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EXISTENCE AND IDENTITY IN QUANTIFIED MODAL LOGICS

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§1. The aim of this paper is to present a way in which philosophical objections to the development of a combined quantification and modal logic based on **S5** can be overcome. In more detail, the objectives are to show that **S5** is immune to criticisms directed at those theorems which distinguish it from **S4** and **T**; that problematic theorems[1] of modalised predicate logic like the Barcan formulae $[\Diamond(\exists x)f(x) \supset (\exists x)\Diamond f(x)]$ and $[\Diamond(\exists t)g(x, t) \supset (\exists t)\Diamond g(x, t)]$ can be appropriately qualified once existence is explicitly treated; that puzzles over identity can be escaped by a more elaborate treatment of identity than the standard treatment; and that difficulties associated with quantification into modal sentence contexts can be cleared away given these treatments of existence and identity. A combination of these moves suffice, so it will be argued, to meet standard objections, most forcefully presented by Quine[2], to quantified modal logics. Admittedly a full elaboration of these moves calls for some sentence/statement distinction, some analytic/synthetic (or necessary/contingent) distinction, and some sense/designation (or connotation/denotation) distinction: but although, consequently, it is not to be expected that a combination of these moves will satisfy Quine, they may satisfy some who have been disturbed by the objections Quine raises.

§2. *Existence in a first-order modalised predicate logic.* A semantical system **S5R*** is obtained by adjoining modal postulates for **S5** (with primitive symbol '$\Box$') to a system **R*** of first-order predicate logic (with primitive quantifier 'Π'). **R*** differs from usual quantification theory in having the predicate constant 'E', read 'exist(s)', added to its primitive symbols, and in interpretation: in place of the frequent interpretation - $[(\Pi x)f(x)]$ is true if f is true of all existent items of the domain selected - the following interpretation - $[(\Pi x)f(x)]$ is true if f is true of all possible (consistently describable or designable) items of the domain selected - is preferred[3]. The postulate set of **R*** is as follows:

R0. *If A is truth-functionally valid, then A is a theorem.*

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R1. $(\Pi x)(A \supset B) \supset . A \supset (\Pi x)B$, *provided individual variable x does not occur free in A.*

R2. $(\Pi x)A \supset \overset{\vee}{S}{}_y^x A|$, *where y is an individual variable or a consistent individual constant.*

RR1. $A, A \supset B \rightarrow B$ (Modus ponens)

RR2. $A \rightarrow (\Pi x)A$ (generalisation)

An individual constant a is *consistent* if 'a' has a possible referent, i.e. 'a' can, logically, have a referent. (With respect to a given domain, a is consistent if $[(\Sigma x)(x = a)]$ is true: see section 4.)

The postulate set for **S5R*** is obtained by adding to the postulate set for **R*** the Gödel postulates:

R3. $\Box A \supset A$
R4. $\sim\Box A \supset \Box \sim \Box A$
R5. $\Box(A \supset B) \supset (\Box A \supset \Box B)$
RR3. $A \rightarrow \Box A$ (necessitation)

A system **S5R$_1$*** is obtained by adding to **S5R*** the axiom:

R6. $(\Pi x)(\sim \Box \sim E(x) \,\&\, \sim \Box E(x))$, i.e. existence of individual items is always contingent.

The syntax of **R*** is simply that of an applied first-order functional calculus: it is as a semantical system that **R*** differs importantly from usual quantificational theories. In terms of 'Π' read 'for all' or 'for all possible' and 'Σ', read 'for some (possible)' and defined:

$$(\Sigma x)A(x) \equiv_{Df} \sim(\Pi x)\sim A(x),$$

quantifiers '∀' read 'for all existing' and '∃', read 'there exists' or 'for some existing', can be defined thus:

$$(\forall x)A(x) \equiv_{Df} (\Pi x)(E(x) \supset A(x))$$

$$(\exists x)A(x) \equiv_{Df} (\Sigma x)(E(x) \,\&\, A(x))$$

(Better definitions of '∀' and '∃' can be obtained using restricted variables but under the limits of **S5R*** these reduce to the above definitions.) The ∀–∃ subsystem of **R*** which has as wff all (definitional abbreviations of) wff of **R*** which contain only quantifiers '∀' and '∃' and which do not (in abbreviated form) contain 'E', and which has as theorems all theorems of **R*** which are wff of the subsystem, coincides with quantification theory as frequently interpreted *when* the further postulate $[(\Pi x)E(x)]$ is added to **R***. But except when, under interpretation, only a restricted class of individual domains is admitted, $[(\Pi x)E(x)]$ and $[(\Pi x) \Box E(x)]$, which would follow using necessitation, are both false.

§2.1 Kripke's quantified extensions of normal modal systems[4] can be developed as subsystems of **S5R***. The postulates of the most comprehensive of these systems, Kripke's quantified **S5**, consist of all Kripke closures of the following schemata (where the *Kripke closure* of A is any wff without free variables obtained by prefixing $\forall$-quantifiers and necessity symbols in any order to A):

K0: Truth-functional tautologies
K1: $\Box A \supset A$
K2: $\Box(A \supset B) \supset. \Box A \supset \Box B$
K3: $A \supset (\forall x)A$, *where x is not free in A*
K4: $(\forall x)(A \supset B) \supset. (\forall x)A \supset (\forall x)B$
K5: $(\forall y)((\forall x)A(x) \supset A(y))$
K6: $\sim\Box A \supset \Box \sim \Box A$

Modus ponens for material implication is the only rule of inference. Existence is introduced by Kripke closures of schemata:

K7: $(\forall x)A(x)\ \&\ E(y) \supset. A(y)$
K8: $(\forall x)E(x)$

Metatheorem: Every theorem of Kripke's quantified **S5** *with existence is (an abbreviation of) a theorem of* **S5R***.

Proof: Since modus ponens is a rule of **S5R*** it suffices to show that the schemes **K0-K8** are derivable in **S5R***. Schemes **K0, K1, K2, K6** are theorem-schemes of **S5R***. **K3** follows from $[A \supset A]$ and the scheme: $(\forall x)(A \supset B) \supset. A \supset (\forall x)B$, provided x is not free in A. This scheme, $(\Pi x)(E(x) \supset. A \supset B) \supset. A \supset (\Pi x)(E(x) \supset B))$, with x not free in A, follows in **S5R*** using commutation and **R1**. **K4** follows from $[(E(x) \supset. A \supset B) \supset. E(x) \supset A \supset. E(x) \supset B]$ by generalisation and distribution of 'Π'.

K5: $\equiv (\Pi y)(E(y) \supset. (\Pi x)(E(x) \supset A(x)) \supset A(y))$
$\equiv (\Pi y)((\Pi x)(E(x) \supset A(x)) \supset. E(y) \supset A(y))$
K7: $\equiv (\Pi x)(E(x) \supset A(x)) \supset. E(y) \supset A(y)$
K8: $\equiv (\Pi x)(E(x) \supset E(x))$.

Also: $A \rightarrow (\forall x)A$

For: $A \rightarrow (\Pi x)A$
$\rightarrow (\Sigma x)E(x) \supset (\Pi x)A$
$\rightarrow (\Pi x)(E(x) \supset A|$, i.e $(\forall x)A$.

Combining this derived rule with the necessitation rule of **S5R*** it follows that all Kripke closures of **K0-K8** are theorems of **S5R***.

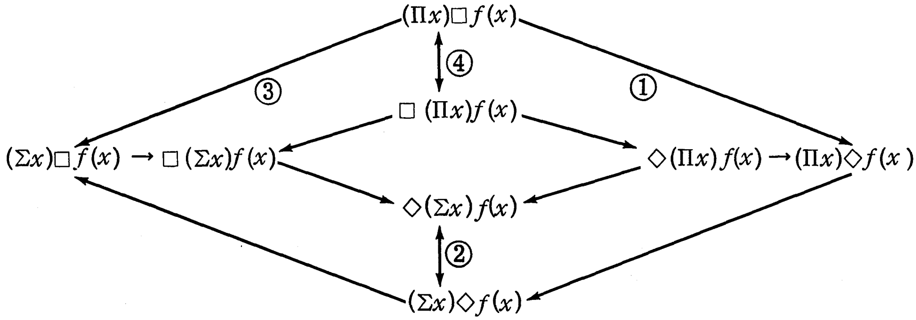
The salient difference appears in axiom scheme **K5.** If null domains were excluded, it would not be necessary to retain closure requirements except as specified in **K5.** So the system could be made to resemble usual systems by replacing taking of Kripke closures by rules of generalisation and necessitation: then the system would be tantamount to a quantified modal logic based on a free logic, for:

$$(\forall y)((\forall x)A(x) \supset A(y)) \equiv. (\forall x)A(x)\ \&\ E(y) \supset. A(y).$$

If, however a null domain[5] (*not* an empty domain) is selected **S5R*** collapses because $[(\Sigma x)\ (f(x) \vee \sim f(x))]$ fails, whereas Kripke's system does not provided further than in **K5** x is free in A. An inclusive logic which holds for null domains as well as for non-null domains can be obtained from **S5R*** in various ways. A way closely resembling Kripke's way leads to a logic $\mathbf{S5R_2}$*. $\mathbf{S5R_2}$* has as its axiom schemata all $\Box$-Π closures of axiom schemata of **S5R*** - with the proviso on **R2** supplemented by: *and x is free in A* - and as its sole rule modus ponens.[5]

Metatheorem: *Every theorem of Kripke's quantified* **S5** *with existence is (an abbreviation of) a theorem of* $\mathbf{S5R_2}$*.

§2.2 The principal relations on combining quantifiers 'Π' and 'Σ' with modal operators '$\Box$' and '$\Diamond$' in **S5R*** can be diagrammed[6]:



The arrows indicate material, or strict, implications. Among examples of ③ might be included: if some item, e.g. Pegasus, is necessarily a horse, then necessarily some item is a horse; among examples of ①: if it is logically possible that every item is thought of then every item is possibly thought of. Moreover neither ② nor ④ seem controversial when quantifiers 'Π' and 'Σ' are used; consider ② which amounts to: some possible items are possibly f if and only if possibly some possible items are f. Relations ② and ④ can be given further defences, by using finite models or, in **S5** systems, by resorting to analyses of quantifiers 'Π' and 'Σ' in terms of more comprehensive quantifiers 'A' and 'S' and the individual predicate '$\Diamond$'[7] or by using reductions of modal operators to quantifiers. For instance, on selecting a finite domain of possibilia, say $a_1, a_2, \ldots a_n$ ④ is replaced by

$$\Box(f(a_1)\ \&\ f(a_2) \ldots \&\ f(a_n)) \equiv \Box f(a_1)\ \&\ \Box f(a_2) \ldots \&\ \Box f(a_n),$$

which holds in virtue of distribution laws for '$\Box$'. The equivalence, would, however, break down if appropriate existence requirements on $a_1, \ldots a_n$ were added: for both implications in the equivalence:

$$E(a_i) \supset \Box f(a_i) \equiv \Box(E(a_i) \supset f(a_i))$$

can be shown to fail. Again, to illustrate the last method, ④ becomes, in a system based on **S5**:

$$(\Pi x)\ (\Pi \omega) f(x;\omega) \equiv (\Pi \omega)\ (\Pi x) f(x;\omega);$$

where 'ω' is a *modal* variable (with certain restrictions on its role in substitutions) which can be variously interpreted as ranging over *possible* worlds, over times, etc. The equivalence follows at once from a theorem on the interchangeability of quantifiers.

Contrast the problematic analogues of the principal relations; viz.:

1' $\Diamond(\forall x)f(x) \supset (\forall x)\Diamond f(x)$
2' $\Diamond(\exists x)f(x) \equiv (\exists x)\Diamond f(x)$
3' $(\exists x)\Box f(x) \supset \Box(\exists x)f(x)$
4' $(\forall x)\Box f(x) \equiv \Box(\forall x)f(x)$

These sentences expand to give three pairs of implications:

(a) 1' $\Diamond(\Pi x)(E(x) \supset f(x)) \supset (\Pi x)(E(x) \supset \Diamond f(x))$
3' $(\Sigma x)(E(x) \,\&\, \Box f(x) \supset \Box(\Sigma x)(E(x) \,\&\, f(x))$

(b) $\frac{1}{2}$2' $\Diamond(\Sigma x)(E(x) \,\&\, f(x)) \supset (\Sigma x)(\Diamond f(x) \,\&\, E(x))$
$\frac{1}{2}$4' $(\Pi x)(E(x) \supset \Box f(x)) \supset \Box(\Pi x)(E(x) \supset f(x))$

(c) $\frac{1}{2}$4' $\Box(\Pi x)(E(x) \supset f(x)) \supset (\Pi x)(E(x) \supset \Box f(x))$
$\frac{1}{2}$2' $(\Sigma x)(E(x) \,\&\, \Diamond f(x)) \supset \Diamond(\Sigma x)(E(x) \,\&\, f(x))$

Members of each pair are logically equivalent. Now none of these implications is a theorem of **S5R***. That none is a theorem can be demonstrated by constructing semantic tableaux for each in turn, in which '*E*' is treated simply as *a* predicate constant, and noting that the respective tableaux constructions are not closed[8] (see 5.2). Alternatively these results could be demonstrated by constructing finite countermodels of the sort indicated above and applying a Skolem-Löwenheim theorem. Thus that none of the implications are theorems can be indicated by picking suitable counterexamples; e.g. the last formula can be refuted by replacing '*f*' by '-*E*', the second last by replacing '*f*' by '*E*', given that there are no special axioms for existence. Now 1'-4' could be reinstated as theorems by adding to **S5R*** as an axiom the unacceptable $[(\Pi x)E(x)]$. But even $[(\Sigma x)E(x)]$, which would also be dubious as an axiom since it would give as a theorem $[\Box(\Sigma x)E(x)]$, would not suffice to reinstate 1'-4'. Indeed the strongest acceptable axioms for '*E*' which can be added to **S5R*** seem to be equivalents of **R5** of **S5R$_1$***. **R6** breaks down into the interpretation axiom $[(\Pi x)\Diamond E(x)]$ and Meinong's axiom $[(\Pi x)\Diamond \sim E(x)]$. A case might be made out for adopting the weaker axiom $[(\Sigma x)\Diamond \sim E(x)]$ in place of Meinong's axiom.

