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SOME EXAMPLES OF DIFFERENT METHODS OF FORMAL PROOFS WITH GENERALIZATIONS OF THE SATISFIABILITY DEFINITION

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This paper[1] is composed of three parts. In the first one we recall my generalization of the usual satisfiability definition, we give a new general variant of my truncated truth definition with it a syntactic picture of sequents; we also construct a generalized diagram introduced here according to the above semantics and [9], and analogous to [5], see also [2]. In the second part generalized sequent proof rules based on their semantics of [3] with their generalized diagram are given, and general decidability possibilities for formulas of the first-order functional calculus are supplied; the last method restricts the number of variables to a finite number but possibly with infinite many monadic relations. The third part includes different examples solved by introduced generalized sequent proof rules. The cited papers with our explanations prove the adequacy of the semantic and syntactic considerations. Certain generalizations of the above results will be included in my future papers.

We use notions and denotations of [3]-[12] and shortly: alternative +; negation '; general quantifier Π; free variables $x, x_1, \ldots$; apparent variables $a, a_1, \ldots$; relations signs $f_1^1, \ldots, f_q^1, \ldots, f_1^t, \ldots, f_q^t$ (f_i^m - of m arguments); expressions $E, F, E_1, F_1, \ldots$; $\{i_l\} = i_1, \ldots, i_l$; $w(E)$ - the maximal number of free variables ($p(E)$ - apparent variables) occurring in E; $\{i_{w(E)}\}$ - sequence of all indices of free variables occurring in E; $i(E) = \max(\{i_{w(E)}\})$; $n(E) = \max\{i(E), w(B) + p(G)\}$, for each alternative indecomposable members G of E^2; $m(E) = i(E) + p(E)$; $\{F_q^t\}$ - the sequence $F_1^1, \ldots, F_q^1, \ldots, F_1^t, \ldots, F_q^t$; $Q, Q_1, \ldots$ - non-empty sets of tables of the same rank; $Q(k)$ - elements of Q have the same rank k; $A, A_1, \ldots$ - sets of indecomposable formulas (i.e. atomic formulas with their negation) whose indices of free variables are $\leq k$ (k is named the rank of the sets) and for such formulas: $E \varepsilon A \;.\!\equiv.\; E' \bar{\varepsilon} A$; $\Gamma, \Gamma_1, \ldots$ - arbitrary sets of formulas; $X, Y, X_1, Y_1, \ldots$ - models $\mathbf{M}$ or sets A described above; $\mathbf{M}/s_1, \ldots, s_k/ = < D_k, \{\phi_q^t\} > \;.\!\equiv.\; (\mathbf{M} = < D, \{F_q^t\} >) \wedge (\phi_j^i(r_1, \ldots, r_i) \;.\!\equiv.\; F_j^i(s_{r_1}, \ldots, s_{r_i}),$

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$i = 1, \ldots, t$ and $j = 1, \ldots, q$); $E \varepsilon A/s_1, \ldots, s_k/ .\equiv. E(x_{s_1}/x_1) \ldots (x_{s_k}/x_k)$ εA^3 ($A/s_1, \ldots, s_k/$ is restricted to indecomposable formulas); $X/\{s_k\} = X/s_1, \ldots, s_k/$; $X\varepsilon Y[k] .\equiv. (\exists \{s_k\}) \{X = Y/\{s_k\}\}$; $C\{E\}$ - the set of all parts of E; $\Gamma(\{i_l\})$ - the set of all formulas belonging to Γ whose free variables have indices $\{i_l\}$; $\mathbf{M}\{E\} = 0$, i.e. E is true in the model $\mathbf{M}$; $\mathbf{M}\{E\{s_k\}\} = 0$, i.e. $\{s_k\}$ are elements of the domain of $\mathbf{M}$, x_j are names of s_j and $\{s_k\}$ do not satisfy E in the model $\mathbf{M}$; T is the description of A iff $T = < D_k, \{F_q^t\} >$, T, A - have the same rank and for each $m_1, \ldots, m_j \leq k$ and $j \leq t$, $i \leq q$: $F_i^j(m_1, \ldots, m_j) .\equiv. f_i(x_{m_1}, \ldots, x_{m_j}) \varepsilon A$. A sequence of formulas is called fundamental iff E and E' occur in one; $R(\mathbf{M}) .\equiv. (s_1)(s_2) \{(\mathbf{M}/s_1/ = \mathbf{M}/s_2/) \rightarrow (s_1 = s_2)\}$.

Of course:

L.1. $X/\{s_k\}/\{j_m\} = X/\{s_{j_m}\}$, see [1].

L.2. If T_1 is the description of A_1 and T_2 is the description of A_2 and both tables have the same rank, then: $T_1/\{j_m\} = T_2/\{j_m\} .\equiv. A_1/\{j_m\} = A_2/\{j_m\}$.

