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MODAL SYSTEMS IN WHICH NECESSITY IS "FACTORABLE"

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We will say that necessity is "factorable" in a modal system S if there are modal functions $X_1 p, \ldots, X_n p$ — L itself being none of the X_i — such that in S the conjunction $KX_1 pKX_2 p \ldots X_n p$ is equivalent to Lp. For the systems discussed in this paper, n in the above formulas will be 2 and $X_1 p$ will be simply p. An obvious example of a system in which necessity is factorable is the system S4.4, which contains as a thesis

(1) $$EKpMLpLp.$$

We shall redirect our attention to S4.4 later on in this paper.

1. S *images in the* S ° *systems*. We shall now show that by considering the operator usually read as "necessity" in the systems S1°-S4° to be a factor of necessity rather than necessity itself, we may find in each of these systems an image of its respective (without the '°') ordinary Lewis-modal system. As bases for S1°-S4°, we may use the C-N-L formulations of [1]; for our present purposes, however, let us employ for these systems the letter Q in place of L, and reserve L for the necessity operator in the "images" we will discover in S1°-S4°. In all of these systems, then, we will define L and M as follows:

Df. L: $L\varphi$ for $K\varphi Q\varphi$
Df. M: $M\varphi$ for $ANQN\varphi\varphi$

Axioms and rules for the systems will be drawn from the following stock, as in [1], with Q read for L:

J1a. $CQCpCqrQCQpCQqQr$
J1b. $CQCpqCQpQq$
J2. $CKQCpqQCqrQCpr$
Ja. If $\vdash\varphi$, then $\vdash Q\varphi$.
Jb. *If* φ *is an axiom or* **PC** *theorem*, $\vdash Q\varphi$.
Jc. *If* $\vdash QC\varphi\psi$, *then* $\vdash QCQ\varphi Q\psi$.
Jd. *If* $\vdash QC\varphi\psi$ *and* $\vdash QC\varphi\psi$, *then* $\vdash QCQ\varphi Q\psi$
Je. *If* $\vdash Q\varphi$, *then* $\vdash\varphi$.

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With "**PC**" as the full classical propositional calculus with detachment and substitution for variables, the bases are:

S1° = **PC** + *J2* + **Jb** + **Jd** + **Je**
S2° = **PC** + *J1b* + **Jb** + **Jc** + **Je**
S3° = **PC** + *J1a* + **Jb** + **Je**
T° = **PC** + *J1b* + **Ja** + **Je**
S4° = **PC** + *J1a* + **Ja** + **Je**.

We first note that in the systems under study, with Df. *L* and *M* as above,

(2) *EMpNLNp*

will clearly be a theorem; thus the standard definition of *M* in an *L*-primitive system holds in these systems. Now let φ be a theorem of one of the systems at hand; in particular, if the system in question is S3° or weaker, let φ be an axiom or **PC** theorem; if the system is T° or stronger, φ may be any theorem. We then have in each of these systems

(3) $\vdash Q\varphi$ — φ, **Ja** or **Jb**
(4) $\vdash L\varphi$ — (3), φ, **PC**, Df. *L*

Rules—call them $\mathbf{Ja}_L$ and $\mathbf{Jb}_L$—like **Ja** and **Jb** except for having *L* for *Q* then are derived rules within these systems, with $\mathbf{Ja}_L$ in T° and S4°, and $\mathbf{Jb}_L$ in the others. Further, in all these systems we have

(5) *CLpp* — **PC**, Df. *L*
(6) *LCLpp* — S1°, Df. *L*, $\mathbf{Jb}_L$

We may note also that whenever $L\varphi$ is a theorem, so too will be $Q\varphi$;

(7) *CLpQp* — **PC**, Df. *L*

is in fact a theorem of S1°. Thus, if we have an S1° theorem of form $LE\varphi\psi$, we will also have

(8) $\vdash QE\varphi\psi$ — Hyp., (7).