That the true statement $[(\Sigma x)E(x)]$ cannot be combined with **S5R*** without leading at once to the unwanted statement $[\Box(\Sigma x)E(x)]$ is a serious defect of **S5R***. It is a defect shared by many logics which combine modal operators with quantifiers. So long as necessitation is a derived rule of such logics the logics cannot serve as satisfactory basic or underlying logics for empirical theories or, more generally, for theories which include true contingent statements among their premiss statements. To rectify this fault of **S5R*** a more comprehensive logic **T***, of which a logic obtained from **S5R$_1$*** by $\Box$-transformation is a subsystem, is developed. A logic **T$_2$** is a $\Box$-transform of **T$_1$** if for every theorem *A* of **T$_1$** there is a theorem $\Box A$ of **T$_2$**: so **S5R*** contains a $\Box$-transform itself. Thus **S2** is a $\Box$-transform of **E2**.

Logic **T*** also has methodological advantages: for it differentiates those rules like modus ponens which apply to all sentences of appropriate form which yield true statements, no matter whether the statements are analytic or not, from those rules like rules of substitution for propositional variables and of necessitation which apply only to sentences which yield analytic statements. But even **T*** has some drawbacks, reflected in the interpretation theorem $[(\Pi x)\Diamond E(x)]$, which emerge in philosophical applications. The interpretation theorem is needed for completeness since it comes out valid under the intended interpretation. Logic **T*** violates a favoured thesis concerning logic, namely that logic is just concerned with what holds in all possible worlds, with what holds simply in virtue of (logical) form.

The postulate set of **T*** is as follows:[9]

Axioms:

T1 $\Box(p \supset. q \supset p)$
T2 $\Box(r \supset (p \supset q) \supset. r \supset p \supset. r \supset q$
T3 $\Box(\sim p \supset \sim q \supset. q \supset p)$
T4 $\Box((\Pi x)(p \supset f(x)) \supset. p \supset (\Pi x)f(x))$
T5 $\Box((\Pi x)f(x) \supset f(y))$
T6 $\Box(\Box p \supset p)$
T7 $\Box(\Box(p \supset q) \supset. \Box p \supset \Box q)$
T8 $\Box(\sim\Box p \supset \Box \sim\Box p)$
T9 $\Box(\Pi x)(\sim\Box\sim E(x)\ \&\ \sim\Box E(x))$
T10 $\sim(\Pi x)\sim E(x)$
T11 $\Box p \supset p$

Transformation rules:

RT1 $A, A \supset B \rightarrow B$ (modus ponens)
RT2 $A \rightarrow (\Pi x)A$ (generalisation)
RT3 $A \rightarrow B$, *where B results from A by substituting for a particular occurrence of C in A,* $\check{S}^x_y\, C|$, *and x is not free in C and y does not occur in C* (change of bound variable)
RT4 $A \rightarrow \check{S}^x_y\, A|$, *where x is an individual variable and y is an individual variable or a consistent individual constant* (substitution for individual variables).
RT5 $\Box A \rightarrow \Box\check{S}^p_B\, A|$ (substitution for propositional variable)
RT6 $\Box A \rightarrow \Box\check{S}^{f(x_1 \ldots x_n)}_B\, A|$ (substitution for functional variables)

Theorems of **T***

(i) $\Box(\Box(p \supset q) \supset (\Diamond p \supset \Diamond q))$
(ii) $\Box(\Box p \supset \Box\Box p)$
(iii) $\Box((\Pi x)\Box A \equiv \Box(\Pi x)A)$

Derived rules of **T***

(i) $\Box A,\ \Box(A \supset B) \rightarrow \Box B$

Proof: $\Box(\Box(A \supset B) \supset. \Box A \supset \Box B)$; by **T7, RT5**

$\Box(A \supset B) \supset. \Box A \supset \Box B$; by **T11**

$\Box A,\ \Box(A \supset B) \rightarrow \Box A,\ \Box A \supset \Box B$; using **RT1**

$\rightarrow \Box B$; by **RT1**

(ii) $\Box A \rightarrow \Box(\Pi x)A$

$\Box A \rightarrow (\Pi x)\Box A$; by **RT2**

Result by theorem (iii) and **RT1**.

(iii) $\Box A \rightarrow \Box B$, with same conditions as in **RT3**.

(iv) $\Box A \rightarrow \Box \check{S}^{x}_{y}A|$, with conditions as in **RT4.**

(v) $\Box A \rightarrow \Box\Box A$ (necessitation)

Proof: Using theorem (ii), **RT5, RT1.**

Metatheorems

(i) *The* $\Box$*-transform of* **S5R$_1$*** *is a subsystem of* **T***.

(ii) *Every theorem of* **S5R$_1$*** *is a theorem of* **T***.

(iii) *Every theorem of* **S5R*** *is a theorem of* **T***.

In applying **T*** restricted variables are often useful. For instance, the statement "All ravens are black", although thought to be true of all actual birds (or even things), presumably does not hold for all possible ravens. So how is the sentence 'all ravens are black' to be symbolised? Using a restricted variable 'w' the sentence can be symbolised:

$$(\Pi w)(\mathrm{rav}(w) \supset \mathrm{bl}(w)).$$

In this case the sentence might well enough be alternatively symbolised: $(\forall x)(\mathrm{rav}(x) \supset \mathrm{bl}(x))$, so that if added to the axioms the generalisation would have as an instantiation:

$$[E(x)\ \&\ \mathrm{rav}(x) \supset. \mathrm{bl}(x)],$$

or

$$[\mathrm{rav}(w) \supset \mathrm{bl}(w)].$$

It is not so simple when physical laws like Newton's first law, formulated 'All bodies not acted on by external forces continue in their state of rest or of uniform rectilinear motion', where referring expressions refer to ideal bodies, are introduced. For here the required referent class is more extensive than the class of all actual bodies (or things) but less extensive than the class of all possible items. To symbolise the law of inertia either the law sentence must somehow be substantially transformed or paraphrased initially, or individual item universes used for interpretation must be appropriately selected, or some new symbolism is needed, e.g. a nomological implication or a symbol like '$\langle p \rangle$' read 'is an actual or ideal physical item'. Given the symbol '$\langle \mathrm{p} \rangle$' semantics of which can be explicated (rather unilluminatingly) using a class of physically possible worlds, the law sentence can be symbolised: $(\Pi x)\ (\langle \mathrm{p} \rangle(x) \supset .\ \mathrm{bod}(x) \supset \mathrm{mov}(x))$. Repetition of the hypothesis '$\langle \mathrm{p} \rangle(x)$' could be avoided with restricted variables.

§3 *Identity in* = **S5R*** *and in* = **T***. When identity is grafted onto a quantified modal logic based on **S5** further difficulties, some attributed to **S5**, some attributed to the combining of modalities with quantifiers, are encountered. If identity is axiomatised using, as is customary, the Leibnizian indiscernibility of identicals principle, two related difficulties arise. First, modal paradoxes, the most famous of which is the morning star paradox, appear (on these paradoxes see section 5). Secondly not only is

(1) $(x = y) \equiv \Box(x = y)$

a theorem - a result which holds in weaker systems based on **T** - but worse

(2) $(x \neq y) \equiv \Box(x \neq y)$

is a theorem.[10] In combating this difficulty various moves are possible:

(A) to eliminate (2) by weakening the modal logic at least to **S4**, but to keep (1). But since defences of (1) have little more plausibility than defences of (2) and most defences of (1) can be transformed into defences of (2), and since even (1) is rejected by philosophers on various grounds[11], the source of the trouble does not appear to be **S5**. And **S5** has not just an alibi but also a good defence (see section 7).

(B) to retain, at least in appearance, the customary (substitution or Leibnizian) identity criterion along with consequences, in an **S5**-modalised theory, like (1) and (2); to argue that (1) and (2) are correct, and that apparent counterexamples are only reached by misconstruing the range-values of variables occurring in (1) and (2). By way of restriction it is proposed either

(B_i) to restrict the class of expressions, which can be substituted in the identity schemes provided by the usual identity criterion, and which can be substituted in (1) and (2), to merely referring or naming expressions; in other words, to narrow drastically both the class of items to which individual expressions '*a*', '*b*', etc. can legitimately refer and the range of individual variables. Recalcitrant expressions which are not merely referring are replaced by definite descriptions. Or

(B_{ii}) to replace (for certain sentence contexts) the items to which individual expressions refer and over which individual variables range, viz. individuals, by different items, e.g. individual concepts.

These moves, which are discussed in more detail below, both in effect *reject* the Leibnizian identity criterion for familiar referring expressions, such as 'Venus' and 'the evening star', which refer to individuals but which do not merely refer. Moreover they are *compatible* with the revision of the Leibnizian criterion as applied to familiar referring expressions.

(C) to revise the identity criterion.[12] After all, why should an analysis of identity, like the substitution analysis, which is carried straight over from extensional logics where all properties are extensional, be expected to hold without qualification for modalised logics? Here the standard analysis of identity in restricted predicate logics is challenged[13] and a different treatment, under which various identity criteria are distinguished, presented. Even so the appearance of the Leibniz principle could be kept by adopting a high redefinition of 'property' under which only extensional attributes qualify as properties. But other than "saving Leibniz" the redefinition lacks virtues; thus a different course is pursued. In a quantified modal logic which contains no intensional operators other than modal operators, two identity criteria are distinguished: extensional identity, and strict identity. A logic =**S5R***, restricted quantified modal logic **S5R*** with extensional and strict identity, will now be developed.

Extensional identity: To **S5R*** is added the binary predicate constant '=', read 'is identical with (under the extensional criterion)' or 'is extensionally identical with', the axiom

=**R1**: $x = x$

and the axiom-scheme

=**R2**: $x = y \supset . A \supset B$, *where x and y are individual variables or constants and B is obtained from A by replacing one particular occurrence of x by y, this particular occurrence of x being neither within the scope of* (Πx) *or* (Πy) *nor modalised, i.e. within the scope of* $\Box$[14].

Call the full proviso on =**R2** *proviso* (α). The last part of the proviso is readily generalized to: provided x is neither within the scope of a quantifier binding x or y nor within the scope of a modal operator. The generalisation follows at once using definitions of other quantifiers and modal operators in terms of 'Π' and '□'. Among results provable for extensional identity are these:

1: $\Box(x = x)$
2: $x = y \supset . y = x$
3: $(x = y) \; \& \; (y = z) \supset . z = x$
4: $(\Pi x)A \; \& \; (\Sigma y)(y = a) \supset . \check{S}^{x}_{a} A|$, *provided x is not modalised*
5: $\check{S}^{x}_{a} A| \; \& \; (\Sigma y)(y = a) \supset . (\Sigma x)A$, *provided x is not modalised*
6: $\Box(x = y) \supset . \Box A \supset \Box B$, with proviso ($\alpha$)
7: $\Box(x = y) \supset . \Diamond A \supset \Diamond B$, with proviso ($\alpha$)
8: $\Box(x = y) \supset . C \supset \Box A \supset . C \supset \Box B$, with proviso ($\alpha$)
9: $\Box(x = y) \supset . \Box A \supset C \supset . \Box B \supset C$, with proviso ($\alpha$)

Strict identity: The binary predicate constant '≡', read 'is identical with (under the strict criterion)' or 'is strictly identical with' is defined:

$$x \equiv y =_{Df} \Box(x = y)$$

Among theorems provable for strict identity are these:

10: $(x \equiv y) \equiv \Box(x = y)$

11: $(x \equiv y) \supset (x = y)$
12: $\Diamond(x \equiv y) \supset (x = y)$
13: $(x \equiv y) \supset (y \equiv x)$
14: $(x \equiv y) \,\&\, (y \equiv z) \supset (x \equiv z)$
15: $(\Pi x)A \,\&\, (\Sigma y)(y \equiv a) \supset. \check{S}^x_a A|$
16: $\check{S}^x_a A| \,\&\, (\Sigma y)(y \equiv a) \supset. (\Sigma x)A$

The individual constants occurring in theorems 4, 5, 15 and 16 need not be consistent constants.