For an arbitrary Q such that $Q(n)$, for an arbitrary formula E, for an arbitrary $T = < D_n, \{F_q^t\} > \varepsilon Q$ and for each k such that $i(E) \leq k$ and $k + p(E) \leq n$ we introduce the following inductive definition of the functional V:

(*1d*) $V\{n, Q, T, k, f_j^m(x_{r_1}, \ldots, x_{r_m})\} = 1 .\equiv. F_j^m(r_1, \ldots, r_m)$,
(*2d*) $V\{n, Q, T, k, F'\} = 1 .\equiv. \sim V\{n, Q, T, k, F\} = 1 .\equiv. V\{n, Q, T, k, F\} = 0$,
(*3d*) $V\{n, Q, T, F + G\} = 1 .\equiv. V\{n, Q, T, k, F\} = 1 \vee V\{n, Q, T, k, G\} = 1$,
(*4d*) $V\{n, Q, T, k, \Pi aF\} = 1 .\equiv. (i) \{(i \leq k) \rightarrow V\{n, Q, T, k, F(x_i/a)\} = 1\}$
$\wedge (T_1)\{(T_1 \varepsilon Q) \wedge (T_1/\{k\} = T/\{k\}) \rightarrow V\{n, Q, T_1, k+1, F(x_{k+1}/a)\} = 1\}$.

D.1. $N(Q, n, G) .\equiv. (k)\{(k + p(G) \leq n) \wedge (i(G) \leq k) \rightarrow$
$(T)(V\{n, Q, T, k+1, G\} = 1 .\equiv. V\{n, Q, T, k, G\} = 1)\}$.
D.2. $E \varepsilon P(n, Q, T, k) .\equiv. (\exists G)\{(G \varepsilon C\{E\}) \ (N(Q, n, G) \rightarrow V\{n, Q, T, k, E\} = 1)\}$.
D.3. $E \varepsilon P\{n\} .\equiv. (Q)(T)\{Q(n) \wedge (T \varepsilon Q) \rightarrow (E \varepsilon P(n, Q, T, i(E)))\}$.
D.4. $E \varepsilon P .\equiv. (\exists n)\{(n \geq m(E)) \wedge (E \varepsilon P\{n\})\}$.

The relation $N(Q, n, G)$ is invariant respectively to the number k and it holds for all quantifierless formulas G.

Definitions (*1d*)-(*4d*) are - in a suitable meaning - generalizations of the satisfiability definition in the domain of natural numbers $1, \ldots, n$; the case is analogous and remains for readers, see [5], [7], [9].

If we assume that Q is one elementing, then (*4d*) is in a certain sense the usual satisfiability definition in the domain $1, \ldots, n$.

If M is a model and $Q = \mathbf{M}[k]$, then elements of Q are submodels of $\mathbf{M}$ in the meaning of homomorphism, the number $i(F) + 1$ in (*4d*) is the name of an arbitrary element of the domain of $\mathbf{M}$.

Of course:

(*4d'*) $V\{n, Q, T, k, \Pi aF\} = 0 .\equiv. (\exists i)\{(i \leq k) \wedge V\{n, Q, T, k, F(x_i/a)\} = 0\}$
$\vee (\exists T_1)\{(T_1 \varepsilon Q) \wedge (T_1/\{k\} = T/\{k\}) \wedge V\{n, Q, T_1, k+1, F(x_{k+1}/a)\} = 0\}$.

(5d') $V\{n, Q, T, k, \Sigma aF\} = 0 .\equiv. (i)\{(i \leq k) \rightarrow V\{n, Q, T, k, F(x_i/a)\} = 0\}$
$\wedge (T_1)\{(T_1 \varepsilon Q) \wedge (T_1/\{k\} = T/\{k\}) \rightarrow V\{n, Q, T_1, k+1, F(x_{k+1}/a)\} = 0\}$.

L.3. If $T/\{k\} = T^0/\{k\}$, then:
$V\{n, Q, T, k, E\} = 1 .\equiv. V\{n, Q, T^0, K, E\} = 1$

The proof of *L.3.* is easy and inductive on the length of the formula E, see, *L.3.* in [7] and *L.2.* in [9].

T.1. If E is an alternative of formulas of the form $\Sigma a_1 \dots \Sigma a_{j-1} \Pi a_j G$, for some quantifierless and variable-free G, $F \varepsilon C\{E\}$, $\mathbf{M}\{E\} = 0$, $k \geq m(E)$, $Q = \mathbf{M}[k]$, $T \varepsilon Q$, $i(F) \leq k$, then:

(1) *If $k + p(F) \leq n$, $\mathbf{M}\{F(1, \dots, k)\} = 0$ and $\mathbf{M}/\{s_k\} = T/\{k\}$, then $V\{n, Q, T, k, F\} = 0$ and for each $H \varepsilon C\{E\}$ we have $N(Q, n, H)$ and there- $E \bar{\varepsilon} P$.*
(2) *If $R(\mathbf{M})$, $\mathbf{M}/\{s_k\} = T/\{k\}$, then for an arbitrary formula F:*
$\mathbf{M}\{F\{s_k\}\} = 0 .\equiv. V\{n, Q, T, k, F\} = 0$.