Whenever, then, we have an "*L*-strict" equivalence in S1°, we will also have the same equivalence "*Q*-strict"; by rule **Jd** we will have in S1° the rule—call it $\mathbf{Jd}_L$—of substitutivity of *L*-strict equivalents.

Easily recognizable as an S1° theorem is

(9) *CQCpqCQCrsQCKprKqs* — S1°
(10) *QCKKCpqCqrKQCpqQCqrKCprQCpr* — (9), S1°
(11) *LCKLCpqLCqrLCpr* = $J2_L$ — (10), S1°, Df. *L*.

With (6), (11), **PC**, and derived rules $\mathbf{Jb}_L$ and $\mathbf{Jd}_L$, it is evident that there is an S1 image in S1° when we employ the earlier stated definitions of *L* and *M*.

We now assume S2°, and further assume

(12)	$\vdash LC\varphi\psi$	Hyp.
(13)	$\vdash QCQ\varphi Q\psi$	(12), (7), **Jc**
(14)	$\vdash QCK\varphi Q\varphi K\psi Q\psi$	(13), (9)
(15)	$\vdash LCL\varphi L\psi$	(14), **PC**, Df. *L*.

Steps (12)-(15) show that there is in *S2°* a derived rule—$\mathbf{Jc}_L$—which is like rule **Jc** except for having *L* where the latter rule has *Q*.

We continue:

(16)	*QCCpqCpq*	S1°
(17)	*QCKCpqQCpqKCQpQpCpq*	*J1b*, (16), (9)
(18)	*LCLCpqCLpLq*	(17), S1°, Df. *L*

S2° is now seen to contain (6), (18), **PC**, and rules $\mathbf{Jb}_L$ and $\mathbf{Jc}_L$; it therefore contains an image of S2 with *L* defined as earlier.

We now assume S3°:

(19)	*CQCqrQCQCpqQCpr*	S3°
(20)	*QCQCpqQCQpQq*	S3°
(21)	*QCQCpqQCpq*	S1°
(22)	*QCKQCpqQCrsQCKprKqs*	S2°
(23)	*QCLCpqQCpq*	(7), S1°
(24)	*QCLCpqQCQpQq*	(19), (23), (20)
(25)	*QCQCtKQCpqQCrsQCtQCKprKps*	(19), (22)
(26)	*CQCpqCQCprQCpKqr*	S1°
(27)	*QCLCpqKQCpqQCQpQq*	(26), (23), (24)
(28)	*QCLCpqQCKpQpKqQq*	(25) *t/LCpq*, *r/p*, *s/Lp*, (27)
(29)	*QCLCpqLCLpLq*	(28), *J1b*, *S1°*, (26), Df. *L*

It should be clear that even without an application of **Jc** the formula

(30) *CLCpqLCLpLq*

may be shown to be an S3° thesis by deductions paralleling those leading to (29); we thus have

(31) *LCLCpqLCLpLq* (29), (30), **PC**, Df. *L*

as an S3° thesis; with (31), (6), **PC**, and rule $\mathbf{Jc}_L$, then, we have an S3 image in S3°. That S4° contains an analogous image of S4 follows immediately, for *S4°* will have the same *S3* image contained in *S3°* plus the unrestricted rule $\mathbf{Ja}_L$. In like manner, strengthening S2° to T° will strengthen the S2 image in S2° to a T image.

2. *Systems in which Q is definable*. We now consider a number of systems in which $Q\varphi$, although a factor of $L\varphi$, might be defined in terms of *L*. Noting the following stock of axioms:

G1: *CMLpLMp*
K1: *CpCMLpLp*

K2: *CpCLMLpLp*
K3: *CLpLMLp*

and definitions:

$\mathrm{Df}_1 Q$: *Q* for *ML*
$\mathrm{Df}_2 Q$: *Q* for *LML*

we may formulate the following *C-N-L* calculi based on standard axiomatizations of the Lewis-modal systems:

S4.4 = S3 + *K1* + Df_1 *Q*
S4.0.4 = S4 + *K2* + Df_2 *Q*
T.4 = S2 + *K1* + Df_1 *Q*
T.0.4 = T + *K2* + *K3* + Df_2 *Q*
T.2 = S2 + *G1*.