Strict identity can be characterised by the theorem and theorem-schema

≡R1: $x \equiv x$
≡R2: $x \equiv y \supset. A \supset B$, *where x and y are individual variables, or constants and B is obtained from A by replacing one particular occurrence of x by y, this particular occurrence of x not being within the scope of* (Πx) *or* (Πy).

Call the full proviso *proviso* (β): it differs from proviso (α) only in not excluding replacements in modal sentence contexts. Theorem ≡**R1** follows at once from *Theorem 1*. Proof of schema ≡**R2**: From 11 and ≡**R2** it follows: $x \equiv y \supset. A \supset B$, with proviso (α). Therefore strict identities may be substituted in all non-modal sentence contexts. It remains to show that strict identities can be substituted in all modal contexts. The proof is by induction over the number n of occurrences of primitive symbols '$\sim$' '$\supset$', 'Π', '$\Box$' occurring in A. Because the schema holds for non-modal contexts a basis for induction, when $n \equiv 0$, is provided. Suppose the result holds when $n \equiv k$. When $n \equiv k + 1$ there are four possible cases to examine. Let A and B be as in proviso (β).
Case 1: A is of the form $\sim A_1$. Then B is of the form $\sim B_1$, and B_1 results from A_1 by replacement of x by y.[15] By hypothesis of induction: $x \equiv y \supset. A_1 \supset B_1$, with proviso (β). Also: $y \equiv x \supset. B_1 \supset A_1$, with proviso (β), since the number of occurrences of primitive symbols in B is the same as the number in A. Case 1 now follows by this argument:

$x \equiv y \supset. y \equiv x$, by *Theorem* 13.
$\supset. B_1 \supset A_1$, with proviso (β)
$\supset. \sim A_1 \supset \sim B_1$, with proviso (β)

Case 2: A is of the form $(A_1 \supset A_2)$. Then B is of the form $(B_1 \supset B_2)$, where B_1 and B_2 result from A_1 and A_2 respectively. By hypothesis of induction:

$x \equiv y \supset. A_1 \supset B_1$, with proviso (β)
$x \equiv y \supset. A_2 \supset B_2$, with proviso (β)
$x \equiv y \supset. B_1 \supset A_1$, with proviso (β)
$x \equiv y \supset. B_2 \supset A_2$, with proviso (β).

From these four premisses: $x \equiv y \supset. (A_1 \supset A_2) \supset (B_1 \supset B_2)$, with proviso (β), follows using propositional calculus. Since substitution occurs only at one place provisos are needed in only two of the four premisses. Con-

sequently the proviso on the conclusion follows at once from provisos on the premisses.
Case 3: A is of the form $(\Pi z)A_1$. Then B is of the form $(\Pi z)B_1$, where B_1 results from A_1 and x and y differ from z. By hypothesis of induction: $x \equiv y \supset . A_1 \supset B_1$, with proviso ($\beta$). Then: $(\Pi z)(x \equiv y \supset . A_1 \supset B_1)$, with proviso ($\beta$), by generalisation. Hence: $x \equiv y \supset . (\Pi z)A_1 \supset (\Pi z)B_1$, with proviso ($\beta$), by theorems of **S5R***, since z differs from x and y.
Case 4: A is of the form $\Box A_1$. Then B is of the form $\Box B_1$, where B_1 results from A_1. By hypothesis of induction: $x \equiv y \supset . A_1 \supset B_1$, with proviso ($\beta$). Then: $\Box(x \equiv y) \supset . \Box A_1 \equiv \Box B_1$, with proviso ($\beta$). Hence: $x \equiv y \supset . A \supset B$, with proviso ($\beta$).
This completes the proof by induction.

More interesting than the theorems of =**S5R*** are the non-theorems. The following, with rejection indicated by a star, are *not* theorems:

*1: $(x = y) \supset (x \equiv y)$
*2: $\sim(\Pi x)(\Pi y)(x = y \supset . x \equiv y)$

That *1 and *2 are not theorems can be shown by constructing semantic tableaux, which prove not to be closed, for *1 and *2. The rules for constructing these tableaux are those of Kripke,[16] except that Kripke's rule I1 is replaced by the following rule:

=1: *If $a = b$ (for some variables a and b) appears in the left column of a tableau, then in both columns of that tableau replace every formulae $A(a, b)$ where a is free and not modalised by $A(b, b)$.*

Rule I1 (or ≡1) for strict identity is a derived rule.

*3: $(\Pi x)(\Pi y)(x = y \supset . x \equiv y)$
*4: $\sim\Box(\Pi x)(\Pi y)(x = y \supset . x \equiv y)$
*5: $(x = y) \equiv \Box(x = y)$

These follow from *1 and *2 by the usual rule of rejection.

*6: $(x \neq y) \equiv \Box(x \neq y)$

For if *6 were a theorem, $\Diamond\sim(x = y) \supset \sim(x = y)$ would be a theorem, using the derived rule: $A \supset \Box B \rightarrow \Diamond A \supset B$ of **S5R***. And then $[x = y \supset . \Box(x = y)]$, i.e. *1, would be a theorem.

*7: $x = y \;\&\; y = z \supset . x \equiv z$
*8: $x \equiv y \;\&\; y = z \supset . x \equiv z$

For replace, in *7 and *8, x by y.

It follows that (1) and (2) fail in =**S5R*** for extensional identity: only analogues of (1) and (2) for strict identity hold. Since the negation of *4, $[\Box(\Pi x)(\Pi y)(x = y \supset \Box(x = y))]$ is, like *4, not a theorem, *4, i.e.:

$$\Diamond(\Sigma x)(\Sigma y)\cdot(x = y) \;\&\; \Diamond(x \neq y)$$

could be added as an axiom to =**S5R*** without rendering the system inconsistent. An even stronger axiom appears =**T***, where =**T*** is a system obtained from **T*** by adding axioms and rules for identity.

Identity in =**T***: To **T*** is added the predicate constant '=', the axioms:

=**T1**: $\Box(x = x)$
=**T2**: $\sim(\Pi x)(\Pi y)(x = y) \supset. \Box(x = y))$

and the transformation rule:

=**RT**: $A \rightarrow (x = y) \supset B$, with proviso ($\beta$)

Metatheorems: (i) *Every theorem of* =$\mathbf{S5R}_1$* *is a theorem of* =**T***
(ii) *Every theorem of* =**S5R** *is a theorem of* =**T***

Unifying identity criteria. So far a theory of identity has been developed only for quantified modal logics. In a logic, based on restricted predicate logic, which included other intensional operators than modal operators, proviso (β) would have to be amended to conclude 'this particular occurrence of x being neither within the scope of (Πx) or (Πy) nor within the scope of an intensional non-modal operator', and proviso (α) amended to conclude 'this particular occurrence of x being neither within the scope of (Πx) or (Πy) nor within the scope of an intensional operator'. Thus an extensional identity permits replacement in extensional sentence contexts only, a strict identity in extensional and modal sentence contexts but not in all sentence contexts. Relative strengths of identity criteria may be compared in terms of the classes of non-extensional sentence contexts with respect to which they allow replacements; e.g. the following criteria (for identity of various items) may be arranged in order of decreasing strength: typographical identity, propositional identity, synonmy, strict identity, extensional identity. Treatment of stronger identity criteria than strict identity is beyond the scope of this paper.

How are these different identity criteria of restricted logics to be unified as *identity* criteria or under a single sense of 'identical'? In this way. A binary predicate constant 'I' is *an identity criterion* in a logic based on restricted predicate logic if it satisfies the condition

(i): $x \, I \, x$

and the open condition scheme

(ii): $x \, I \, y \supset. A \supset B$, *where* x *and* y *are individual variables or constants and* B *is obtained from* A *by replacing an occurrence of* x *by* y, *this particular occurrence of* x *not being within the scope of* (Πx) *or* (Πy), *at least where* x *is not within the scope of an intensional operator* (*or connective*).

That scheme (ii) holds at least for all extensional contexts is an important point since this is sufficient to guarantee a minimum condition for identity: sameness of referents (under appropriate criteria) of individual referring expressions. Identity criteria of restricted logics are unified, as identity criteria, by satisfying the conditions on 'I'. If you like then, there is one sense of 'is identical with' specified, in outline, by the above conditions on 'I'; but there is an indeterminacy in the sense conditions owing to the appearance of the phrase 'at least', and depending on how this indeterminacy is taken up various related identity criteria result. Consider for instance how criteria for sameness of type of expressions are distinguished from criteria for sameness of token. Whether, of course, 'is identical with' has one sense or several itself depends on how criteria for identity of sense are finally adjudicated.

A binary predicate constant 'I' is an identity criterion in an unrestricted predicate logic if it satisfies the condition:

$$u\ I\ v \equiv (\Pi f)\ (f \epsilon Q\ .\supset. f(u) \equiv f(v)),$$

where Q represents *at least* the class of all extensional predicates or sentential functions. Also formation rules may be so laid down that 'uIv' is wf only if 'u' and 'v' are of the same sort. Details of requisite modifications of the Leibniz identity definition (now regularly used in higher predicate logics) so as to obtain different identity criteria and details of restrictions on the class of primitive predicates of =**S5R*** are reserved for another paper[17].

§4. *On modal paradoxes and Quine's objections to quantifying into modal sentence contexts.* Criteria for transparency and opacity of sentence contexts should be distinguished according as identity criteria. In what follows they are distinguished just for extensional and strict identity. A particular occurrence of a referring expression $^{<}x^{>}$ in a sentence context $^{<}f^{>}$ is *referential* if truth-value is preserved under replacement of $^{<}x^{>}$ by any $^{<}y^{>}$ such that $y = x$, i.e. if $[(\Pi y)\ .\ x = y \supset. f(x) \equiv f(y)]$; *modal* if truth-value is preserved under replacement of $^{<}x^{>}$ by any $^{<}y^{>}$ such that $y \equiv x$, i.e. if $[(\Pi y)(x \equiv y \supset. f(x) \equiv f(y))]$[18]. A sentence context $^{<}h^{>}$ is *r-transparent* if for every singular referring expression $^{<}x^{>}$, if an occurrence of $^{<}x^{>}$ is referential in $^{<}f(x)^{>}$ (i.e. in context $^{<}f^{>}$), then that occurrence of $^{<}x^{>}$ is referential in $^{<}h(f(x))^{>}$; i.e. if $[(\Pi x)(\Pi f)((\Pi y)(x = y \supset. f(x) \equiv f(x)) \supset (\Pi y)(x = y \supset. h(f(x)) \equiv h(f(y)))]$; otherwise $^{<}h^{>}$ is *r-opaque*.[19] A sentence context (of sentences) $^{<}h^{>}$ is *m-transparent* if for every singular referring expression $^{<}x^{>}$, if an occurrence of $^{<}x^{>}$ is modal in $^{<}f(x)^{>}$, then that occurrence of $^{<}x^{>}$ is modal in $^{<}h(f(x))^{>}$; otherwise $^{<}h^{>}$ is *m-opaque*. All extensional sentence contexts are r-transparent; but the converse does not hold.

Sentence contexts of the form '$\Box(\ldots)$' and '$\Diamond(\ldots)$', where no intensional functors occur within the brackets, are r-opaque but m-transparent. These features provide the genesis of modal paradoxes. To illustrate with a typical modal paradox: It is true that

(1): $\sim\Box(\#\mathrm{pl} > 7)$

where '#pl' abbreviates 'the number of major planets'. But using the true extensional identity

(2): $\#\ \mathrm{pl} = 9$

and substituting identicals in the truth

(3): $\Box(9 > 7)$

it follows

(4): $\Box(\#\,\mathrm{pl} > 7)$

Since (1) and (4) are inconsistent we are landed with a paradox. Substitution using (2) is not truth preserving in (3) but it is truth preserving in

(5): $(9 > 7)$.

Therefore the sentence context '(. . .)' is r-opaque. What follows from the paradox and r-opacity? As with most paradoxes, quite diverse conclusions have been drawn. In particular, given supplementary assumptions, these conclusions have been reached:

(I) The Leibniz identity criterion is inadequate in intensional sentence contexts. What the r-opacity and paradox arguments show is that

(6): $x = y \supset . \Box f(x) \supset \Box f(y)$

is invalid. My thesis, reinforced given the feasibility of logics like =**S5R*** in which (6) is not valid, is that only substitutions based at least on strict identities, *not* substitutions based on extensional identities, are permissible in such sentence contexts.