The inductive prrof of *T.1.* is almost identical with the proof of *T.2.* in [8] and *T.6.* in [9] and remains for readers. In *T.5.* we also use the property of the inductive proof of (*1*) that in this proof it sufficies to use the second member of the alternative at (*4d'*).

T.2. If $E_1, \dots E_r$ is a formalized proof of the formula E, and $n \geq \max\{m(E_1), \dots, m(E_r)\}$, then $E_j \varepsilon P\{n\}$, $j = 1, \dots, r$.

The proof of *T.2.* is almost identical with the proof of *T.2.* in [9] and analogous to the proof of *T.3.* in [7]. From *T.1.* and *T.2.* follows (see also the construction of Skolem's normal forms):

T.3. A formula E is a thesis iff $E \varepsilon P$.

For normal forms we received a more strong theorem given in [8], [9], namely that for ones we can replace *D.2.* by:

D.2'. $E \varepsilon P(n, Q, T, k) .\equiv. N(Q, n, E) \rightarrow V\{n, Q, T, k, E\} = 1$

and the second equivalence in *D.1.* we can replace by the implication.

In order to give sequent proof rules we introduce certain additional definitions, see [6].

For each k, n, Γ, F: x means the first variable x_i such that $i \leq n$ and $F(x_i/a) \bar{\varepsilon} \Gamma$.

We consider equivalence sequent proof schemas—proof rules—which we read in an usual manner with certain generalizations, see Figure I; we read e.g.:

$$\frac{\Gamma_1}{\dfrac{\Gamma_2}{\dfrac{\Gamma_3}{\cdots\cdots}}}$$

- from Γ_1 follows Γ_2 and from Γ_2 follows $\Gamma_3, \dots$; such schemas are called diagrams and we assume $\Gamma_1, \Gamma_2, \Gamma_3, \dots$ non-empty;

$$\frac{\Gamma}{\Gamma_1 \mid \Gamma_2}$$ - from Γ follows Γ_1 or Γ_2; the rule determines two diagrams; the prolongation of Γ in the first diagram is Γ_1 and in the second one is Γ_2.

All above schemas determine one last line and the following:

$$\frac{\Gamma}{\Gamma_1 \mid \Gamma_2 \mid \ldots}$$ - determines many last lines according to the number of $\mid$ and it means: from Γ follows Γ_1 and Γ_2 and ...; the prolongation of Γ is Γ_1 and the prolongation of Γ_i is Γ.

Composition of such proof rules according to a diagram is called a generalized diagram or a generalized tree; we apply all proof rules in a generalized diagram which describes their use, see Figure I ... According to the interpretation E and 0 and according to the generalized satisfiability definition given above, we apply to an arbitrary formula E - called a topformula - the following sequent proof rules depending on a given number n:

$$(\mathbf{A})\ \frac{\Gamma,\ F+G}{\Gamma,\ F,\ G};\quad (\mathbf{K})\ \frac{\Gamma,\ (F+G)'}{\Gamma,\ F' \mid \Gamma,\ G'};\quad (\mathbf{N})\ \frac{\Gamma, F''}{\Gamma, F};$$

(Π_1) $$\frac{\Gamma,\ (\Pi aF)'}{\Gamma,\ (\Pi aF)',\ F'(x_i/a)}$$ - if $i = n - p(F) + 1$, then we do not apply further the rule to the formula $(\Pi aF)'$ with explanations given below;

(Π_2) $$\frac{\Gamma,\ \Pi aF}{\Gamma \mid \Gamma,\ F(x_k/a) \mid \ldots}$$ - $k = i(F) + 1,\ i(F) + 2, \ldots, n - p(F) + 1$; last lines of columns Γ, $F(x_k/a)$ here considered must be equal with Γ on free variables with indices $1, \ldots, k - 1$ (it suffices to assume the property for last lines of these columns).

A generalized diagram is correct if for two arbitrary columns $\mathcal{T}_1$, $\mathcal{T}_2$ and each formula F:

1. If $\mathcal{T}_1/\{k\} = \mathcal{T}_2/\{k\}^4$, $i(F) \leq k$, $k + p(F) \leq n$, then F occurs in the column $\mathcal{T}_1$ iff F occurs in the column $\mathcal{T}_2$.
For each $F \varepsilon C\{E\}$, if $m(F) \leq n$, then either F belongs to $\mathcal{T}_1$ or F' belongs to $\mathcal{T}_1$.