We observe first that in the field of S1°, with *K1* as an added axiom we have:

(32)	*LCNpCMpLMp*	
(33)	*LCNpCMLpLMp*	(32), S1
(34)	*LCpCMLpLMp*	*K1*, S1
(35)	*LCMLpLMp* = *LG1*	(33), (34), S1°
(36)	*LLCLpMp*	(35), S2°

In the field of S2, then, *K1* yields *G1* and so (36), and so—as is wellknown—the rule to infer $L\varphi$ from any theorem φ. Therefore, S3 + *K1* contains S4 and so is S4.4 [2], and T.4 and T.2 contain T. Clearly, S4.4 contains all the above-mentioned systems; S4.0.4 contains T.0.4 and S4; T.4 contains T.2. That S4.4 contains S4.0.4 *properly* and that T.4 is not contained in T.0.4 is shown by Matrix I (due to Parry [3]):

p =	1*	2	3	4	5	6	7	8
Lp =	1	6	7	8	5	6	7	8
Mp =	1	2	3	4	1	2	3	8

(Matrices referred to in this paper are assumed to include the standard 2 tables for *C* and *N*; designated value is 1.) Matrix I validates S4 and *K2*, but fails to validate *G1* (and so, of course, *K1*). T.4 by the same considerations is seen not to be contained in T.0.4.

T.4 is clearly not a subsystem of S4; that it is independent of S4 is shown by Matrix II:

p =	1*	2	3	4	5	6	7	8
Lp =	1	6	3	8	8	8	8	8
Mp =	1	1	1	1	1	6	3	8

Matrix II validates T and *K1*, but fails to do so for S4. This matrix also shows that T.0.4 is not contained in T.4, and so that these systems are independent, for it fails to validate *K3*. We may point out, by the way, that the addition of the Brouwerian formula

C12. *CpLMp*

or its dual

(37) *CMLpp*

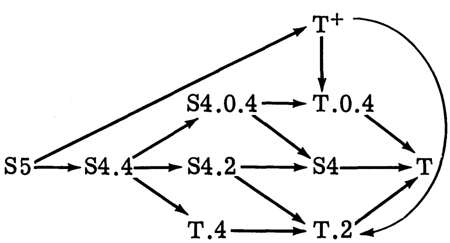
to T.4 yields S5. The addition of C12 (or (37)) to T, of course, gives the system T^+, which is independent of S4; in T^+ we have:

(38)	*CLMLpLp*	(37), T
(39)	*CpCLMLpLp* = *K2*	(38), PC
(40)	*CLpLMLp* = *K3*	C12; *p*/*Lp*;

T^+ thus includes T.0.4.

(41)	*CMLpLMp* = *G1*	C12, (38), PC;

T^+ then also includes T.2, which as a subsystem of T.4 not contained in T.0.4 is independent of T.0.4; T^+ itself is, of course, independent of T.4. We note here, by the way, that S4.0.4 contains S4 properly, for if it did not, S4 and so S4.2 would contain *K2*, which in the field of S4.2 is deductively equivalent to *K1*. The relationships between the systems we have been discussing are illustrated in the following diagram; the arrows point from properly containing to properly contained systems.



3. *The above systems with Q primitive*. We shall now present bases for the systems of factorable necessity of section 2 having as primitive modal operator not *L* or *M* as is usual, but *Q*, which might be read as the sign of "possible necessity" or of "necessarily possible necessity" depending on the system involved. In all cases, *L* and *M* will be defined as they were in section 1 of this paper, and we will draw from the following stock of axioms, as well as from the axioms and rules of section 1, for our formulations.