(II) The Leibniz criterion is correct but cannot be applied unrestrictedly in r-opaque contexts like (3) because these contexts are impure, i.e. they contain quotation essentially. R-opaque sentences, which are really *verbal*, really about expressions, contain when expanded quoted expressions; e.g. (3) expands to

(3'): $9 > 7$ and '$9 > 7$' is analytic

and (1) expands similarly to (1'). Since (1'), (3') and (5) are mutually consistent, paradox is beaten. A Pyrrhic victory. For first, given the standard theory of quotation, (6) is rejected under (II) as not universally valid: the correctness of (I) is thereby virtually admitted. Second, verbal interpretations qualify, as well as the Leibniz criterion, several other logical principles, e.g. universal instantiation and existential and particular generalisation, and in general, block substitution within and quantification into r-opaque contexts. These heavy sacrifices - though insisted upon by Quine and others - are not at all satisfactorily substantiated and seem unwarranted. Third, given a non-standard but more plausible theory of quotation[20], (6) does hold under verbal interpretations but these interpretations then fail to eliminate modal paradoxes unless coupled with an approach like (I), (III) or (IV). Fourth, verbal interpretations of intensional functors have not been vindicated and remain open to serious objections.[21]

(III) In order to retain the Leibniz criterion the class of individual expressions which can replace individual variables is severely curtailed. Consider the typical restriction, proposed in (B_i) of §3, where individual expressions are narrowed to merely referring expressions. The test for whether an expression is merely referring in a context is whether the scope of its associated description matters, that is affects truth-value, in that context: it is *merely referring* only if scope does not matter. The *associated description* of a name <m> is <the item which is m>, i.e. <$(\iota x)m(x)$>, and of a description is the description itself. If scope of the expression is not indifferent in its sentence context, so that the expression is not merely referring, the expression is replaced by its associated description and the description has in that context a sufficiently wide scope, that is a scope under which truth-value is unaffected by taking a wider scope if there is one. A sufficiently wide scope can always be found. In the setting of **=S5R***, an expression is merely referring in a sentence context if it is referential in that context.

To illustrate the method consider the resulting solution of modal paradoxes. (3) is (replaced by)

(3''): $[(\iota x)(x = 9) \,.\, \Box((\iota x)(x = 9) > 7)$

i.e.: $(\Sigma z)((\Pi y)(y = 9 \equiv.\ y = z)\ \&\ \Box(z > 7))$. Using **=R2** and (2) there follows:

$$(\Sigma z)((\Pi y)(y = \#\mathrm{pl} \equiv.\ y = z)\ \&\ (z > 7)) \ ,$$

i.e.

(4''): $[(\iota x)\ \mathrm{pl}(x)].\Box((\iota x)\#\mathrm{pl}\ (x) > 7)$,

where '$(\iota x)\#\mathrm{pl}(x)$' is the associated description of '# pl'. But (4'') (i.e. (4) according to (III)) is not inconsistent with

(1''): $[(\iota x)\#\ \mathrm{pl}(x)]\ .\ {\sim}\Box((\iota x)\#\ \mathrm{pl}(x) > 7)$

i.e. with (the replacement of) (1). What amounts to this method, a method which is a straightforward extension of Russell's technique for dealing with names and descriptions which lack actual referents and which fits nicely into the framework of *Principia Mathematica*, is advocated by Smullyan and by Prior[22].

The Smullyan-Prior technique succeeds formally because it is parasitic on solution (I), because it replaces a modal sentence context where substitution of 'b' for 'a' using an extensional identity $[a = b]$ would go bad by an extensional substitution context. If 'a' is not modalised then in the relevant logics 'a' occurs in an extensional context. Then in general the scope of the associated description of a is indifferent - by

(7): $(\Sigma!x)\ f(x) \supset.\ (\Pi p,q)\ (p \equiv q \supset.\ g(p) \equiv g(q)) \supset.$

$$g\{[(\iota x)f(x)].\ h((\iota x)f(x))\} \equiv [(\iota x)f(x)]\ .\ g\{h((\iota x)f(x))\},$$

an analogue of *Principia Mathematica* ***14.3** with a similar proof - and 'b' can replace 'a' in virtue of the extensional identity criterion. If 'a' is modalised then either the scope of its associated description is indifferent or it is not. If the scope is indifferent, then a wider scope can be selected such that the relevant substitution position occurs in an extensional context. But it will not happen with the usual logical modalities (except for special combinations) that scope is indifferent. If the scope of the associated description is not immaterial then the expression substituted for is brought into an extensional context by an adaption of the usual method of replacing a non-extensional context by an extensional context, viz. using identity and quantification. Thus substitution is not really made within a modal context. The Smullyan-Prior technique is tantamount to narrowing the class of individual names so that all but logically proper names need occur only in extensional contexts. Hence *the technique conforms to solution (I)*. Indeed (4'') follows at once from (3) and

(2''): $9 = (\iota x) \# \mathrm{pl}(x)$,

a relation obtained from (2) by replacing '# pl' by its associated description, using a derived rule of =**S5R***, viz:

$$B(y),\, y = (\iota x)A(x) \rightarrow B((\iota x)A(x)),$$

provided that the scope of the description includes all modal (intensional) operators in B.[23]

The Smullyan-Prior technique amounts to a special application of the usual technique for replacing intensional contexts by equivalent extensional ones, together with a restriction on the interpretation of variables so that a variable can only go proxy for merely referring expressions or logically proper names. Other singular referring expressions are replaced under the interpretation by descriptions, the role of which is regulated by new scope conventions. To illustrate consider a generalisation of (3), viz:

$$\Box(x > 7).$$

To ensure that the variable ('x') on which replacement is made occurs in a non-modal context this is transformed into the logically equivalent:

$$(\Sigma z)(x = z \;\&\; \Box(z > 7)).$$

Since now replacement using an extensional identity such as:

$$x = (\iota x) \# \mathrm{pl}(x)$$

is permissible it follows:

$$(\Sigma z)(z = (\iota x) \# \mathrm{pl}(x) \;\&\; \Box(z > 7))$$

and therefore:

$$[(\iota x) \# \mathrm{pl}(x)].\ \Box((\iota x) \# \mathrm{pl}(x) > 7),$$

i.e. (4'').

Although the Smullyan-Prior technique is as formally satisfactory as the theory of descriptions on which it depends, that is not enough. Difficulties are simply transferred to the interpretation of the symbolism. For under interpretation it raises acutely all the difficulties raised by Russell's sharp distinction between proper names and definite descriptions and by Russell's and Wittgenstein's theories of logically proper names, difficulties intensified, once the motley of intensional operators is admitted. For instance if 'Lesbia' and 'Clodia' were logically proper names not only [$\Box$(Lesbia = Clodia)] but worse [$(\Pi x)\ K_x$ (Lesbia = Clodia)] would be true.

It is a short route to the conclusion that there are in English no logically proper names and can be none: the variables have no English substitution values.

(IV) To guarantee the Leibniz principle the items to which individual expressions relate or refer and over which individual variables range designationwise, viz. individuals, are replaced by different items, e.g. individual concepts. Compare (B_{ii}). This procedure, pursued according to Quine[24] by Frege, Church and Carnap, though it might, after refinement, suffice for a theory of individual concepts, bypasses the main problems at hand, problems as to the criteria for the (contingent) identity of *individuals*. The procedure becomes practically unworkable when the full spectrum of intensional functors is introduced. And as stressed by Quine[24], even when only modal functors are added the procedure is not, *on its own*, going to solve problems raised by identity relations and quantifiers in modal sentence contexts: for consider such identities as $y = (\iota x)(p \,\&\, (x = y))$ where p is contingently true. Distinctions between various identity relations, or else distinctions between equalities or equivalences of various strengths (the course adopted by Carnap in explications of the issues), still have to be made. But if these distinctions are made, there is no need to limit or change designation ranges of variables. Because such distinctions are made and substitutions in intensional sentence contexts are restricted in what follows, variables are *not* there limited to intensional values or required simply (or even at all) to designate intensional objects (in some sense).

(V) The Leibniz criterion is correct: but certain laws of classical logic, in particular existential generalisation (*E.G.*) and universal instantiation (*U.I.*), must be abandoned when non-extensional predicates or contexts are admitted; and more generally binding of variables in modal contexts by quantifiers, since not significant, must be given up. This is the course advocated by Quine. Quantification into non-extensional sentence contexts is impermissable, i.e. variables occurring within such contexts cannot, legitimately or significantly, be bound by quantifiers occurring outside the context.

It is easy to plot out routes by which Quine arrives at his conclusions:

(i) His strictures on quantification and rejection of fully quantified modal logics would follow at once using the verbal interpretation explained in (II). And in exposition[25] Quine often reaches his position by carrying over results supposed to follow from the verbal interpretation to non-verbal construals of modalities. But not only is the verbal interpretation open to the criticisms levelled in (II); more important the extrapolation is not warranted.

(ii) Quine is forced - on pain of inconsistency - to abandon *U.I.* in modal contexts. For Quine maintains both that the Leibniz identity principle is correct for all contexts, not just for extensional contexts, and that modal contexts are referentially opaque; from which it follows that *U.I.* is false. Moreover the modal paradoxes can be blocked by abandoning *U.I.* (and the related *E.G.*). For in order to use (2), to make a replacement according to the Leibniz principle in (3) and so to get (4), *U.I.* is needed. Thus given that the full identity principle is secure and that designation ranges of variables are not to be tampered with, modal paradoxes can be re-employed as reductio arguments against use of *U.I.* and *E.G.* in modal contexts. Such reductio arguments are not fully convincing on their own especially when the assumed premisses are not well secured.

The stock argument to secure a full-strength (substitutivity of) identity principle, like the indiscernibility of identicals, runs as follows:[26] If a and b are identical then a and b are one; therefore whatever can be truly said of or about a should equally be true of or about b. Unless a purely referential theory of identity, to the effect that identity and difference sentences relate just to the *referents* of expressions standing on each side of identity and difference signs, is adopted, the argument is not cogent. To illustrate, $\Box(a = a)$ and $B_x(a = c)$, i.e. x believes that $a = c$, are true but $\Box(a = b)$ and $B_x(b = c)$ may be false, even when a and b are in fact one. The argument also fails when quotation occurs in test sentences. In the face of this failure, qualifications are frequently imposed on the substitution principle with respect to sentence contexts containing quotes, e.g. the principle is said to apply only to first-order contexts. But, in spite of the similarities, analogous qualifications are not usually imposed on sentence contexts containing intensional operators. Why, in such cases, is the indiscernibility of identicals principle adhered to so tenaciously? Because, it would seem, of reliance on a purely referential theory of identity, a theory typically reinforced by a denotation theory of meaning. I take the inadequacy of an unqualified denotation theory of meaning to have been demonstrated: the same arguments undermine a purely referential theory of identity. The inadequacy of a purely referential theory has been elaborated by Frege[27]; in particular he stresses that on such a theory it would be impossible to explain differences between $a = a$ and $a = b$ when both are true. On the contrary, a solid case can be put up for claiming that with an identity sentence '$a = b$' not only the referents of 'a' and 'b' but also their senses are relevant. For instance, in 'Necessarily $a = b$' what is said is said not just about the referent of 'a', if any, but also about the sense of 'a'. Then, however, the conclusion of the stock argument does not ensue. Truth will only be preserved under substitution of (extensional) identicals where only referents are in question, i.e. in *extensional* contexts. A linguistic surrogate of the full substitutivity principle can be kept by the terminological strategy of suitably narrowing the application of 'property', 'condition' or 'trait' so that sentence contexts or sentential functions containing intensional or modal operators do not specify properties or traits. But there does not appear to be much justification for this piece of legislation: it seems methodologically much preferable to distinguish sorts of properties.

(iii) Quine does take more direct routes. His initial strategy then consists in showing that modal contexts are r-opaque. But the argument only shows that either (6) is invalid or that *U.G.* has to be qualified or.... It is important to emphasize that *on its own* demonstration of r-opacity of modal sentence contexts establishes nothing except this. It goes no distance towards establishing one of (I)-(IV). It does, however, point to a deficiency in some standard quantified modal logics with identity, where no provision is made for the symbolisation or treatment of contingent identities like (2); where provision is only made for strict identities like:

$$3^2 \equiv 9.$$

Using such identities replacements can, of course, be made in (3) in virtue of the theorem:

$$x \equiv y \supset . \Box f(x) \supset \Box f(y).$$

If, however, the unqualified Leibniz identity requirements from which these standard treatments begin are kept, all contingent identities vanish in quantified modal logics. A demonstration of this point may be used in a reductio ad absurdum of the full Leibniz requirement.

Quine's main direct arguments are designed to show that no variables within a modal context (or, more generally, no variables within an opaque construction) can be bound by an external operator or quantifier, that quantification into modal sentence contexts is not possible. There is, however, nothing to stop us particularising[28] on (3) to obtain the apparent truth

(8): $(\Sigma x)\Box(x > 7)$

or to stop us from discussing the truth or falsity of

(9): $(\exists x)\Box(x > 7)$.

