}$. Then there is an assignment μ into $\mathfrak{M}$ such that, for some j, $\mu_j(C\beta\alpha) = 0$; whence $\mu_j(\beta) = 1$ and $\mu_j(\alpha) = 0$. It follows that $\mu_j(\beta_i) = 1$ for all i, $1 \leq i \leq m$. Then, for each γ_i, there exists some $k(i) \geq j$ such that $\mu_{k(i)}(C\gamma_i L\gamma_i) = \mu_{k(i)}(CN\gamma_i LN\gamma_i) = 1$. From this we clearly get $\mu_{k(i)}(\gamma_i) = 1$ iff $\mu_f(\gamma_i) = 1$ for all $f \geq k(i)$, and indeed, if we let $k = \max\{k(i)\colon\ 1 \leq i \leq m\}$, then $\mu_k(\gamma_i) = 1$ iff $\mu_f(\gamma_i) = 1$ for all $f \geq k$.

Now let μ^* be the assignment into $\mathfrak{M}^*$ such that, for each propositional variable p, $\mu_j^*(p) = \mu_{k-j}(p)$ when $0 \leq j \leq k$ and $\mu_j^*(p) = \mu_j(p)$ otherwise. We show by induction on the complexity of γ_i that $\mu_j^*(\gamma_i) = \mu_{k-j}(\gamma_i)$ for all j, $0 \leq j \leq k$. This is true by the definition of μ^* for complexity 1, so suppose γ_i is of complexity $n + 1$ and assume the hypothesis true for complexity $\leq n$. We shall consider here only the case $\gamma_i = L\gamma_h$, the cases $\gamma_i = N\gamma_h$ and $\gamma_i = K\gamma_f\gamma_h$ being perfectly straightforward. If $\mu_{k-j}(L\gamma_h) = 1$, then $\mu_f(\gamma_h) = 1$ for all $f \geq k - j$ and in particular for those f such that $k - j \leq f \leq k$. But then, by hypothesis, $\mu_f^*(\gamma_h) = 1$ for all f such that $0 \leq f \leq j$. Hence, $\mu_j^*(L\gamma_h) = 1$. Conversely, if $\mu_j^*(L\gamma_h) = 1$, then $\mu_f^*(\gamma_h) = 1$ for all f such that $0 \leq f \leq j$, and so by hypothesis, $\mu_f(\gamma_h) = 1$ for all f, $k - j \leq f \leq k$. But now since $\mu_k(\gamma_h) = 1$, we also have $\mu_f(\gamma_h) = 1$ for all $f \geq k$; and hence $\mu_{k-j}(L\gamma_h) = 1$. This completes the induction.

To complete the proof of the lemma, we recall that $\mu_j(\alpha) = 0$ for some j, $0 \leq j \leq k$. Therefore, by the result just established, $\mu_{k-j}^*(\alpha) = 0$, making μ^* an assignment into $\mathfrak{M}^*$ which falsifies α.

That $D^* \subseteq S$, and thus $S = K3.1 = D^*$, follows at once from Lemmas 1 and 2, together with the fact that $\mathfrak{M}$ is characteristic for D. In light of Makinson's result for D*, this means that the systems K1, K2, K3, K1.1, K2.1 and K3.1 of [4] all have infinitely many modal functions, i.e., non-equivalent formulas in a single propositional variable.

Finally, we remark that D* (K3.1) has no finite characteristic matrix; the proof is a simple adaptation of Dugundji's proof [1] that there are no finite characteristic matrices for S1-6.

NOTE

1. Cf. [3], p. 176, for this formulation of D. Throughout the present paper, we assume S4 and its extensions to be axiomatized with the rule to infer $\vdash L\alpha$ from $\vdash \alpha$.

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