J1c. *CQpQQp*
J3. *CCpqCQCpqCpCQpQq*
J4. *CQpp*
J5. *CQpNQNp*
J6. *CQCQpNpCQpp*
J7. *CQpLMLp*
J8. *CNQNQpQp*
J9. *CQNQNQpQp*

J10. *CNQNLpQp*
J11. *CLMLpQp*
J12. *CNQNQpp*

Following will be our Q-primitive bases; each system will include the **PC** and rule **Ja**, and in addition, the indicated axioms.

QS5 : *J1b* + *J4* + *J8*
QS4.4 : *J1b* + *J1c* + *J5* + *J8*
QS4.0.4 : *J1b* + *J1c* + *J5* + *J9*
QT.4 : *J3* + *J6* + *J10*
QT^+ : *J3* + *J7* + *J11* + *J12*
QT.0.4 : *J3* + *J7* + *J11*.

We shall show in this section that the above Q-systems are equivalent to their respective L-primitive systems described in the previous section. We have here included S5 and T^+ as systems in which necessity is factorable; they do include, respectively, the theses *ELpKpMLp* and *ELpKpLMLp*, but they contain them in a manner different from that in which the other above systems contain them. S5 contains the law *ELpMLp* and T^+ has *ELpLMLp*; from these theorems follows trivially the factorability of necessity in these systems. The p in the conjunctions *KpMLp* and *KpLMLp* contributes nothing whatsoever to the interpretation of these conjunctions as *Lp* in S5 and T^+ respectively. This is not the case for the other systems discussed above; for them, both of the conjuncts as factors of necessity are needed for the interpretation of the formula as *Lp*. We may accordingly say that in systems like S5 and T^+ necessity is "improperly factorable," while in systems like S4.4 and the others, it is "properly factorable."

It should be clear that **PC** + **Ja** + *J1b* + *J1c* is a subsystem of QS5 as well as of QS4.4 and QS4.0.4. Many formulas will be easily recognizable as provable within this subsystem; such formulas we will justify simply by the words "*J1* base"; processes of deduction clearly permitted by this subsystem will also be so designated. We observe now that in all the above systems, rule **Ja** and **PC** permit us to state "*If* $\vdash\varphi$, *then* $\vdash K\varphi Q\varphi$," which with Df. L is

$\mathbf{Ja}_L$: *If* $\vdash\varphi$, *then* $\vdash L\varphi$."

Also, in all of these systems, by **PC** and Df. L we have the following two theses:

(42) *CLpp*
(43) *CpCQpLp* .

In the systems containing the *J1* base, we will have, by methods paralleling those of section 1, formula (31)—*LCLCpqLCLpLq*—as a thesis. By (31), (42), and $\mathbf{Ja}_L$, then, systems QS5, QS4.4, and QS4.0.4 contain S4. So far as the other systems—containing the weaker *J3*—are concerned:

(44)	*CCpqCQCpqCpCQpq*	PC
(45)	*CCpqCQCpqCpCQpKqQq*	(44), *J3*, PC
(46)	*CLCpqCLpLq*	(45), PC, Df. *L*

By (46), (42), and $\mathbf{Ja}_L$ the systems QT.4, QT^+, and QT.0.4 then contain T. We now go on to show that the definitions of *Q* in the *L*-primitive systems hold in the respective *Q*-primitive systems. Working in QS4.4, we have:

(47)	*QCQpCpKpQp*	*J1* base
(48)	*CQpQCpKpQp*	(47), *J1* base
(49)	*CQpCNQNpNQNKpQp*	(48), *J1* base
(50)	*CQpNQNLp*	(49), *J5*, Df. *L*, PC
(51)	*CQpMLp*	(50), Df. *M*, PC
(52)	*CNQNLpNQNQp*	Df. *L*, *J1* base
(53)	*CNQNLpQp*	(52), *J8*, PC
(54)	*CANQNLpLpQp*	(53), Df. *L*, PC
(55)	*CMLpQp*	(54), Df. *M*.