So it is possible to do what Quine says it is not. This is not what Quine means. What his claims regarding quantification into modal sentence contexts boil down to can be put: sentences like (8) and (9) are senseless, nonsense, improper, lack a clear interpretation; so assessment of their truth or falsity is likewise non-significant. I submit that these sentences are significant, are intelligible and understood by most students of logic, and have as clear an interpretation as some sentences of restricted predicate calculus. I further submit that Quine's arguments fail to show that they are not significant. Quine's direct arguments to show that something or other is wrong with quantification into r-opaque contexts follow similar lines. They can be illustrated using example (8). Quine asks[29]: What is this number which according to (8) is necessarily greater than 7? According to (3) from which it is inferred, it is 9, *that is* the number of major planets. But to suppose that it is would conflict with the falsity of (4). In the sense of 'necessarily' in which (8) is true, (4) has to be reckoned true along with (3). Therefore with (8) we wind up either with nonsense or else with unintended sense. Quine's argument is fallacious, given that extensional and strict identity criteria can be distinguished in approximately the way they were distinguished in section 3. *Quine's argument rests on an equivocation on 'that is'* (in later versions on an equivocation on 'i.e.') *as between extensional and strict identity*. For the number of planets *is*, in fact but *not* necessarily, nine. If the identity in question were strict then substitution in the instantiation of (8) would be admissable and would not lead to attribution of inconsistent truth-values to (4). But the identity (2) is not strict, so its truth does not conflict with the falsity of (4) unless the non-theorem

(10): $x = y \supset . \Box f(x) \supset \Box f(y)$

is assumed. Using (10) Quine's reduction argument may be represented:

(1) & (2) & (3) & (10)	; premisses
(8)	; from (3) by particularisation, i.e. by (i) note 28.
$(\Sigma x)\ \Box(x > 7) \supset (\Sigma x)(\Pi y)(x{=}y \supset \Box(y > 7))$	; from (10)
$(\Sigma x)(\Pi y)(x{=}y \supset . \Box(y > 7))$;	; using (8)
$(\Pi y)(9{=}y \supset . \Box(y > 7))$	; since 9 is such a number
$(2) \supset (4)$	; by *U.I.*
(1) & (4), i.e. (4) & -(4).	

Quine, exporting, concludes that *U.I.* and *E.G.* must be qualified, and somehow also concludes that (8) (got from (3) by what amounts to *E.G.*) is not significant! At this stage there are serious and irreparable gaps in his argument; for instance his argument by no means establishes that (8) is not significant. For present purposes, however, these gaps may be disregarded: for as the argument uses the incorrect (10), it does not call into question the truth of (8), and it fails to impugn quantification into modal contexts.

Nor therefore does retention of (8) force us to change or limit the designation range of individual variables, or to introduce a domain of individual items (and expressions) in which items if identical at all are strictly identical. Contrary to Quine's claim[30] it does not follow from the true premisses:

$$(\Sigma x)(x = \# \text{ pl} \ \& \ \Box(x = 9))$$
$$(\Sigma x)(x = \# \text{ pl} \ \& \ - \Box(x = 9))$$

because the matrices under the quantifiers yield contraries, that there must be at least *two* items x such that

(11): $x = \# \text{ pl}$

is true. Such a conclusion would only follow given (what does *not* hold for extensional identity but only for strict identity):

$$(g(x) \ \& \ \Box f(x)) \ \& \ (g(y) \ \& \ -\Box f(y)) \supset . x \neq y.$$

The argument merely shows that there are at least two items which are not strictly identical such that (11) holds.

Since: $y = (\iota x)(p \ \& \ (x = y))$, but: $y \not\equiv (\iota x)(p \ \& \ (x = y))$, when p is not necessary, the same moves (as above) can be repeated to block Quine's objection[31] to limiting variables to (consistent) intensional items such as individual concepts. To so limit ranges of individual variables is quite unnecessary. Inessential also in rebutting Quine's arguments is use of quantifiers 'Π' and 'Σ'. The points made against Quine so far hold even if '∀' and '∃' quantifiers are used and designation-ranges of variables are limited to items which actually exist - whichever these are.

The same error that features in Quine's 'that is' argument is sometimes smuggled in by way of a neutral items shuffle. It is suggested to us that the morning star is identical with the (description) neutral item, Venus, and that the neutral item is identical with the evening star, and that identity is transitive. Then we are presented with an argument something like this: The morning star is necessarily the same as the morning star. The morning star is however identical with the neutral item (or the item itself, Venus). Thus the morning star is necessarily the same as the neutral item. And so on. The argument fails: for the identity of the morning star with the description neutral item, in this case the planet Venus, is contingent only, and not sufficient to warrant substitutivity in all modal contexts. The notion of a description neutral item is itself confused. Though items are to a large extent independent of descriptions, descriptions, since sensed expressions, are not modally neutral. 'The description neutral item' is yet another modally non-neutral description.

(iv) Perhaps Quine's main argument should be expanded in this rather different way: *U.I.* and *E.G.* are already suspect because of existence presuppositions. When modal functors are introduced the situation deteriorates further. Because of failure of substitutivity of contingent identities in modal contexts it is not clear which individual(s), if any, the term generalised upon, in quantifying into modal contexts like (2), refers to; it is not even clear that the term specifies a definitely existing individual. Until this obscurity is cleared up, we are not entitled to argue:

$$\frac{\Box(9 > 7)}{\therefore (\exists x)\ \Box\ (x > 7);}$$

any more than we are entitled to argue

$$\frac{-E(\text{Pegasus})}{\therefore (\exists x)\ \sim E(x).}$$

Certainly neither of these inferences is valid. But is the first inference any more problematic than:

$$\frac{9 > 7}{\therefore (\exists x)(x > 7)\ ?}$$

Is the indefiniteness of reference of (8) any more worrying than the indefiniteness of reference of $[(\Sigma x)(x > 7)]$?

The failure of the first inference, like that of the third, *is* not *a consequence* of the failure of substitutivity of extensional identities in modal contexts, but *of inadequate existential premisses*. And the worry over indefiniteness stems at least partly from ensuing difficulties in guaranteeing existential premisses. Moreover quantification does not have to be independent of or neutral with regard to means of specifying substitutions for variables right up to contingent identities. Quine seems to suppose that it does; for he claims[32] that the crux of the trouble with (8) is that a number x may be uniquely determined by each of two conditions which are not strictly equivalent. But results 4, 5, 15, 16, A5 of section 3 show clearly enough that introduction and elimination of quantifiers is not independent of whether constants are identified using extensional or strict identities, and hence is not independent of whether determining conditions are extensionally or strictly equivalent.

Doesn't all this indicate a departure from purely extensional quantification theory? Maybe it does[33]; maybe such a departure is inevitable when quantification theory is extended to include non-extensional functors. This depends on criteria adopted for a "purely extensional quantification theory". At any rate variables do not do a purely referential job: they go proxy for expressions with both sense and designation. We are not thereby engulfed in Aristotelian essentialism, an emendation Quine thinks needed[34] to refloat quantified modal logic, Essentialism would result only if we were to revert to, what we have already rejected, a purely reference theory of identity and of the possession of properties or traits, e.g. that if a possesses properties g and $\Box h$ then b also possesses these properties if $b = a$. On the contrary, *what properties and relations a has depends not merely on the reference of 'a', but also, and crucially in the case of non-extensional properties, on the sense or meaning of 'a'*; consider, e.g., the attribute of a of being strictly identical with b.

Quine's question[35] designed to evoke bewilderment, as to modal properties of the cycling mathematician, c, only gets its point when we are *not* concerned purely with the referent of 'c'. Even then it is important to remove a familiar ambiguity, which Quine so works into the premisses as to increase the confusion. For the premisses could be represented (using obvious abbreviations, 'rat' for '(is) rational', 'twl' for '(is) two-legged') either:

1a. $(\Pi x)(\Box(\text{math}(x) \supset \text{rat}(x)) \;\&\; \sim \Box(\text{math}(x) \supset \text{twl}(x)))$
2a. $(\Pi x(\Box(\text{cyc}(x) \supset \text{twl}(x)) \;\&\; \sim \Box(\text{cyc}(x) \supset \text{rat}(x)))$

or:

1b. $(\Pi x)(\text{math}(x) \supset. \Box\, \text{rat}(x) \;\&\; \sim \Box\, \text{twl}(x))$
2b. $(\Pi x)(\text{cyc}(x) \supset. \Box\, \text{twl}(x) \;\&\; \sim \Box\, \text{rat}(x))$.

From the much more plausible a-premisses it follows, using: $\text{math}(c)$ & $\text{cyc}(c)$, that: $\text{rat}(c) \;\&\; \text{twl}(c) \;\&\; \sim \Box\text{rat}(c) \;\&\; \sim \Box\text{twl}(c)$. Hence: $\nabla\, \text{rat}(c)$ & $\nabla\, \text{twl}(c)$, i.e. c is contingently rational and contingently two-legged. It also follows that it is contingently true that c is rational and two-legged. These are (the) modal properties of the cycling mathematician c. But from the implausible b-premisses it follows that: $\sim \Diamond (\Sigma x)(\text{math}(x) \;\&\; \text{cyc}(x))$, i.e. it is impossible that anyone is both a mathematician and a cyclist.

The same modal fallacy principle, $\Box(p \supset q) \supset. p \supset \Box q$, which leads from a-premisses to b-premisses is needed to get from the correct

(12): $(\Pi w)(f(w) \equiv . \ w = x) \ \& \ (\Pi w)g(w) \equiv . \ w = x) \supset . \Box (\Pi w)(f(w) \equiv g(w))$

to

(13): $(\Pi x)(f(w) \equiv . \ w = x) \ \& \ (\Pi w)(g(w) \equiv . \ w = x) \supset . \Box (\Pi w)f(w) \equiv g(w)$[36],

the disastrous assumption Quine considers needed in order to interpret fully quantified modal logic. But (13) is invalid as counterexamples readily show; e.g. take 'f' to be 'is Venus' and 'g' 'is the morning star'. Also (13) is demonstrably not a theorem of =**S5R***: since $[p \supset \Box p]$, which (13) implies, is rejected so is (13). Why the modal-flattening assumption (13), as opposed to (12), is supposed to be needed for interpreting quantified modal logics is not made clear. In fact it is plausible only in the context of essentialism. If earlier arguments are cogent assumption (13) is definitely not required.

§5 *On a semantics for* =**S5R*** *and* =**S5R$_1$***

§5.1 *Semantics of* =**S5R***[37] A *model structure* (*m.s.*) S is an ordered quadruple $\langle K, G, R, d \rangle$, where K is a set, $G \varepsilon K$, R is an equivalence relation on K, d is a function which assigns for every member H of K a set $d(H)$ (at least one of which is non-null) called the *domain of individuals* of H. K represents a set of logically possible worlds or structures. G, which represents the (an) actual world, is a distinguished element of K. R represents an alternativeness relation between possible worlds; e.g. if H_1 and H_2 are possible worlds of set K, then $H_1 \ R \ H_2$ holds if H_2 is a modal alternative to H_1, i.e. logically possible relative to H_1, so that every statement true given H_1 is possible given H_2 and every individual existent given H_1 is possibly existent given H_2. To gain an **S5** modal logic the modal alternativeness relation R is required to be an equivalence relation. Semantics for systems based on weaker modal logics are reached by appropriately relaxing requirements on R; e.g. for system =**TR*** it is only required that R is a reflexive relation. Since it can be shown that for satisfiable wff only connected modal structures are needed, R can be omitted without loss when only **S5** systems are under discussion.[38]

A *superset* U of all possible distinct individual items (of all possibilia) is defined as the union over all $H \varepsilon K$ of $d(H)$, where x and y are distinct elements of U if they are distinct in some domain of individuals, i.e. the union of all strictly distinct possibilia. For instance, for suitable S Scott and the author of Waverley would be distinct in U even though they coincide in G (as the Scott-author of Waverley individual), U^n is the n^{th} Cartesian product of the set U with itself. An *augmented domain* $d^+(H)$ of H is defined as the set obtained from U by identifying those elements of U identified in $d(H)$, i.e. $x \ \varepsilon \ d^+(H)$ iff either $x \ \varepsilon \ d(H)$ or $x \ \varepsilon \ U$ but does not coincide with a (complex) individual in $d(H)$.

An *extensional model* (*e.m.*) m on m.s. S is defined as a function $m(f^n, H)$, $n \geq 0$, where f^n ranges over n-adic primitive predicates when $n \geq 1$ and over atomic sentences (propositional expressions) when $n = 0$, which assigns the following designation-values:

P(i) An element of U to each (consistent) individual variable. Thus individual variables are variables having elements of U as their designation-range.

P(ii) One of the truth possibilities (relative truth values) $\not t$ or $\not f$ to each propositional variable. That is, if $n = 0$, $m\ (f^n, H)$, i.e. $m\ (p, H)$ has value $\not t$ or $\not f$ ($= \not t$ or $\not f$). Thus propositional variables are variables having as their designation-range $\not t$ and $\not f$.

P(iii) A subset of U^n to each n-adic predicate variable. That is, if $x \geqslant 1$, $m(f^n, H)$ is a subset of U^n. Thus n-adic predicate variables are variables having as their designation-range ordered n-tuples of elements of U.