The above points *1* and *2* mean that if for a generalized diagram points *1* and *2* are not fulfilled, then we add to suitable columns respective formulas, i.e. in point *1* we add the formula F and in point *2* forumlas F or F' and afterwards we act according to the introduced sequent rules; the point *1* means also that if ΠaG occurs in a certain column, then the rule (Π_2) must be fulfilled for this column.

In the following we consider only correct diagrams. In the classical case we assume that all columns are equal, i.e. in the classical diagram we have only one column; thus all assumptions about columns are less here and we recieve an usual sequent proof, see [2]; the choosing of the column in this case, i.e. the choosing of the number k in (Π_2) is classical; therefore our proof rules are generalizations of classical ones. According to the considered proof rules each formula E determines a generalized diagram composed of columns with the main top E:

Work scheme of a generalized diagram

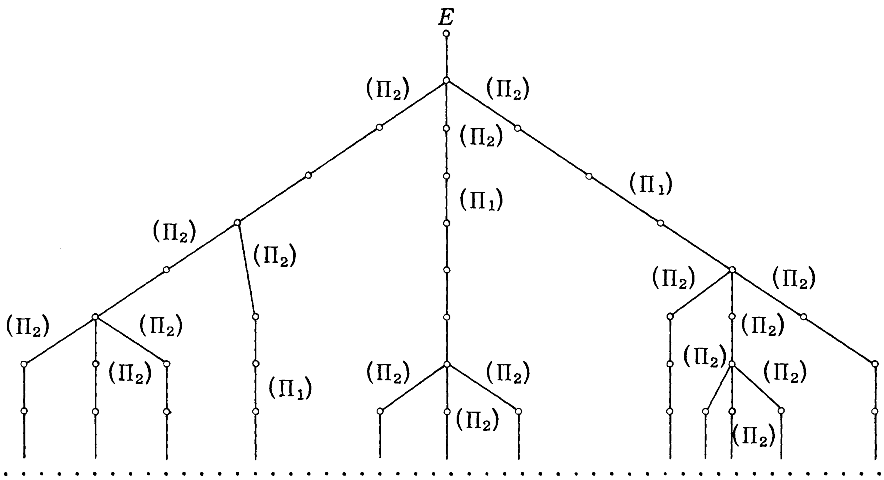

Figure I

Each column determines a new last line; a line is denoted by a circle. Signs (Π_1) and (Π_2) on the figure denote the application of rules (Π_1) and (Π_2) respectively; every new circle is generated by application of a certain proof rule. Dots denote prolongation of the diagram according to considered sequent proof rules and properties *1-2* of a correct diagram.

T.4. If for each $n \geq m(E)$ all lines of each column of a certain generalized diagram are not fundamental, then E is not a thesis.

Proof. In order to prove *T.4.* for a given formula E we consider a natural number $n \geq m(E)$ and the generalized diagram of E with properties described in the theorem. Each last line we consider as a set A of formulas of the rank n (completion of the last line to the set A of the rank n is here arbitrary) and to each set A we attribute the description T of negated indecomposable formulas belonging to A (thus A and T have the same rank n) and the family of all such T's creates the set Q of tables of the rank n^5. We point out each last line A determines the described table T and a column $\mathcal{J}$ with the basis A and the top E. We prove by induction on the length of a formula H:

(*1*) If $H \varepsilon \mathcal{J}$, then $V\{n, Q, T, k, H\} = 0$, for each k such that $i(H) \leq k$, $k + p(H) \leq n$.

For atomic formulas and their negation, (*1*) holds by the assumption. Let (*1*) hold for formulas of the length $<r$; we shall prove it for formulas H of the length r.

We consider here three cases:

1. $H = F + G$, for some F, G;
2. $H = F'$, for some F;
3. $H = \Pi aF$, for some F.

In the case $H = F + G \varepsilon \mathcal{T}$ by virtue of (**A**) we receive $F, G \varepsilon \mathcal{T}$; therefore by the inductive assumption $V\{n, Q, T, k_1, F\} = 0$, for each $k_1 \geq i(F)$, $k_1 + p(F) \leq n$, and $V\{n, Q, T, k_2, G\} = 0$, for each $k_2 \geq i(G)$, $k_2 + p(G) \leq n$; therefore by (*3d'*) $V\{n, Q, T, k, F + G\} = 0$, for each $k \geq i(F + G)$, $k + p(F + G) \leq n$, which proves (*1*) in the first case.

In the case $H = F'$ we consider three cases:

(1^0) $F = F_1'$;
(2^0) $F = F_1 + G_1$;
(3^0) $F = \Pi aF_1$.

In the case $F = F_1'$ we have by assumption $H = F_1'' \varepsilon \mathcal{T}$; therefore by (**N**) we have $F_1 \varepsilon \mathcal{T}$. Hence by means of the inductive assumption $V\{n, Q, T, k, F_1\} = 0$, for each $k \geq i(F_1)$, $k + p(F_1) \leq n$, and because $i(F_1) = i(H)$, we have also $V\{n, Q, T, k, H\} = 0$, for each $k \geq i(H)$, $k + p(H) \leq n$, which proves (*1*) in the case (1^0).