Since the interchangeability of even material equivalents holds in the *J1* base (actually, in all of our systems) by (51) and (55) Df_1 *Q* holds in QS4.4. We also have

(56)	*CpCMLpLp* = *K1*	(43), (55), PC;

S4.4 is therefore contained in QS4.4.

Working in the other direction, we have (assuming S4.4)

(57)	*CMLCpqMLCpq*	PC
(58)	*CMLCpqMCMLpMLq*	(57), S4°
(59)	*CMLCpqCLMLpMMLq*	(58), S1°
(60)	*CMLCpqCMLpMLq*	(59), S4.2.

But with Df_1 *Q*, (60) is axiom *J1b*. This points up an interesting and indeed characteristic feature of S4.2, by the way—in this system, *ML* distributes over implication. Easily recognizable as S4 theses are

(61)	*CMLpMLMLp*
(62)	*CNMLNMLpMLp* (*CLMMLpMLp*)

which with the application of Df_1 *Q* become axioms *J1c* and *J8*, respectively. An obvious S4.2 thesis is

(63)	*CMLpNMLNp*

All the axioms of QS4.4, then, are S4.2 theses. Equivalences corresponding to the definitions of *L* and *M* in the *Q*-primitive systems are—in the presence of Df_1 *Q*—theorems of S4.4; QS4.4 and S4.4 are then equivalent systems.

If *L* were written for *Q* in the axioms and rule of QS5, we would have a basis for *L*-primitive S5. It is then obvious that the system QS4.4 is contained in QS5; so too then is S4.4 a subsystem of QS5. But by *J4* and

(55) we then have (37), *CMLpp*, as a thesis of QS5; (37) in the field of S4 yields S5, which is then contained in QS5. S5, containing S4.4, also contains QS4.4; by Df_1 *Q*, it also contains *J4*. S5 and QS5 are then equivalent.

We now assume system QS4.0.4; note that the steps leading to the proof of formula (51) may be performed in this system as in S4.4; we have:

(64)	*CQpQMLp*	(51), *J1* base
(65)	*CQpLMLp*	(51), (64), PC, Df. *L*
(66)	*CQpNQNQp*	*J5*, *J1* base
(67)	*QCANQNQpQpNQNQp*	(66), *J1* base
(68)	*CQMQpQNQNQp*	(67), *J1* base Df. *M*
(69)	*CQMQpQp*	(68), *J9*, PC
(70)	*CLMLpQp*	(69), Df. *L*, PC

By (65) and (70), the equivalence corresponding to Df_2 *Q* holds in QS4.0.4.

(71)	*CpCLMLpLp* = *K2*	(43), (70), PC

As noted previously, QS4.0.4 contains S4; containing *K2*, then, it also contains S4.0.4. Working in the other direction:

(72)	*CLMLCpqLMCLpLq*	S2°
(73)	*CLMLCpqLCLpMLq*	S4°, (72)
(74)	*CLMLCpqLCMLpMLq*	(73), S4°
(75)	*CLMLCpqCLMLpLMLq*	(74), S1°.

LML then distributes over *C* in S4; with Df_2 *Q*, (75) is axiom *J1b*. We also have

(76)	*CLMLpLMLLMLp*	S4
(77)	*CLMLpNLMLNp* (*CLMLpMLMp*)	S4
(78)	*CLMLNLMLNLMLpLMLp* (*CLMLMLMLMLpLMLp*)	S4.

With Df_2 *Q*, the above three formulas are respectively axioms *J1c*, *J5*, and *J9*. All the axioms of QS4.0.4 are, then, S4 theses. By *K2*, the proper axiom of S4.0.4 and Df_2 *Q*, the equivalences corresponding to the *Q*-primitive definitions of *L* and *M* will be S4.0.4 theses. QS4.0.4 is then contained in S4.0.4, and the two systems are equivalent.