A *valuation* (or truth possibility assignment) t is defined as a ternary function $t(m, A, H)$ which, given m and H, assigns a truth possibility to each sentence A (of a given set). The definition of t is recursive on the length of sentences. It is convenient to abbreviate '$t(m, A, H)$' as '$m(A, H)$'. (Alternatively a valuation t (or m) on model m would be defined as a function $t\,(A, H)$ which assigns...) $m(A, H)$ is defined inductively, for every wff A and every $H \varepsilon K$, so that:

T(i). $m(p, H) = \not t$ or $\not f$ according as p is assigned $\not t$ or $\not f$ by $m(p, H)$ in P(ii); i.e. $t\,(m, p, H) = m(p, H)$.

T(ii). For an atomic wff $f^n(x_1, \ldots x_n)$, $x \geqslant 1$, given an assignment of elements $a_1, a_2, \ldots a_n$ of U to $x_1, x_2, \ldots x_n$ respectively, $m(f^n(x_1, ..x_n), H) = \not t$ or $\not f$ according as the n-tuple $(a_1, .. a_n)$ is or is not a member of $m(f_n, H)$.

T(iii). For an atomic wff $E(x)$, given an assignment of element a of U to x, $m(E(x), H) = \not t$ or $\not f$ according as a is or is not a member of $d(H)$.

T(iv). For an atomic wff $(x_1 = x_2)$, given an assignment of elements a_1, a_2 of U to x_1, x_2 respectively, $m((x_1 = x_2)H) = \not t$ or $\not f$ according as a_1 and a_2 are or are not the same element of $d^+(H)$, i.e. according as x_1 and x_2 are or are not assigned the same element of $d^+(H)$.

T(v). $m(\sim A, H) = \not t$ or $\not f$ according as $m(A, H) = \not f$ or $\not t$.

T(vi). $m(A \supset B,)H) = \not f$ if both $m(A, H) = \not t$ and $m(B, H) = \not f$; and $m(A \supset B, H) = \not t$ otherwise.

T(vii). $m(\Box A, H) = \not t$ if $m(A, H') = \not t$ for every $H' \varepsilon K$ for which $H R H'$; otherwise $m(\Box A, H) = \not f$.

T(viii). $m(\Pi x)A, H) = \not t$ if $m(A, H) = \not t$ for every assignment of an element z of U to x, i.e. for every designation-value of x; $m((\Pi x)A, H) = \not f$ otherwise.

Some *derived valuations* under t are worth recording. As before they are relative to a given assignment of elements of U to free variables of A.

T(ix). $m((x_1 \equiv x_2), H) = \not t$ if x_1 and x_2 are assigned the same element of $d^+(H')$ for every $H' \varepsilon K$ for which $H R H'$; otherwise $m((x_1 \equiv x_2), H) = \not f$.

T(x). $m((\forall x)A, H) = \not t$ if $m(A, H) = \not t$ for every assignment of an element a of $d(H)$ to x (including the null assignment); $m((\forall x)A, H) = \not f$ otherwise.

T(xi). $m(\Diamond A, H) = \not t$ if $m(A, H') = \not t$ for some $H' \varepsilon K$ for which $H R H'$; otherwise $m(\Diamond A, H) = \not f$.

In systems based on **S5**, A is possible iff $m(A, H) = \not t$ for some $H \varepsilon K$. Thus relative truth-values amount to truth possibilities. Also in such systems x possibly exists iff $m(E(x), H) = t$ for some $H \varepsilon K$. A *holds in* H if $m(A, H) = t$; roughly if A would be true in world H.

An *absolute valuation* $m(A)$ under m for A is defined: $m(A) = t$ iff $m(A, G) = t$; $m(A) = f$ iff $m(A, G) = f$. For example, $E(a)$ has value t under m if $a \varepsilon d(G)$; and $a = b$ has value t under m if a and b are assigned the same value in $d(G)$.

A *system* is an ordered pair (m, S), where m is an extensional model on m.s. S. A is *true in* system (m, S) if $m(A) = t$; false in (m, S) if $m(A) = f$. A is *valid* iff A is true in every system; *satisfiable* if A is true in some system.

When specific individual and predicate constants are added to the logic, as in applied logics, further sets of conditions, e.g. meaning postulates, have to be added to the interpretation to show inter-dependence relations. For instance, strictly identical individuals are identified in U.

The interpretation is given in the material mode. Reference is repeatedly made to possible items, to possibilia, and to possible worlds. To avoid difficulties such talk is thought to encounter, the interpretation could be presented in the formal mode. Then value-ranges of variables would be replaced by substitution-ranges consisting of classes of expressions[39]. This re-presentation would hardly improve matters, and it would have to surmount formal obstacles. Thus, unless expressions were used autonomously, the interpretation would be complicated by quotation devices. In any case the formal mode interpretation would appear inadequate, e.g. because the class of expressions is at most denumerable whereas the class of possibilia is presumably non-denumerable.

Alternatively talk of possibilia and possible worlds can be dispensed with in favour of set-theoretical jargon. But this strategy raises a serious dilemma for the normal modal logician: either his set-theoretical "interpretation" is a merely formalistic one, or it must use non-existential quantifiers. If, on the one hand, talk of possibilia and possible worlds is replaced by talk of uninterpreted elements (representable, e.g., by numbers) and uninterpreted sets of these, the "interpretation" is replaced by a bloodless formalism; it loses its semantical punch and its semantical links, and much of its point except insofar as it provides a decision procedure for certain classes of wff and a device for proving metatheorems of the logic. Such an abstract set-theoretical model fails to go any distance towards explaining - what is often thought to be the main object of a semantics - the (denotational) meaning of modal expressions. Such a failure is all the worse in the present case, since many philosophers think the meaning of modal expressions is radically unclear.

If on the other hand, the set-theoretical terminology is interpreted, over sets of individuals, then several of the individuals must be possibilia, and several of the sets, or worlds, must be *merely* possible. The world containing as well as Pegasus the contingently distinct individuals Scott and the author of Waverley is a merely possible set; it is possible but it does not exist. Not all logically possible worlds can exist. The modal logician, in interpreting his logic using rules like T(viii) and T(xi), is obliged to quantify over such merely possible worlds. If in doing so he is not to resort to Platonistic double talk according to which there exist worlds or sets which are merely possible - and consequently shouldn't exist - he must revise his quantification theory. Thus the modal logician who wants a genuine interpretation of his logics of the possible-worlds sort is forced into adopting non-existential quantifiers, like 'Π' and 'Σ'.

Once these quantifiers are selected the semantics of **S5R*** can be developed much more simply on the basis of the Leibnizian scheme:

$$\Box A \text{ iff } (\Pi w)(A; w) ,$$

i.e. A is logically necessary iff for all possible worlds w, A is true in w. Then T(v) is replaced by

T*(v): $(\sim A; w)$ iff $\sim(A, w)$;

T(viii) by

T*(viii): $((\Pi x)A; w)$ iff $(\Pi x)(A; w)$

and so on.

§5.2 *Results on* **S5R*** *-E*, i.e. on the system obtained from **S5R*** by deleting the predicate constant 'E' or, better, by construing 'E' as a predicate variable. These results are established by showing that results obtained by Kripke[40] for his system **S5*** can be transferred to **S5R***-E. First, rules can be prescribed for replacing any wff A of **S5R***-E by a wff A_1 of **S5*** and conversely. The relation is one-to-one.

1. *A is a theorem of* **S5R***-*E iff A_1 is a theorem of* **S5***.
2. *A of* **S5R***-*E is valid iff A_1 of* **S5*** *is universally valid.*

Proof sketch: R can be suppressed because only connected models need be considered. Correlate U with Kripke's D. Then A is valid iff $m(A, G) = \not t$ for every system (m,S), i.e. iff A is assigned $\not t$ by G for every system (m,S), i.e. iff A is assigned $\not t$ by G for every Kripke model (G,K) of A over U (i.e. over non-empty D), i.e. A is universally valid in **S5***. That every element $H \varepsilon K$ in a Kripke model (G, K) of A must agree with G in assignments for free individual variables of A results in no loss of generality for an **S5** system when *every* Kripke model (G, K) of A over D is taken; for then the set of worlds got from the set of systems for A coincides with the set of worlds got from the set of Kripke models of A.

Second, semantic tableaux constructions are defined, and when constructions are closed is explained[41]. A *construction for A* is a construction begun by putting A on the right of the main tableaux of the construction.

3. *A is valid iff the construction for A is closed.*

Proof. Semantic tableaux constructions for A of **S5R***-E are the same as tableaux constructions for A_1 of **S5***. Thus the result follows from Kripke's *Theorem 1* (*JSL*, repaired) by 1. and 2.

4. *A is a theorem of* **S5R***-*E iff A is valid.*

This result follows from Kripke's *Theorem 7* (*JSL*) for **S5*** using 1, 2, and 3.

5. *A is a theorem of* **S5R***-*E iff the construction for A is closed.*

From 3 and 4.

When '*E*' is introduced as a specific predicate constant with the valuation given in T(iii) these metatheorems fail; for it is not true that if the tableaux construction for *A* is not closed that *A* is not valid. For instance, $[\Diamond E(x)]$ is valid but its tableau is not closed. These troubles are rectified in $\mathbf{S5R_1}$*.

§5.3 *Results on* =**S5R***-*E*. The connexion between =**S5R***-*E* and =**S5*** is not so straightforward. *D* has to be split up into a set of (possibly overlapping) subdomains *d*(*H*) such that $D \doteqdot \bigcup_{H \varepsilon k} d(H)$. Also tableaux construction rule =1 is introduced and so proofs in each of the lemmas needed for main results which concern rule II have to be replaced by proofs for =1. The changes to be made in the critical lemmas 1, 2 and 4 of Kripke (*JSL*) when =1 replaces II are now sketched.

Lemma 1. If a = b appears in the left column of a table, then a = b is assigned $\dotplus$*; therefore a and b are assigned by G the same element of d(G).*

(Similarly for auxiliary tableaux with some member of *K*). This validates rule =1.

Lemma 2. (as repaired in *ZML*). *A (counter-) model m for a wff A which is not valid is so defined that* $m(p, H) = \dotplus$ *(p atomic, H ε K) iff p appears on the left of the tableau H associated with H, and* $m(p, H) = \not f$ *otherwise.*

And that every free variable in *H* is assigned a distinct value unless eliminated later in the construction by =1 in which case it is assigned the value of the variable which replaces it. If $a = b$ appears on the left of *H* then ultimately by application of =1, $b = b$ appears on the left. Hence by definition of *m* and T(iv) $m(a = b, H) = \dotplus$. If $a = b$ appears on the right of *H*, since the construction is not closed, *a* and *b* remain distinct after all replacements by =1. Hence by T(iv) $m(a = b, H) = \not f$, as required.

Lemma 4. Rule =1, *in contrast with* II, *applies to only one tableau of a set.*

It is justified at once by schema =**R2**.

Using the modified lemmas and the results of §5.2 it follows:

1. *A is a theorem of* =**S5R***-*E iff A is valid*
2. *A is a theorem of* =**S5R***-*E iff the construction for A is closed.*

§5.4 *A semantics for* =$\mathbf{S5R_1}$* *and results*. In order to guarantee Meinong's axiom it is required of an =$\mathbf{S5R_1}$* m.s. *S* that for every $x \varepsilon d(G)$ some *H* occurs in *K* such that *G R H* and $x \not\varepsilon d(H)$. (For the weaker axiom $(\Sigma x) \sim \Box E(x)$ it need only be required for any m.s. *S* that for some $x \varepsilon d(H)$, where *G R H* and $H \neq G$, $x \not\varepsilon d(G)$.) The interpretation axiom is already guaranteed. Apart from the above restriction on =$\mathbf{S5R_1}$* m.s. *S*, *exten-*

sional model, valuation, system and *valid* are defined as for the semantics of =**S5R***. But closure definitions are amended. A tableau is closed iff some wff A appears on both sides of the tableau or $a = a$ for some parameter a appears in its right column. A set of tableaux is *closed* iff some tableau in it is closed or $E(b)$ for some parameter b appears on the left of every tableau of the set or else appears on the right of every tableau of the set. A system of tableaux is *closed* iff each of its alternative sets is closed. A construction is *closed* iff at some stage a closed system of alternative sets appears.

Results on =**S5R***-E are extendable to =$\mathbf{S5R}_1$*. The lemmas cited in §5.3 are further amended.

Lemma 1. When the construction is closed there are further cases to consider. Every alternative set either contains a tableau which either has a wff in both columns or has $a = a$ on the right, or else consists of a set which has $E(b)$ on the left of every tableau of the set or has $E(c)$ on the right of every tableau of the set. But then some member of K either assigns both t and f to some wff or assigns f to $a = a$ or assigns t to $\Box E(b)$ or assigns t to $\Box \sim E(a)$. The valuation rules render these alternatives impossible. Consider the last two. If $b \varepsilon d(G)$ then $m(E(b),G) = t$ and for some H, by requirements on the m.s., $m(E(b),H) = f$. Hence $m(\Box E(x),G) = f$. If $b \not\varepsilon d(G)$ then $m(E(b),G) = f$, and thus $m(\Box E(b),G) = f$. Therefore for every $x \varepsilon U$, $m(\Box E(x),G) = f$. Also for every $x \varepsilon U$, there occurs some H such that $x \varepsilon d(H)$. Then $m(E(x),H) = t$ and $m(\Diamond E(x),G) = t$. Hence for every $x \varepsilon U$, $m(\Box \sim E(x),G) = f$. Since the alternatives are impossible, the reductio argument is complete.