In the case $F = F_1 + G_1$ we have by assumption $H = (F_1 + G_1)' \varepsilon \mathcal{T}$; therefore by (**K**) or $F_1' \varepsilon \mathcal{T}$ or $G_1' \varepsilon \mathcal{T}$.

We consider here the case $F_1' \varepsilon \mathcal{T}$; the case $G_1' \varepsilon \mathcal{T}$ is analogous.

From the above by means of the inductive assumption we have $V\{n, Q, T, k_1, F_1'\} = 0$, for each $k_1 \geq i(F_1')$, $k_1 + p(F_1') \leq n$; therefore by (*2d*) $V\{n, Q, T, k_1, F_1\} = 1$, for each $k_1 \geq i(F_1)$, $k_1 + p(F_1) \leq n$, and by (*3d*) and (*2d*) we obtain respectively $V\{n, Q, T, k, (F_1 + G_1)'\} = 0$, for each $k \geq i(F_1 + G_1)$, $k + p((F_1 + G_1)') \leq n$, i.e. $V\{n, Q, T, k, H\} = 0$, for each $k \geq i(H)$, $k + p(H) \leq n$, which proves (*1*) in the case (2^0).

In the case $F = \Pi aF_1$ we have by assumption $H = (\Pi aF_1)' \varepsilon \mathcal{T}$; therefore by the property (*1*) of the correct diagram and (Π_1) for each $k \geq i(F_1)$, $k + p(F_1) \leq n$, for every $i \leq k + 1$ and for each $\mathcal{T}_1$ if $\mathcal{T}_1/\{k\} = \mathcal{T}/\{k\}$, then $(\Pi aF_1)' \varepsilon \mathcal{T}_1$ and $F_1'(x\ /a) \varepsilon \mathcal{T}_1$; hence by the construction of Q, *L.2.* and the inductive hypothesis for each $k \geq i(F_1)$ $k + p(F_1) \leq n$, for every $i \leq k + 1$ and for each $T_1 \varepsilon Q$, if $T_1/\{k\} = T/\{k\}$, then $V\{n, Q, T_1, k + 1, F_1'(x_i/a)\} = 0$ and $V\{n, Q, T_1, k + 1, F_1'(x_i/a)\} = 1$. Therefore by virtue of (*4d*) $V\{n, Q, T, k, \Pi aF_1\} = 1$, for each $k \geq i(F_1)$, $k + p(F_1) \leq n$, and therefore by (*2d*) we have $V\{n, Q, T, k, H\} = 0$, for each $k \geq i(H)$, $k + p(H) \leq n$, which proves (*1*) in the case (3^0).

In the last case $H = \Pi aF \varepsilon \mathcal{T}$. Hence in view of the construction of the generalized diagram and (Π_2) for each $k \geq i(F)$, $k + p(F) \leq n$, there exists $\mathcal{T}_1$ such that $\mathcal{T}_1/\{k\} = \mathcal{T}/\{k\}$ and $F(x_{k+1}/a) \varepsilon \mathcal{T}_1$.

Hence in view of the definiton of Q, *L.2.*, and the inductive hypothesis for each $k \geq i(F)$, $k + p(F) \leq n$, there exists T_1 such that $T_1/\{k\} = T/\{k\}$ and $V\{n, Q, T_1, k + 1, F(x_{k+1}/a)\} = 0$. Thus by virtue of (*4d'*) $V\{n, Q, T, k, \Pi aF\} = 0$, for each $k \geq i(\Pi aF)$, $k + p(\Pi aF) \leq n$, and also $V\{n, Q, T, k, H\} = 0$, for each $k \geq i(H)$, $k + p(H) \leq n$, which proves (*1*) in the last case *3*.

Thus we closed the inductive proof of (*1*). Therefore for formulas H belonging to the generalized diagram we proved $N(Q, n, H)$. If now $H \varepsilon C\{E\}$, $m(H) \leq n$, then in view of the construction of a correct generalized diagram, property *2*, either H belongs to each column of the generalized diagram or H' belongs to the same column. Therefore in view of the above we have $N(Q, n, H)$ for each $H \varepsilon C\{E\}$, $m(H) \leq n$. Because E belongs to the diagram, therefore even for each $T \varepsilon Q$, we have $V\{n, Q, T, k, E\} = 0$, for each $k \geq i(E)$, $k + p(E) \leq n$, and therefore $E \bar{\varepsilon} P\{k\}$. From the above and the assumption we obtain $E \bar{\varepsilon} P$ and therefore in view of *T.2.* E is not a thesis.

According to our explanation, see footnote 5 . . . , the proof also holds in the classical case; then we only have one column, but we must consider also $n = \aleph_0$.