We now assume the system QT.4; we then have:

(79)	*CANQNLpLpQp*	*J10*, PC, Df. *L*
(80)	*CMLpQp*	(79), Df. *M*
(81)	*CpCMLpLp* = *K1*	(43), (80), PC
(82)	*CQpCQNKpQpp*	*J6*, PC
(83)	*CQpQNQNLpp*	(82), PC, Df. *L*
(84)	*CQpANQNLpQp*	PC
(85)	*CQpMLp*	(83), (84), PC, Df. *L*, *M*

Formulas (80) and (85) show that Df_1 *Q* holds in QT.4; with (81) and the previously established fact that QT.4 contains T, we have T.4 as a subsystem of QT.4.

In the system T.4, on the other hand, we have

(86)	*CLCpqCLpMLq*	S1
(87)	*CKCpqMLCpqCKpMLpMLq*	(86), T.4
(88)	*CCpqCQCpqCpCQpQq* = *J3*	(87), **PC**, Df_1 *Q*
(89)	*CLMLpMLp*	S1
(90)	*CNQNLpQp* = *J10*	(89), S1, Df_1 *Q*.
(91)	*CLMpLMp*	**PC**
(92)	*CKMpMLMpLMp*	(91), T.4
(93)	*CMLNLpCNLpLMNp*	(92) *p/Np*, **PC**, Df. *M*
(94)	*CMLNLpCMLpLp*	(93), S1°
(95)	*CQNKpQpCQpLp*	(94), Df_1 *Q*, T.4
(96)	*CQCQpNpCQpp* = *J6*	(95), S1.

By (88), (90), and (96), all the axioms of QT.4 are theorems of T.4; the equivalences for the *Q*-primitive definitions of *L* and *M* in QT.4 are characteristic T.4 theses, in the presence of Df_1 *Q*. QT.4 and T.4 are thus equivalent systems.

We now assume system QT.0.4; here we have immediately with axioms *J7* and *J11* the formulas needed to prove Df_2 *Q*; by (43), then, *K2* will be a QT.0.4 theorem, and

(97) *CLpLMLp* = *K3*

follows immediately by Df. *L* and axiom *J7*. T.0.4 is thus contained in QT.0.4.

Assuming T.0.4, we have

(98)	*CLCpqCLpLMLp*	S1°, *K3*
(99)	*CKCpqLMLCpqCKpLMLpLMLq*	(98), T.0.4
(100)	*CCpqCQCpqCpCQpQq* = *J3*	(99), Df_2 *Q*, **PC**

Clearly, *J7* and *J11* are T.0.4 theorems immediately by Df_2 *Q*; the definitions of *L* and *M* in QT.0.4 are, again, characteristic T.0.4 theses, since T.0.4 employs Df_2 *Q*. T.0.4 and QT.0.4 are then equivalent.

The addition of *J12* to QT.0.4 gives us QT^+; in the field of QT.0.4, *J12* yields:

(101)	*CMLMLMLpp*	
(102)	*CMLpMLMLMLp* T.0.4 (*CLpLMLp*, *CMpMLMp*)	
(103)	*CMLpp*	(101), (102), **PC**.

But (103) in the field of T yields T^+. T^+ is thus included in QT^+. In T^+, we have—by *CLMLpLp* and *CMLpp*:

(104)	*CMLMLMLpp*	
(105)	*CNLMLNLMLpp*	(104), S1
(106)	*CNQNQpp* = *J12*	(105), Df_2 *Q*.

T^+ thus contains *J12* and so—since it also contains T.0.4—it is equivalent to QT^+.

REFERENCES

[1] Zeman, J. Jay, "Lemmon-style bases for the systems S1°-S4°" to appear in *The Journal of Symbolic Logic*.

[2] Sobociński, B., "Modal system S4.4," *Notre Dame Journal of Formal Logic*, v. 5 (1964), pp. 305-312.

[3] Parry, W. T., "Modalities in the *Survey* system of strict implication," *The Journal of Symbolic Logic*, v. 4 (1939), pp. 137-154.

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