Lemmas 2 & 4. as before. Note that because of conditions on model structures $\Box E(x)$ and $\Box \sim E(x)$ cannot be assigned t in the countermodel.

Using these lemmas and the results of §5.3 it follows:

1. *A is a theorem of* =$\mathbf{S5R}_1$* *iff A is valid*

2. *A is a theorem of* =$\mathbf{S5R}_1$* *iff the construction for A is closed.*

To prove these theorems some alterations must be made in Kripke's theorems (*JSL*), in particular to *Theorem 5.* There the further possibilities resulting from the amended closure definitions have to be considered.

§6 *On certain reduction principles in quantified modal logics*

§6.1 *In defence of an* **S5** *modal basis* Recently **S5** with its strong principles for reducing iterated modalities has been defended on the ground of its comparative strength compared with other normal modal systems, on grounds of completeness and of simplicity, and on the ground that it has the most natural modal-theoretic treatment[42]. While some of these arguments carry some weight, they are not going to succeed so long as reduction theorems of **S5** like

(1): $\Diamond p \supset \Box \Diamond p$
(2): $\Diamond \Box p \supset \Box p$

remain dubious under the intended interpretation. In fact (2) has sometimes been rejected on the basis of considerations like: (empirical) sentences such as ''Gold dissolves in aqua regia', 'Water freezes at 32°F' are not necessary but it is possible that they are necessary; and (1) on the basis of examples like: it is possible that there is a non-white swan but it is not logically necessary that this is possible. To vindicate choice of **S5** - a choice needed in §2 - these considerations must be met, and (1) and (2) must be so interpreted as to be rendered plausible. To achieve this the interpretation should be further fixed by these two measures:

(A) adoption of a sentence/statement (proposition) distinction under which a statement is invariant both under change of time and under change of sense of related sentences. Thus if a sentence representing a statement changes in sense it simply ceases to represent the statement in question. Exact details of the distinction are not crucial for its use here. To carry this measure through in detail quantified logic should be set up for propositional expressions and functions, not (except derivatively) for sentences and sentential functions.

(B) imposition of certain conditions on the *logical* modality-values, e.g. logical necessity, logical contingency, of *statements*; specifically that they are modally invariant, where a value of a statement is modally invariant if it is not possible that it change even if actual matters or states of affairs were (or became) different. For other ''modality-values'' of statements such as provability or epistemic possibility (defined in terms of 'it is known that' as: $\sim K \sim$) an **S5**-structure does not obtain.

Those enemies of **S5** who do not regard the introduction of modalities into logic as inadmissable may be divided into the **S4, T** or **S2** supporters who hesitate over (1) and (2) and the conventionalists, who to be consistent should reject not just **T** but even **S2.** Measure (B) is directed against those who support some *Lewis* modal logic weaker than **S5**, and against the conventionalists, measure (A) against the conventionalists.

Because of (A) the predicates '(sentence)' ... 'yields a logically necessary statement (expresses a necessary proposition)' (symbolised '$\dashv \Box$') and '(statement) ''....'' is logically necessary' ('$\Box$') must be distinguished. For '$\dashv \Box$' is a time-dependent predicate, and so not a predicate of statement expressions, whereas '$\Box$' is a time-independent predicate. It seems that conflation of these predicates and of sentential variables with propositional variables is a chief reason for rejection of theorems special to **S4** and **S5**. (2) is conflated with

(3): $\Diamond[\dashv \Box p] \supset [\dashv \Box p]$

which is certainly not analytic. Even if the sentence 'Gold dissolves in aqua regia', for example, does yield a contingent statement, not a necessary one, it is still possible that it yields a necessary statement: but only because it is possible that the senses of some words in the sentence change and thus that the sentence comes to yield a different statement. Indeed if '$\Box$' is replaced by '$\dashv \Box$' not only (1) and (2) but the characteristic theses of **S4** fail.[43] More important, the same or similar arguments to those used to repel counterexamples to, and to defend the characteristic theses of **S4** against conventionalism on the basis of a sentence/proposition distinction can be employed to defend typical theorems of **S5**[44]. Typical theorems of **S4** and **S5** can be said to distinguish the modal logic (with respect to logical modalities) of propositions. The close connexions between **S4** and **S5** in this respect can be brought out by comparing theorems which distinguish them from **S2** or from Feys' system **T**; and few comparisons are more revealing than that between

(4): $[p \circ q] \ominus \Diamond [p \ominus q]$,
(5): $[p \ominus q] \ominus \Diamond [p \circ q]$,

where these are added to **T** or **S2** reformulated with consistency connective 'o' as modal primitive. Connective '$\ominus$' symbolises 'is logically inconsistent with'. Thus according to (4) the logical consistency of two propositions is inconsistent with their possible inconsistency, while according to (5) the inconsistency of two propositions is inconsistent with their possible consistency. Although these theses look as if they stand or fall together addition of (4) strengthens **S2** or **T** to **S5**, but of (5) only to **S4**.

(A) does not, on its own, provide nearly as strong a case as can be made out for (1) and (2). Consider, to illustrate, the following argument for (2). Suppose (2) is false, i.e. $[\Diamond \Box p \ \& \sim \Box p]$. If p is impossible then it is not possible that p is necessary; a consequence which follows even in **T**. But if p is contingent it is not necessary. Moreover it is not possible that it is necessary; for if it were it would have to be possible for p to vary in modality-value, which is impossible if (B) is correct. If the value did vary a different statement would result. Conditions like (B), however, need some support. Supporting arguments can be divided, very roughly, into two main classes: those which appeal to some analysis or account of logical modality-values, e.g. in terms of analyticity, logical truth or sense, and those which do not.

Now the first class of arguments run has been set out explicitly in the case when logical necessity is explicated in terms of L-truth by Carnap[45]. Similar arguments can be used to show that other analyses of logical necessity, which are less problematic, also lead to the distinctive theorems of **S4** and **S5**. Consider, for instance, the following non-conventionalist analysis of the truth-conditions for the logical necessity of statement p; namely, statement p is logically necessary iff p is true and the truth of p is a consequence of the logical content of p (or, alternatively, of the sense of qu(p)). Suitable theories of consequence and of logical content can be adopted from the work of Tarski and Carnap.

The second class of arguments presuppose less. Consider two such arguments: (i) Although it is possible that states of affairs be different so that statements which are true would if matters were different be false, it is not possible that logically possible matters might have been different. This marked contrast between truth-values and modality-values can be stressed by contrasting

(6): $p \ominus \Diamond \sim p$,
(7): $p \ominus \Diamond \sim p$,

provided p is modalised, i.e. occurs within the scope of a modal operator. (6) is false[46]: if adopted it would collapse modal logic into propositional logic. But (7), of which (4) and (5) are instances, seems to be true, and if added to **T** strengthens it to **S5.** But **T** without (7), like **S4** without all cases of (7), only incompletely formulates propositional modal logic. (7) can be taken as saying that a statement's having a given modality is inconsistent with the possibility of its not having that modality, that a statement's modalities are modally invariant. In contrast the rejected (6) can be taken to claim in addition that a statement's truth-values are modally invariant. To put the point another way, the truth of a contingent statement is contingent, but the contingency of a contingent statement is not contingent, i.e. $\sim\nabla\nabla p$. The modal invariance of modal propositional functors can in fact be represented by any one of the following theses: $\sim\nabla\Box p$, $\sim\nabla\Diamond p$, $\sim\nabla\nabla p$. Any one of these anti-conventionalist theses added to **S3** or **T** leads to an **S5** modal structure. Aren't these theses very plausible: isn't the necessity of a proposition a non-contingent matter?

(ii) Modal invariance of modality-values can also be supported by consideration of relations between possible worlds. A possible world is a modal alternative (see §5.1) to the real world or to some other possible world. Now it can be convincingly argued that the intuitive relation of modal alternativeness is not only reflexive as is required for **T** and **S2**, but also symmetric and transitive. But a structure where modal alternativeness is an equivalence relation can, given usual value assignments, only be completely axiomatised by an **S5** system. To illustrate: if it is possible that there is a non-white swan then in some possible world it is true that there is a non-white swan. Then, given, what is usually assumed, that the alternativeness relation is an equivalence one and is connected, it is true for all possible worlds that in some possible world there is a non-white swan; so it *is* necessary that it is possible that there is a non-white swan.

§6.2 *On the reduction of problematic modal expressions of* **S5R*** The *problematic* modal expressions of **S5R*** are expressions of the form: δA, where 'A' is a predicate expression containing free variables and 'δ' is a modal functor. Whereas the modal functors of other - non-problematic -- modal expressions of **S5R*** have a fairly straightforward *de dicto* rendering, the modal functors of problematic modal expressions are sometimes supposed to represent *de re* modalities (in one sense of this dubious[47] medieval distinction); actually they also have a *de dicto* reading. At any rate there are more difficulties about how problematic expressions such as '$\Box f_0(x)$' are to be construed than there are about non-problematic expressions or about expressions which don't contain modal functors such as '$f_0(x)$'. Thus it is an important question whether problematic modal expressions can be eliminated in favour of at least logically equivalent non-problematic expressions or in favour of sets of such expressions. Since all iterated modalities collapse in **S5R*** and since all variables can be bound there are only four main problematic modal schemes to consider: these can be typified using the sample predicate 'f' by:

$$(\Pi x) \Box f(x),\ (\Sigma x) \Diamond f(x),\ (\Pi x) \Diamond f(x),\ (\Sigma x) \Box f(x)$$

Now the first two can be eliminated using equivalences ④ and ② of §2. Can the last two be eliminated? Both von Wright and Kneale claim that in the case of logical modalities they can. If they can be eliminated not only is **S5R*** defective under interpretation because it contains too many distinctions[47]; also combination of modalities with quantification loses some of its interest[49].

Von Wright bases his elimination proposal on his principle of predication[49], a principle which can be formulated, when significance conditions are omitted, as,

(8): $(\Sigma x)(\Diamond f(x) \supset \Box f(x)) \supset (\Pi x)(\Diamond f(x) \supset \Box f(x))$[50]

On the strength of this principle von Wright divides attributes into two classes: logical and descriptive. Then separate elimination schemes are suggested for problematic modal expressions according as the property specified is logical or descriptive, e.g. '$(\Sigma x)\Box f(x)$' is eliminated using: if f is logical, $(\Sigma x)\Box f(x) \equiv (\Sigma x)f(x)$; if f is descriptive, $(\Sigma x)\Box f(x) \equiv \not f\ (\equiv \Box (\Sigma x)f(x))$. Von Wright does not propose, contrary to what Prior[48] is inclined to suggest, a single unconditional elimination scheme such as is illustrated by

(9): $(\Sigma x)\Box f(x) \equiv \Box(\Sigma x)f(x)$

However given a very plausible condition on logical properties (9) does follow from von Wright's elimination schemes. Now not only is (9) false: it does not follow from (8) as can be shown by construction of the relevant semantic tableau. Principle (8) does not appear to provide a single elimination scheme. Any scheme it did furnish would be as unsatisfactory as the principle (8) on which it is based. Principle (8), since a *some* to *all* implication, is not valid in **S5R***; also it fails for higher order properties[48]. But worse, the principle and the proposed elimination schemes are implausible once a purely referential theory of the role and meaning of variables and constants is abandoned. Whether a property belongs necessarily to a subject which has it does not as a rule depend just on the sort of attribute; it also depends (except in the case of L-empty predicates and their negations) on the description or mode of referring to the subject[51]. As soon as it is admitted that true ascriptions of modal properties to subjects depend also on the *sense* of the subject expressions that classification of properties as logical or descriptive which rests on the principle of predication breaks down. Thus the dichotomy essential for von Wright's replacement of problematic modal expressions is destroyed. Furthermore even when f is an example of what von Wright would class as a descriptive property, e.g. a simple colour property, $(\Sigma x)\Box f(x)$ is not automatically false. The same mistaken assumptions are made in the principle of predication as are made in some of Quine's arguments (in §4) and in modal paradoxes.

Kneale distinguishes two interpretations of '$(\exists x)\Box f(x)$'. Under the first essentialistic, and inadequate, interpretation, as 'there is something which under any description is necessarily *f*', the statement is reckoned to be equivalent to $\Box(\forall x)f(x)$. The second interpretation of '$(\exists x)\Box f(x)$', as 'there is something which under some description is necessarily *f*', is more important. Then, Kneale writes[52], '. . . $(\exists x)f(x)$ cannot express a true proposition unless there is something which among its permissible descriptions has one entailing the predicate $f(x)$. But this is as much as to say that the disputed formula is equivalent to $(\exists x)f(x)$. Therefore $(\exists x)\Box f(x)$ cannot on either interpretation represent a new kind of proposition'. The argument is invalid. Kneale asserts what is tantamount to: $(\exists x)\Box f(x) \supset (\exists x)f(x)$, which is correct. But this is *not* to say: $(\exists x)f(x) \equiv (\exists x)\Box f(x)$ as Kneale claims. Kneale does not show how: $(\exists x)f(x) \supset (\exists x)\Box f(x)$ follows. It does not: it is not a theorem and not valid under the second intended interpretation. There are critical limitations on the ways an item may be correctly described.