T.5. If a line of a certain column of each generalized diagram of E for certain $n \geq m(E)$ is fundamental, then E is a thesis.

Proof: To the contrary, if E is not a thesis, then according to *T.1.* $E \varepsilon P$; therefore for each $n \geq m(E)$ there exists such set Q_1 of tables of the rank n and there exists a table $T \varepsilon Q_1$ such that for each $k \geq i(E)$ we have $V\{n, Q_1, T, k, E\} = 0$ and for each $G \varepsilon C(E)$ we have $N(Q_1, n, G)$ (according to the remark given in the description of the inductive proof of *T.1.*, using (*4d'*) we use here only the second member of the alternative of it). Then the generalized satisfiability definition determines here a generalized diagram analogous to sequent proof rules which correspond to the satisfiability definition; according to the above, the diagram has no fundamental line contrary to the assumption of *T.5.*

From *T.4.* and *T.5.* follows:

T.6. A formula E is a thesis iff each of its generalized diagrams has a fundamental line for certain $n \geq m(E)$.

We proved analogical theorems in [5] by replacing $i(E)$ by means of $\{i_{w(E)}\}$; the proof was also analogous. In order to give sequent proof examples we shall introduce also a second generalized diagram based on [3] but first of all we recall certain definitions, lemmas and theorems with their stronger modification and without additional explanations:

D.5. $m(Q, k) \;.\!\equiv\!.\; Q(k) \wedge (\{t_k\})(T)\{(t_1, \ldots, t_k \leq k) \wedge (T \varepsilon Q) \rightarrow (T/\{t_k\}\, \varepsilon Q)\}$

D.6. $N(Q, k) \;.\!\equiv\!.\; m(Q, k) \;\; (t)(T_1)(T_2)(\exists T_3)\; (t + 2 \leq k)\, \{(T_1,\ T_2 \varepsilon Q) \wedge$
$(T_1/\{t\} = T_2/\{t\}) \rightarrow (T_3 \varepsilon Q) \wedge (T_3/\{t + 1\} = T_1/\{t + 1\}) \wedge (T_3/\{t\}, t + 2/$
$= T_2/\{t\}, t + 2/)\}$.

L.4. If $N\{Q, k\}$, then for an arbitrary permutation[6] $s_1, \ldots, s_r, s_{r+1}, \ldots, s_{k-1}, s_k$ *of natural numbers $\leq k$ we have:*

$(T_1)(T_2)(\exists T_3)\, \{(T_1/\{s\} = T_2/\{s_r\}) \wedge (T_1,\ T_2 \varepsilon Q) \rightarrow (T_3 \varepsilon Q) \wedge (T_3/\{s_{k-1}\} =$
$T_1/\{s_{k-1}\}) \wedge (T_3/\{s_r\},\ s_k/ = T_2/\{s_r\},\ s_k/)\}$.

L.5. If $Q = \mathbf{M}[k]$, then $m(Q, k)$.

D.7. $R(\mathbf{M})$.≡. $(t)(j)\{(\mathbf{M}/t/ = \mathbf{M}/j/) \rightarrow (k = j)\}$

L.6. Each model $\mathbf{M}_1 = \langle D, \{F_j^i\}\rangle$ *may be extended to a model* $\mathbf{M} = \langle D, \{F_j^i\}, \{G_r^1\}\rangle$ *such that* $R(\mathbf{M})$ [*we extended the model by means of monadic relations* $\{G_r^1\}$].

L.7. If $R(\mathbf{M})$ *and* $Q = \mathbf{M}[k]$, *then* $N(Q, k)$.

L.8. If $\mathbf{M} = \langle D, \{F_j^1\}\rangle$ *is a monadic model and* $Q = \mathbf{M}[k]$, *then* $N(Q, k)$.

D.8. $T^0 \varepsilon Q|k|$.≡. $(\exists n)(\exists T)\{(n \geq k) \wedge Q(n) \wedge (T \varepsilon Q) \wedge (T^0 = T/1, \ldots, k/)\}$.

L.9. If $N(Q, n)$, $n \geq k$ *and* $Q^0 = Q|k|$, *then* $N(Q^0, k)$.

L.10. If $k \leq n$, $N(Q^0, k)$, *then* Q^0 *may be extended to such minimal* Q (*using only the property defined in D.6.*) *that* $Q^0 \subset Q|k|$ *and* $N(Q, n)$.

We shall call the extension considered in *L.10.* the minimal extension of Q^0 respective to the property $N(Q^0, k)$.

Assuming $Q(n)$ we recall the finite interpretation of the general quantifier of [3] and give a more strong definition of true formulas used in the last paper, see footnote 5, p. 202 of [3]:

(*d4*) $W\{Q, T, \Pi aF\} = 1$.≡. $(i)(T_1)\{(i \leq n) \wedge (T_1 \varepsilon Q) \wedge (T_1/\{i_{w(F)}\} = T/\{i_{w(F)}\}) \rightarrow W\{Q, T_1, F(x_i/a)\} = 1\}$.

D.9. $E \varepsilon P\{n\}$.≡.[7] $(Q)\{N(Q, n) \rightarrow (F \varepsilon P(Q, n))\}$

D.10. $E \varepsilon P$.≡. $E \varepsilon P\{n(E)\}$.

P is the class of true formulas; we note here that though all Q's have infinite many elements but with a certain regularity, see *L.6.* and *L.7.*

(*d4'*) $W\{Q, T, \Pi aF\} = 0$.≡. $(\exists i)(\exists T_1)\{(i \leq n) \wedge (T_1 \varepsilon Q) \wedge (T_1/\{i_{w(F)}\} = T/\{i_{w(F)}\}) \wedge W\{Q, T_1, F(x_i/a)\} = 0\}$

L.11. Let E^0 *result from* E *by replacing free variables with indices* $\{i_{w(F)}\}$ *correspondingly by free variables with indices* $\{j_{w(E^0)}\}$ *and* $w(E) = w(E^0)$ (then E results from E^0 by an inverse substitution).

Let $T, T^0 \varepsilon Q$, $m(Q, k)$ and $T/\{i_{w(E)}\} = T^0/\{j_{w(E^0)}\}$; then:

$W\{Q, T, E\} = 1$.≡. $W\{Q, T^0, E^0\} = 1$.

L.12.' Let $m(Q, k+1)$, $m(Q^0, k)$,[8] $k \geq n(E)$, $Q^0 = Q|k|$, $T \varepsilon Q$, $T^0 \varepsilon Q^0$ *and* $T^0 = T/1, \ldots, k/$; *then:*

$W\{Q, T, E\} = 1$.≡. $W\{Q, T^0, E^0\} = 1$.

L.12. Let $N(Q^0, k)$ *and let* Q *be of the rank* $k+1$ *and be the minimal extension of* Q^0 *respective to the property* $N(Q^0, k)$ (then according to *L.10.* also $N(Q, k+1)$) $k \geq n(E)$, $T \varepsilon Q$, $T^0 \varepsilon Q^0$, $T^0 = T/1, \ldots, k/$; *then:*

$W\{Q, T, E\} = 1$.≡. $W\{Q^0, T^0, E\} = 1$.

T.7. If E *is a thesis, then* $E \varepsilon P$.

T.8. If E has Skolem's normal form for theses, $F \varepsilon C\{E\}$, $\mathbf{M}\{E\} = 0$, $n \geq n(E)$, $Q = \mathbf{M}[n]$, $T \varepsilon Q$; then:

If $\mathbf{M}(F\{s_{i_{w(F)}}\}) = 0$, $\mathbf{M}/\{s_{i_{w(F)}}\} = T/\{i_{w(F)}\}$, then $W\{Q, T, F\} = 0$.

T.9. The formula E is a thesis iff $E \varepsilon P$.

T.10. The formula E is true iff $E \varepsilon P$.

T.11. (Analogue to Skolem-Löwenheim's theorem) *If $\mathbf{U}$ is an arbitrary class of formulas, then $\mathbf{U}$ is a class of true formulas in some model iff for each finite sequence $E_1, \ldots, E_r \varepsilon \mathbf{U}$ there exists $n = max\ \{n(E_1), \ldots, n(E_r)\}$ and Q such that $N(Q, n)$ and $E_1, \ldots, E_r \varepsilon P(Q, n)$. If for each $E \varepsilon \mathbf{U}$, $n(E) \leq n$, then we can also assume that in the sequence $E_1, \ldots, E_r$ all elements of $\mathbf{U}$ occur.*

The definition (*d4*) creates sequent proof rules analogous to the case of (*4d*); namely to an arbitrary formula F and $n = n(E)$ we apply the sequent proof rules (**A**), (**K**), (**N**), (Π_1) till $i = n$ and the following:

$$(\Pi^2)\ \frac{\Gamma,\ \Pi a F}{\Gamma \ \underset{\circ}{\rfloor}\ \Gamma,\ F(x_i^0/a)},\ \text{where } \Gamma(\{i_{w(F)}\}) = (\Gamma,\ F(x_i^0/a))(\{i_{w(F)}\})^9$$

they determine the third type of generalized diagrams of the paper.