NOTES

1. On the defects and difficulties of such theorems see, e.g., W. and M. Kneale, *The Development of Logic*, Oxford (1962), and A. N. Prior, *Time and Modality*, Oxford (1957).

2. See, especially, W. V. Quine: "The problem of interpreting modal logic," *The Journal of Symbolic Logic*, vol. 12 (1947), pp. 43-48; *From a Logical Point of View*, Revised edition, Cambridge, Mass. (1961); *Word and Object*, New York (1960).

3. For fuller elaboration of the interpretation of **R*** and details of the consequences of these changes, see R. Routley, "Some things do not exist," *Notre Dame Journal of Formal Logic*, vol. 7 (1966), pp. 251-276. The interpretation of **S5R*** is more fully explained below.

4. S. A. Kripke, "Semantical considerations on modal logic," *Proceedings of a Colloquium on Modal and Many-Valued Logics*: *Acta Philosophica Fennica*, Fasc. XVI, Helsinki (1963), pp. 89-90.

5. Compare Routley, *op. cit.*

6. Adapted from Kneale, *op. cit.*, p. 614.

7. For preliminary explanations of these symbols see Routley, *op. cit.*, pp. 260-261. Details of the analysis using unlimited quantifiers '*A*' and '*S*' are given in my "Exploring Meinong's Jungle" (unpublished).

8. Rules for constructing semantic tableaux for quantified modal formulae of **S5R***-*E* (i.e. without the specially interpreted predicate '*E*'), and a metatheorem that *A* is a theorem of **S5R***-*E* if and only if its tableau construction is closed, can be adapted from S. A. Kripke, "A completeness theorem in modal logic," *The Journal of Symbolic Logic*, vol. 24 (1959), pp. 1-14 (abbreviated *JSL*). Details of the semantics of **S5R*** are presented in § 5.1. The relevant theorems are established in § 5.2.

9. The postulate set is based on Church's set for F^1_2: see A. Church, *Introduction to Mathematical Logic: Part I*, Princeton (1956), pp. 218-219. The substitution notation used throughout, with the exception of '$\check{\mathsf{S}}^x_y A|$' is that of Church. The definition of '$\check{\mathsf{S}}^x_y A|$' parallels Church's explanation of '$\check{\mathsf{S}}^p_B A|$'. The system is really a quantificational extension of a rule simplified formulation of Lewis' **S5**. For philosophical purposes Lewis' formulations of modal logics are superior to the weakly equivalent Gödel formulations.

10. See, e.g., A. N. Prior, *Formal Logic*, pp. 205-6. Prior's attempt to explain away the incredibility which (1) produces is unconvincing.

11. See, especially, work on brain process theories of mind and on scientific reductions: for instance, text of and references in J. J. C. Smart, *Philosophy and Scientific Realism*, London (1963), and V. Macrae and R. Routley, "On the identity of sensations and physiological occurrences," *American Philosophical Quarterly*, vol. 3 (1966), pp. 87-110.

12. This move is not new. A more elaborate treatment than the standard formal treatment has been repeatedly indicated by philosophers: for recent examples see G. E. Hughes, "Mr. Martin on the Incarnation," *Australasian Journal of Philosophy*, 40 (1962), p. 208; N. Malcolm, "Scientific Materialism and the Identity Theory," *The Journal of Philosophy*, LX, 22 (1963), p. 663; P. T. Geach, *Mental Acts*, Routledge & Kegan Paul, London (1957), p. 84. That the Leibniz definition needs *some* qualifications has been recognised for a long time: see W. E. Johnson, *Logic*, Pt. I, Cambridge (1921), and in particular, A. N. Whitehead and B. Russell, *Principia Mathematica*, Vol. I, Cambridge (1911), p. 61.

    What is done to amend the Leibniz criterion resembles what has periodically been suggested by logicians, and in particular what was early recommended by Miss Barcan and has more recently been developed by her, in a second-order predicate logic, as a distinction between sorts of equality: see *Proceedings of a Colloquium on Many-Valued and Modal Logics*, *op. cit.* Some of the formal work has been anticipated by J. Hintikka, who builds up a good case for qualifying substitutivity of identity: see *Knowledge and Belief*, Cornell (1962), pp. 132-136. Hintikka introduces rules allowing for substitutivity of identity only for certain classes of primitive expressions of his modal systems. A similar move could be made within usual presentations of restricted predicate logics, and the conditions on extensional identity then derived by an induction argument.

13. For reasons for the challenge see Macrae and Routley, *op. cit.*

14. Compare A. Church, *op. cit.*, p. 281. The axiom scheme differs from Church's scheme only in the qualification, 'nor modalised'. Church's schema coincides with that for strict identity, provided the only intensional operators are modal operators.

15. Initial steps in each case rely upon preliminary lemmas. These lemmas are analogous for **S5R*** of Church's **313, **314, **315, and proofs are analogous to those of **226, **227; see A. Church, *op. cit.*

16. S. A. Kripke, *JSL*, *op. cit.* That tableaux constructed using $=1$ serve to show that *1 and *2 are not theorems only follows given modifications both of Kripke's definitions and of his theorems 1 and 7. For the requisite changes and theorems see §5.3.

17. "Non-existence does not exist," *Notre Dame Journal of Formal Logic* (forthcoming).

18. Angle quotes represent the quotation function '**qu**' of L. Goddard and R. Routley, "Use, Mention & Quotation," *The Australasian Journal of Philosophy*, vol. 44 (1966), pp. 1-49.

19. These definitions are got from Quine's definitions in *Word and Object, op. cit.* (abbreviated *WO*), by replacing '∀' by 'Π' and distinguishing identity criteria. Quine's definitions are not unambiguous; e.g. a more satisfactory definition of *r-transparency* uses:

$$(\Pi f, x, y) \cdot (x = y \supset \cdot f(x) \equiv f(y)) \supset \cdot x = y \supset \cdot hf(x) \equiv hf(y).$$

20. For such a theory, see L. Goddard and R. Routley, *op. cit.*

21. See, e.g., A. Church, "On Carnap's analysis of statements of assertion and belief," *Analysis*, 10 (1950), pp. 97-99.

22. A. F. Smullyan, "Modality and description," *JSL*, vol. 13 (1948), pp. 31-37; A. N. Prior, "Is the concept of referential opacity really necessary?", *Proceedings of a Colloquium on Modal and Many-Valued Logics*, *op. cit.*

23. For the theory of descriptions assumed, see Routley, "Some things do not exist," *op. cit.*

24. See Quine, *From a Logical Point of View* (abbreviated *LP*) *op. cit.*, pp. 152-4 for references and criticism. It is at least very dubious whether Carnap pursues the course attributed to him by Quine, whether Carnap's variables are limited to intensional values. Those formal techniques outlined in *Meaning and Necessity*, which are designed to divert modal paradoxes, and which are independent of the (inadequate) analysis of analyticity in terms of *L*-truth and ultimately in terms of state descriptions, are similar to some of those explained in § 3. But not only do the interpretations differ markedly. Further, whereas the solution proposed in (I) specifically qualifies Leibniz's criterion and applies *directly* to puzzles concerning identity, Carnap's "solution" is much less specific and direct: it requires "translation" of the paradoxes into the notation of his semantical systems. Also Carnap's exposition of some vital notions, e.g. of 'individual concept' or as it should be 'self-consistent individual concept' and of '*x* is the same individual as *y*' in rule of truth *3-3*, is insufficiently explicit. Very roughly Carnap's "solution" is the formal mode analogue of the solution proposed in (I).

25. See, especially, W. V. Quine, "Three grades of modal involvement," *Proceedings of XI'th International Congress of Philosophy*, vol. 14, Brussels (1953), pp. 65-81.

26. Quine relies on this sort of argument to get his critique of modality moving: see *LP*, p. 139.

27. P. Geach and M. Black (eds). *Translations from the Philosophical Writings of Gottlob Frege*, Oxford (1960), pp. 56-57. Quine comes close to repeating some of Frege's points when he writes 'Being necessarily or possibly thus and so . . . depends on the manner of referring to the object' *LP*, p. 148. The resulting undermining of the full-strength identity principle has not often been noticed and is not indicated by Frege though his identity principle is effectively qualified through the theory of change of references in oblique contexts.

28. In place of Quine's intuitive criterion (ii), in *JSL*, 1947, the following principles, which accord with earlier theorems, are used:

(i) A particular quantification is true if for some consistent constant '*c*' the substitution of '*c*' for the variable of quantification would render the matrix statement true.

(ii) An existential quantification is true if for some constant, '*c*', $E(c)$ is true, and the substitution of '*c*' for the variable of quantification would render the matrix statement true.

29. *LP*, p. 148; *WO*, p. 147.

30. Made in *JSL*, p. 47.

31. *LP*, pp. 152-3.

32. *LP*, p. 152.

33. That range or designation values of variables are intensions is *not* established by the following invalid argument (effectively that used by Quine against Carnap in *Meaning and Necessity*, *op. cit.*, pp. 196-7):

We have that (Πx) $(x \equiv x)$, i.e. every item (entity) is strictly identical to itself. This is the same as saying that items between which strict identity fails are distinct items—a clear indication that the *values* of variables are intensions, e.g. individual concepts rather than individuals.

For saying that every item is strictly identical with itself is not the same as saying that items between which strict identity fails are distinct items: they may in fact, be (extensionally) identical. (Πx) $(x = x)$ is also true.

34. *LP*, pp. 155-6; *WO*, p. 199. By 'Aristotelian essentialism' is here meant: that essentialism, attributed by Quine to Aristotle, under which 'an object of itself and by whatever name or none, must be seen as having some of its traits necessarily and others contingently, despite the fact that the latter traits follow just as analytically from some ways of specifying the object as the former traits do from other ways of specifying it' (*LP*, p. 155). Rejection of this essentialism is consistent with retention of intensional and modal properties, and also with recognition that objects may have some properties such as being either green or not green necessarily.

Quine is right in rejecting essentialism. Essentialism does collapse modality.

35. *WO*, p. 199.

36. Approximately Quine's assumption (4), *WO*, p. 198.

37. The account given is based on that of Kripke: S. A. Kripke, "Semantical considerations on modal logic," (*SC*), *op. cit.* It is also influenced by the work of A. Church, *Introduction to Mathematical Logic*, *op. cit.*; R. Carnap, *op. cit.*; and J. Hintikka, *op. cit.*

38. See, S. A. Kripke, "Semantical Analysis of Modal Logic I: Normal Modal Propositional Calculi," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 9 (1963), pp. 67-96.

39. See, e.g., Routley, *op. cit.*

40. S. A. Kripke, *JSL*. It is assumed that appropriate repairs are made to the defective lemma 2 of *JSL* along lines elaborated by Kripke, *ZML*.

41. The procedures, rules and definitions are those of Kripke's *JSL* as improved in *ZML*. More details of tableaux constructions and the rules for '$\supset$' appear in E. W. Beth, *The Foundations of Mathematics*, Amsterdam (1959), section 68.

42. See, e.g., A. N. Prior, *Formal Logic*, *op. cit.*; R. Montague, "Syntactical treatment of modality," *Proceedings of a Colloquium on Modal and Many-Valued Logics*, *op. cit.*, p. 161.

43. If mixed modalities are introduced systems with formal affinities to Lewis systems **S7** and **S8** result, e.g. $\Diamond(\neg\Diamond p)$ is a thesis, and also $\Box\Diamond(\neg\Diamond p)$ if an **S5** structure holds for propositional modalities.

44. For a detailed presentation of these arguments see A. Pap, *Semantics and Necessary Truth*, New Haven (1958), pp. 119-127.

45. R. Carnap, "Modalities and Quantification," *JSL*, 11 (1946), pp. 34-36.

46. Though apparently acceptable to Peirce for possibility in his substantive sense of 'possible': see C. Hartshorne and P. Weiss (eds.), *Collected Papers of Charles Sanders Peirce*, Vol. III, Cambridge, Mass. (1933), 3.527. (6) can be retained at the expense of weakening some of the normal modal connections.

47. For elaboration see Kneale, *op. cit.*, p. 616.

48. Prior, *Formal Logic*, *op. cit.*, pp. 211-214.

49. G. von Wright, *An Essay in Modal Logic*, North-Holland, Amsterdam (1951), p. 27.

50. This formulation is equivalent (given appropriate readings of quantifiers) to Prior's formulation; *ibid.*, p. 211.

51. Kneale drives this point home beautifully: *op. cit.*, p. 616. Von Wright cites as typical logical properties arithmetical properties. But as Kneale says: 'Being less than 13 is an arithmetical attribute, and we may, if we like, say that it belongs necessarily to the number 12; but it is false that the number of apostles is necessarily less than 13, although the number of apostles is undoubtedly 12.'

52. Kneale, *op. cit.*, p. 618. I have replaced 'Fx' by '$f(x)$'.

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