To the definition of a correct diagram we add here that the set Q, constructed in *T.4.* to a given formula E, and its generalized diagram, must also have the property $N(Q, n(E))$; if it has not the property, then we add all suitable lines to the diagram one after another and in such way we rebuild one, beginning with the greatest possible part of the diagram. The easy translation of the property $N(Q, n(E))$ to the language of indecomposable formulas remains for the readers. Then, of course, remain here true theorems *T.4.*, *T.5.* and *T.6.* with the additional assumption $n = n(E)$. The above, especially with the last sequent proof rules, give also the decidability of the monadic first-order functional calculus and a general decidabling possibility for arbitrary formulas; another stronger one built also on sequent proof rules is given in [10], [12]. The picture of the considerations in many valued propositional logic is also indicated in my papers, see [3], [11]. Explanations to short writing of examples we leave for the readers:

1) $n(E) = 1$:

$$\frac{\dfrac{E = \{\Pi a f(a) \supset f(x_1)\}}{f(x_1),\ (\Pi a f(a))'}}{f(x_1),\ (\Pi a f a)',\ f'(x_1)}\ \text{- contradiction}$$

Because the first generalization includes usual sequent proof rules then simple examples as above have an usual sequent proof.

In two following examples we name free variables by their indices:

2) $n = n(E) = 2$

$$E = \{\Sigma a_1 \Pi a_2 f(a_1, a_2) \supset \Pi a_2 \Sigma a_1 f(a_1, a_2)\}$$

$$\Pi a_1(\Pi a_2 f(a_1, a_2))', \Pi a_2 \Sigma a_1 f(a_1, a_2)$$

$$\mid \ (\Pi a_2 f(1, a_2))', \Pi a_2 \Sigma a_1 f(a_1, a_2)$$

$$f'(1, 1), f'(1, 2), \Pi a_2 \Sigma a_1 f(a_1, a_2)$$

$$f'(1,1), f'(1, 2) \ \mid \ \Sigma a_1 f(a_1, 1)$$

$$f(1, 1), f(2, 1)$$

the last line $\mid$ $f'(1, 1), f(2,2)$/D.5-6/

$f'(1, 1), f(2, 2), (\Pi a_2 f(1, a_2))'$/correct/

$$f'(1, 1), f(2, 2), f'(1, 2)$$

the last line $\mid$ $f(1, 1), f'(2, 2), f'(2, 1)$/D.5-6/

the last line, $\Sigma a_1 f(a_1, 1)$/correct/

$$f(1, 1), f'(2, 2), f'(2, 1), f(2, 1)$$

contradiction

3) $n(E) = 2$

$$E = \{\Pi a_2 \Sigma a_1 f(a_1, a_2) \supset \Sigma a_1 \Pi a_2 f(a_1, a_2)\}$$

$$(\Pi a_2 \Sigma a_1 f(a_1, a_2))', \Sigma a_1 \Pi a_2 f(a_1, a_2)$$

$$\Pi a_1 f'(a_1, 1), \Pi a_1 f'(a_1, 2), \Sigma a_1 \Pi a_2 f(a_1, a_2)$$

$$\mid \ f'(2, 1), \Sigma a_1 \Pi a_2 f(a_1, a_2) \ \mid \ f'(1, 2), \Sigma a_1 \Pi a_2 f(a_1, a_2)$$

$f'(2, 1), \Pi a_2 f(1, a_2), \Pi a_2 f(2, a_2)$ analogic

$f'(2, 1), \Pi a_2 f(2, a_2) \mid f(1, 2)$ . . . .

$$f'(2, 1) \ \mid \ f(2, 1) \qquad f(1, 2) \quad f'(1, 2) \quad f(2, 1) \quad f(1, 2)$$

Therefore we received the following line-tables $< f(1, 2), \ldots >$, $< f(2, 1), \ldots >$, $< f'(1, 2), \ldots >$, $< f'(2, 1), \ldots >$; the closing of the sequence of line-tables under the operations *D.5-D.6.* gives also:

$$< f(1, 1), f(2, 2), \ldots >, \quad < f(1, 1), f'(2, 2), \ldots >, \quad < f'(1, 1), f(2, 2), \ldots >$$

with their permutation (with reiterations). In such way we described a diagram without a fundamental line and it means that the formula in the last example is not a thesis (true). Other examples remain for the readers, see [10].

NOTES

1. The paper is connected with my lectures on J. Słupecki's seminar and on meetings of Polish Mathematical Society at Wrocław in 1955-7, [3].

2. e.g., if $E = E_1 + (E_2 + E_3)$ and E_1, E_2, E_3 are not alternatives of at least two formulas, then E_1, E_2, E_3 are alternative indecomposable members of E and $E_2 + E_3$.

3. $E(x/y)$ - substitution x for y with known restriction.

4. i.e. columns $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal on indecomposable formulas with indices $\leq k$.

5. In the classical case we must also consider $n = \aleph_0$ and then the diagram has only one column and T is the description of negated indecomposable formulas belonging to the column.

6. A stronger lemma for permutation with reiterations is given in [3], proofs of the paper hold without the stronger lemma.

7. We assume here the formal definition of $P(Q, n)$ given in *D.2*.

8. If $m(Q^0, k)$, then the construction of Q with the property $m(Q, k+1)$ is immediate.

9. $(\Gamma, F(x_i^0/a))$ denotes the second column.

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